

Multiplicity-one fusion rings of ranks nine and ten

Evidence, constructions, and exact classification ledger

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Abstract

This package studies the supplied 142 rank-nine and 194 rank-ten multiplicity-one based fusion rings over the complex numbers. It combines standard obstructions, a signed-sphericalization reduction, exact constraints on restrictions of central simple objects, Morita-theoretic arguments, and explicit categorical constructions. Every accepted outcome is attached to the original input line, a tensor hash, a proof, and a deterministic replay. The counts below are generated from the accompanying ledger. An existence entry means that at least one unitary categorification is constructed; it does not classify all categorifications of that ring. Every supplied multiplicity-one ring at ranks nine and ten is decided. In particular, the cyclic rank-ten ring is realized inside the classical conformal-embedding module category for $SU(4)_6 \subset SU(10)_1$.

Rank	Inputs	Excluded over $\mathbb{C}$	Unitary models	Unresolved
9	142	102	40	0
10	194	136	58	0
Total	336	238	98	0

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1 Scope and interpretation

The identifiers $R9_1_i$ and $R10_1_i$ refer to physical lines i in the two supplied compressed multiplication-table files. The tensor convention is $X_i X_j = \sum_k N_{ij}^k X_k$, with the unit in position zero. The input audit checks integrality, nonnegativity, the multiplicity-one bound, unit, duality, Frobenius reciprocity, all associativity identities, and absence of duplicate based rings. The supplied tables agree, up to based isomorphism, with the corresponding AnyonWiki multiplicity-one census snapshot; the crosswalk is included. This audits the supplied census, rather than repeating the original enumeration of every possible based ring of these ranks.

An exclusion is a statement about all fusion categories over $\mathbb{C}$. A test requiring a spherical or pseudo-unitary category is used only when a separate argument supplies that hypothesis. Weak integrality supplies pseudo-unitarity by ENO. For the indicated irrational cases, the signed-sphericalization certificates and an exact character analysis supply a Galois conjugate with positive categorical dimensions. A unitary model is an existence proof, not an inference from successful necessary tests.

The machine-readable ledger is authoritative for the current accepted status. The supplementary directories contain full mathematical proofs and explicit phase data. Searches that fail to find a model, floating-point optimization failures, and experimental induction matrices are not treated as exclusions. The repository and website publish the same accepted ledger, with individual case pages linking to the full evidence preserved here.

2 Standard criteria and their scope

The baseline replay accepts 474 obstruction receipts on 203 distinct input rings; overlapping receipts are not added to the case count. It verifies each certificate directly against the original tensor. The following families are used.

Criterion	Precise role and scope
Formal-codegree d -numbers	General fusion categories. The regular Casimir has eigenvalues $\dim(E)f_E$; the d -number property is unchanged by a nonzero rational factor. An irreducible polynomial gives an exact divisibility failure.
Cyclotomicity	Applied to commutative based rings, where irreducible characters are scalar. Exact polynomial and prime witnesses rule out an abelian splitting field.
Drinfeld-center dimension integrality	Applied to integral rings, whose categorifications would be pseudo-unitary, including a separately checked noncommutative implementation.
Near-group restrictions	Used with their exact hypotheses, including the divisibility condition for irrational two-value dimension profiles and the abelian stabilizer condition for Tambara–Yamagami fusion rules.
Excluded fusion subrings	A listed closed based subring carries its own direct obstruction certificate. Absence from a realization catalogue is never sufficient.
Involution module constraints	Exact nonnegative integer module-matrix identities attached to the specified invertible object.
Two-dimension induction	The required central dimensions cannot be written in the nonnegative integral dimension semiring with the prescribed unit multiplicity.
Zero-spectrum and one-spectrum criteria	Direct multiplicity-free pentagon obstructions, valid for general complex fusion categories.

The report deliberately does not use a unitary Schur-product obstruction, an unproved

universal Isaacs property, or a low-rank modular classification as an exclusion of arbitrary fusion categories of ranks nine and ten. The survey and reference scope checks are in `work/literature/notes.md`. The 40 subring receipts include two rank-six subrings for which the replay supplies a direct dimension-lattice witness, so no catalogue inference remains.

3 A unified central-restriction obstruction

Suppose $\mathcal{C}$ is pseudo-unitary. Let $D = \text{FPdim}(\mathcal{C})$, $d_i = \text{FPdim}(X_i)$, and let $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ be the forgetful functor with adjoint I . For each irreducible representation E of the complexified Grothendieck ring, put $e_E = \dim E$ and let f_E be its formal codegree. There is a distinct simple central object A_E occurring in $I(1)$ with multiplicity e_E , with

$$\beta_E = \dim A_E = D/f_E, \quad v_{E,i} = [F(A_E) : X_i], \quad v_{E,0} = e_E.$$

Consequently v_E is a nonnegative integer vector, its class is central, and $\sum_i d_i v_{E,i} = \beta_E$. Let

$$B_{ij} = [FI(X_j) : X_i] = \sum_{a,k} N_{aj}^k N_{k,a^*}^i.$$

If the columns of the full restriction matrix are v_Z , then $B = \sum_Z v_Z v_Z^t$. For any chosen subset of known central simples,

$$B - \sum_E v_E v_E^t$$

must be entrywise nonnegative and positive semidefinite. These are necessary conditions, and their use does not assert that a surviving matrix is categorifiable.

For scalar characters χ, ψ , the symmetry of the central modular S -matrix yields the particularly useful condition

$$\beta_\chi \chi(v_\psi) = \beta_\psi \psi(v_\chi).$$

For higher-dimensional E , the character normalization is $S_{A_E, Z} = \beta_E \chi_E(FZ)/e_E$; omitting e_E would be incorrect. The final rank-ten exclusion replay uses only independently verified scalar characters, so it requires neither a complete character table nor this higher-dimensional extension. Ostrik's induction theorem [3, Theorem 2.13] and Deshpande's character formula [4, Equations (35)–(36)] provide the categorical input.

This single system unifies dimension-lattice, central-support, reciprocity, and induction-Gram obstructions. A failure at any stage is a rigorous exclusion after the required pseudo-unitarity reduction. The exact arithmetic is performed in rational number fields or cyclotomic fields, with no numerical tolerance in the final decisions.

3.1 Twist moments in integral dimension sixty

For a nonunit simple X of dimension d , put $J = I(X)$, so $\dim J = Dd$. The summands A_E of $I(1)$ have twist one. If their forced total dimension inside J is M , the remaining twist eigenvalues lie on the unit circle. The indicator identities [13] give

$$\text{Tr}(\theta_J) = 0, \quad \text{Tr}(\theta_J^2) = D\nu_2(X), \quad \nu_2(X) \leq 1.$$

Writing real parts of the residual eigenvalues as x , their weighted first moment is $-M$ and their weighted second moment is $(Dd + D\nu_2(X) - 2M)/2$. Cauchy–Schwarz implies

$$M \leq \frac{Dd(d + \nu_2(X))}{3d + \nu_2(X)} \leq \frac{Dd(d + 1)}{3d + 1}.$$

The denominator is positive for $d \geq 1$. The last bound is increasing in $\nu_2(X)$, so it remains valid for non-self-dual simples, whose indicator is zero. For the four rings $R9_1_2, 48, 84, 101$, exact enumeration of the low central columns leaves 404 candidate frames, and each violates this bound for a specified simple. The replay checks every branch and every moment inequality. Two of these four inputs already had baseline exclusions; only two are new cases in the total.

4 A sign-associativity criterion and a rank-nine exclusion

All categories in this section are fusion categories over $\mathbb{C}$. Let R be a multiplicity-free fusion ring whose distinguished basis $b_0 = 1, b_1, \dots, b_{r-1}$ is self-dual. An *admissible sign table* is an associative integral multiplication table T with unit b_0 , $T_{ij}^0 = \delta_{ij}$, full symmetry of T_{ij}^k in i, j, k , and $|T_{ij}^k| = N_{ij}^k$. A basis-sign change has the form $T_{ij}^k \mapsto s_i s_j s_k T_{ij}^k$, where $s_0 = 1$ and $s_i \in \{1, -1\}$.

Proposition 4.1 (Sign-associativity criterion). *If every admissible sign table of R is a basis-sign change of R , then every fusion categorification of R admits a spherical structure. Consequently, if no formal codegree D of R has D/f an algebraic integer for every formal codegree f , then R has no categorification.*

Proof. Let $\mathcal{C}$ be a hypothetical categorification and take its canonical sphericalization $\tilde{\mathcal{C}}$ [1, Remark 3.1][3, Section 5.1]. The two lifts of each simple object differ by the sign object ε , of spherical dimension -1 . Every lift of a self-dual simple is self-dual: its dual has the same nonzero spherical dimension, whereas the other lift has the opposite dimension. Choose a unit-preserving set of lifts. In $K_0(\tilde{\mathcal{C}})/(\varepsilon + 1)$, their structure constants form an admissible sign table. Self-duality makes this quotient commutative, and Frobenius reciprocity gives full symmetry. Multiplicity one makes each nonzero coefficient equal to 1 or -1 .

By hypothesis, tensoring some selected lifts with ε makes all nonzero quotient coefficients positive. The product of two such lifts has exactly the selected lift in every nonzero channel: the total multiplicity in the two lifts is one and their difference is $+1$. Hence the selected lifts span a spherical fusion subcategory of $\tilde{\mathcal{C}}$. The forgetful functor restricts to a tensor equivalence with $\mathcal{C}$, proving sphericity.

For a spherical categorification, its dimension D is the formal codegree of its dimension character. Every D/f is an algebraic integer by [3, Theorem 2.13]. The last assertion follows. $\square$

Corollary 4.2. *The supplied ring $R9_1_41$, equivalently the AnyonWiki ring $\text{FR}_{38}^{9,0}$, has no fusion categorification over $\mathbb{C}$.*

Proof. Use the zero-based basis order in the supplied tensor. There are 47 nonzero unordered triples of non-unit labels. Basis-sign changes allow the following eight coefficients to be set equal to $+1$:

$$122, \quad 228, \quad 236, \quad 237, \quad 245, \quad 247, \quad 256, \quad 334.$$

Here ijk denotes T_{ij}^k . Independence of these eight gauge conditions is an elementary rank-eight calculation over $\mathbb{F}_2$. Thirty-six two-term consequences of associativity give the following table; the remaining nonzero coefficients are all $+1$:

a	$335, 347, 366, 446, 455, 567, 777$
b	$336, 344, 357, 456, 477, 555, 667$
c	$337, 346, 355, 444, 457, 566, 677.$

The b_4 coefficient in $(b_3 b_3) b_4 = b_3 (b_3 b_4)$ then reads

$$a + ab + c = 3.$$

Since $a, b, c \in \{1, -1\}$, this forces $a = b = c = 1$. Thus every admissible sign table is gauge-equivalent to R . The supplementary certificate lists each of the 36 source associativity identities and checks every deduction by exact integer arithmetic. No indicator normalization is used.

The characteristic polynomial of the regular Casimir matrix is

$$(x - 15)^6 (x - 2) (x^2 - 120x + 1200).$$

Thus the possible formal codegrees D are 2, 15, and $60 \pm 20\sqrt{6}$. The first two fail because $2/15$ and $15/2$ are nonintegral rationals. For the last two,

$$N_{\mathbb{Q}(\sqrt{6})/\mathbb{Q}}\left(\frac{60 \pm 20\sqrt{6}}{15}\right) = \frac{16}{3},$$

so the ratios are not algebraic integers. The proposition excludes general categorification. $\square$

Relation to the literature. Vercleyen–Slingerland, *On Low-Rank Multiplicity-Free Complex Fusion Categories*, Communications in Mathematical Physics **407**, 231 (2026), Remark 4.3, DOI 10.1007/s00220-026-05705-2, record the pivotal exclusion of this ring. The sign-associativity argument above supplies the missing reduction from general categorification. It is a new application of sphericalization to this particular tensor, not a claim that the pivotality conjecture is known in general.

5 Oriented signed sphericalization beyond the self-dual case

For non-self-dual rings, unrestricted permutation symmetry of a sign table would be wrong. The general checker instead uses the unit, dual reversal, and the oriented Frobenius-reciprocity symmetries $T_{ij}^k = T_{j,k}^{i*}$ together with dual-compatible basis signs $s_i = s_{i*}$. Associativity supplies exact parity equations and then the remaining integer constraints. The alternative signed tables and their scalar characters are enumerated completely modulo the recorded gauge choices.

If a sign table is gauge-equivalent to the original positive table, the chosen lifts close under tensor product in the sphericalization, furnishing a spherical tensor section. Other sign branches are eliminated only by their own exact dimension-character and formal-codegree arguments. A spherical character is promoted to pseudo-unitarity only after an explicit Galois and pivotal-sign transformation makes every simple dimension positive. The audit replays 69 oriented receipts, 40 self-dual receipts, 22 alternative signed-character inventories, and 77 original-character inventories. It certifies the reductions needed in this report. This is a finite, tensor-specific argument, not a general solution of the pivotality conjecture.

6 Eight rank-nine exclusions from central restrictions

All fusion categories in this section are over $\mathbb{C}$. The independent signed-sphericalization certificates show that a categorification of any of the eight rings considered here has a Galois conjugate with a spherical structure whose simple dimensions are its Frobenius–Perron dimensions. It therefore suffices to exclude pseudo-unitary categorifications.

Write $F: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ for the forgetful functor and I for its adjoint. Put $D = \text{FPdim}(\mathcal{C})$. For each irreducible representation E of $K_0(\mathcal{C}) \otimes \mathbb{C}$, Ostrik’s Theorem 2.13 in *Pivotal fusion categories of rank 3* (Moscow Math. J. **15** (2015), 373–396; [arXiv:1309.4822](#)) gives a simple object A_E of the center such that

$$\dim A_E = D/f_E, \quad [F(A_E) : \mathbf{1}] = \dim E.$$

Here f_E is the formal codegree. The class $[F(A_E)]$ is central in $K_0(\mathcal{C})$, since a half-braiding gives isomorphisms $F(A_E) \otimes X \cong X \otimes F(A_E)$. For one-dimensional characters χ, ψ , the modular character formula and symmetry of the unnormalized S -matrix also give

$$\frac{D}{f_\chi} \chi([F(A_\psi)]) = \frac{D}{f_\psi} \psi([F(A_\chi)]). \quad (1)$$

Indeed the character of $K_0(\mathcal{Z}(\mathcal{C}))$ obtained by composing χ with F is $[B] \mapsto S_{A_\chi, B} / \dim A_\chi$. This is the correspondence preceding Ostrik’s Theorem 2.13.

6.1 A dimension-lattice obstruction

Let L be the additive subgroup of $\mathbb{R}$ generated by the simple Frobenius–Perron dimensions of $\mathcal{C}$. Every D/f_E belongs to L : it is the dimension of the object $F(A_E)$. This gives a necessary condition stronger than algebraic integrality of D/f_E .

For each of

$$R9_1_15, \quad R9_1_68, \quad R9_1_88, \quad R9_1_116,$$

the dimension multiset is $(1^4, 2, a^2, b^2)$, with $a \simeq 3.603875$ and $b \simeq 4.493959$; their minimal polynomials are

$$a^3 - 2a^2 - 8a + 8 = 0, \quad b^3 - 4b^2 - 4b + 8 = 0.$$

The three numbers $1, a, b$ are rationally independent, and $D = 24 + 4a + 8b$. The formal codegree 8 occurs. Hence a hypothetical categorification would satisfy

$$3 + \frac{1}{2}a + b = \frac{D}{8} \in \mathbb{Z} + \mathbb{Z}a + \mathbb{Z}b,$$

contradicting rational independence. The verifier reconstructs the Frobenius–Perron character, the degree-three field, the independence, and the codegree directly from each input tensor.

6.2 A three-character obstruction

Put $d = 2 + \sqrt{6}$, so $d^2 = 4d + 2$. Each of

$$R9_1_17, \quad R9_1_83, \quad R9_1_122$$

has four invertibles, one simple Y of dimension 2, and four simples of dimension d . Its global dimension is $D = 16(1 + d)$. There are two one-dimensional characters χ_+, χ_- of codegree 16 and one character ψ of codegree 8 with the following properties. Both $\chi_{\pm}$ take value 1 on every invertible and -2 on Y . The four dimension- d simples split into two pairs, on which χ_+ takes values $\sqrt{2}$ and $-\sqrt{2}$, respectively; $\chi_- = -\chi_+$ on these four objects. The character ψ vanishes on one pair and is nonzero on both members of the other pair. These assertions are verified as exact character identities against all fusion products; no inferred character-table completeness is used.

The central objects A_+, A_- corresponding to χ_+, χ_- have dimension $1 + d$ and contain the unit with multiplicity one under F . Rational independence of $1, d$ therefore forces

$$[F(A_+)] = 1 + X, \quad [F(A_-)] = 1 + Z$$

for dimension- d simples X, Z . Equation (1) for this pair gives $\chi_+(Z) = -\chi_+(X)$, so X, Z lie in opposite pairs. The central object A_{ψ} has dimension $2 + 2d$ and unit multiplicity one. Thus its restriction has exactly two invertible summands and two dimension- d summands, counting multiplicities, and no copy of Y . Consequently

$$(\chi_+ + \chi_-)([F(A_{\psi})]) = 4.$$

Summing (1) for (χ_+, ψ) and (χ_-, ψ) , and using $D/8 = 2D/16$, gives

$$4 = 2(2 + \psi(X) + \psi(Z)).$$

Hence $\psi(X) + \psi(Z) = 0$. But exactly one of these two values is nonzero, since X, Z lie in opposite pairs. This contradiction excludes all three rings. It uses only three central simple objects and three characters, with no enumeration of induction matrices.

6.3 A central-support obstruction

The remaining ring $R9_1_120$ has the same dimension multiset $(1^4, 2, d^4)$ and global dimension $16(1+d)$, but is noncommutative. It has a one-dimensional character of formal codegree 16 (its values on the dimension- d simples are $\pm i\sqrt{2}$). The corresponding central simple again restricts to $1 + X$ for a single dimension- d simple X . None of the four possible X is central in the fusion ring. Indeed, in the original zero-based input order, X_1, X_3 fail to commute with X_2 , while X_2, X_4 fail to commute with X_1 . Thus $1 + X$ cannot be the restriction of a central object. The certificate records a differing coefficient for each of the four commutators. This proves the fourth quadratic-family exclusion.

Verification files. `cubic_lattice.py` verifies the four lattice obstructions; `quadratic_exact.py` verifies the three-character and central-support obstructions. Both reconstruct their input tensors from the supplied archives and bind their receipts to SHA-256 hashes. Neither final verifier uses floating-point decisions, optimization, or the exploratory induction-matrix computations.

7 Ten rank-ten exclusions from central reciprocity

The rings

$$R10_1_k, \quad k \in \{65, 89, 90, 91, 120, 143, 144, 147, 162, 182\},$$

have no categorification over $\mathbb{C}$. The separately verified signed/Galois reduction in `work/general_signed` shows that a hypothetical complex categorification of each listed based ring would give a pseudo-unitary categorification of the same based ring. The following obstruction rules out that consequence.

Let $\mathcal{C}$ be pseudo-unitary, $D = \text{FPdim}(\mathcal{C})$, $F : Z(\mathcal{C}) \rightarrow \mathcal{C}$ the forgetful functor, and I its adjoint. For each one-dimensional representation χ of its complexified fusion ring, let $f_\chi = \sum_i \chi(X_i)\chi(X_i^*)$ be its formal codegree. There is a distinct simple $A_\chi \subset I(1)$ with

$$[F(A_\chi) : 1] = 1, \quad \dim A_\chi = \beta_\chi = D/f_\chi.$$

Moreover, for every central simple Z ,

$$S_{A_\chi, Z} = \beta_\chi \chi(F(Z)).$$

These are Ostrik's induction/formal-codegree theorem and the identification of the corresponding central character (Ostrik, *Pivotal fusion categories of rank 3*, arXiv:1309.4822, Example 2.9 and the discussion preceding Theorem 2.13; equivalently Deshpande, arXiv:1611.04486v1, equations (35)–(36)). Symmetry of S gives the necessary equations

$$\beta_\chi \chi(v_\psi) = \beta_\psi \psi(v_\chi), \quad v_\chi = [F(A_\chi)]. \quad (2)$$

Only scalar characters are used, also for the eight noncommutative rings; no noncommutative character recovery or normalization is needed.

Write N_{ij}^k for the supplied based tensor and define its exact induction Gram matrix by

$$B_{ij} = \sum_{a,k} N_{aj}^k N_{ka}^i.$$

Adjunction gives $B = MM^\top$, where the columns of M are the restriction vectors of all simples of $Z(\mathcal{C})$. Hence any distinct chosen v_χ satisfy

$$v_\chi \in \mathbb{Z}_{\geq 0}^{10}, \quad (v_\chi)_0 = 1, \quad d \cdot v_\chi = \beta_\chi, \quad 0 \leq (v_\chi)_i \leq \min(B_{0i}, \lfloor \sqrt{B_{ii}} \rfloor),$$

and $Q = B - \sum_{\chi} v_{\chi} v_{\chi}^{\top}$ is entrywise nonnegative and positive semidefinite. Also $\sum_{\chi} v_{\chi} \leq B_{0,*}$ coordinatewise, since every selected column has zeroth coordinate one. The verifier uses only the entrywise and 2×2 principal-minor consequences of positivity. These weak consequences already suffice.

For every listed ring, the dimension vector consists of four 1's, two $2 + 2\sqrt{2}$'s and four $3 + 2\sqrt{2}$'s, and $D = 96 + 64\sqrt{2}$. All scalar characters and codegrees lie in $\mathbb{Q}(\sqrt{2}, i)$. The certificate stores them using the primitive element $t = \sqrt{2} + i$, whose minimal polynomial is $t^4 - 2t^2 + 9$. The independent verifier decodes them into the rational basis $(1, \sqrt{2}, i, i\sqrt{2})$ and checks each of the 100 multiplicativity equations for each distinct character. It checks $f_{\chi}\beta_{\chi} = D$ exactly. It independently regenerates every possible v_{χ} from the two rational dimension equations and the displayed finite bounds. It then exhausts equations (2), the Gram bounds, and first-row inequalities. All ten finite searches are empty, after only three or five search nodes each. It does not assume that the stored scalar table is complete, nor does it impose first-row equality.

The machine receipt is `exclusion_receipts.json`. The initial searches are `reciprocal_partial.py` for the eight noncommutative rings and `reciprocal.py` for the two commutative rings 89 and 147. The separate, standard-library-only replay is `work/audit_new_models/verify_rank10_reciprocity.py`; its complete output is `work/audit_new_models/rank10_reciprocity_audit.json`. The signed reduction was replayed separately using the verifier in `work/general_signed`; it is not part of this standard-library scalar-character verifier. Thus the exclusions concern arbitrary complex fusion categories, not just pseudo-unitary or spherical categories assumed in advance.

8 Structural exclusions in dimensions forty and one hundred sixty

8.1 A Morita obstruction in dimension forty

The two rings $R10_1_72$ and $R10_1_127$ have dimension profile $(1^4, 2^4, \sqrt{10}^2)$. Their invertibles form V_4 . There is an involution h fixing each dimension-two simple on both sides, with exact stabilizer $\{1, h\}$, and exchanging the two irrational-dimensional simples. Some dimension-two simple is not self-dual. These conditions alone give the following exclusion, without a pivotal assumption.

Let $\mathcal{D}$ be the integral subcategory, of dimension twenty. Since h fixes a simple, its pointed associator is trivial in cohomology: the fixed simple gives a rank-one module category over $\langle h \rangle$. Thus $A = 1 + h$ is a connected separable algebra. There are ten simple right A -modules in $\mathcal{D}$, all of underlying dimension two: two free orbits of invertibles, and two module structures on each fixed simple. For each such module M , its internal endomorphism object has dimension two and contains the unit once. It is therefore $1 + g$ for an invertible $g \neq 1$ fixing M . The underlying stabilizers force $g = h$. Every module consequently has two simple left A -structures, and ${}_A\mathcal{D}_A$ is pointed of dimension twenty.

Write this pointed category as $\text{Vec}(G, \eta)$. The Morita module has rank ten, so $\mathcal{D}$ is a group-theoretical category $\mathcal{C}(G, \eta, K, \psi)$ with $|K| = 2$. The groups of order twenty are the two abelian groups, D_{20} , Dic_5 , and $F_{20} = C_5 \rtimes C_4$ with faithful action. A normal order-two K would make every simple invertible, excluding the abelian and dicyclic possibilities. In D_{20} a nonnormal K is a reflection subgroup; every relevant double coset is inverse-stable with trivial stabilizer, so every dimension-two simple is self-dual, independently of η, ψ . Hence $G = F_{20}$.

The full Morita dual ${}_A\mathcal{C}_A$ has twenty invertibles and two simples of dimension $\sqrt{10}$. Its noninvertible stabilizer has order ten and must be abelian by Liptrap's generalized Tambara–Yamagami theorem. But F_{20} has a unique index-two subgroup, the nonabelian D_{10} . This contradiction proves both exclusions. The proof allows arbitrary associators throughout; it does not assume that a known untwisted model is the only possible categorification of the integral

subring.

8.2 Forced commutativity in dimension one hundred sixty

The two rings $R10_1_83$ and $R10_1_177$ have dimensions $(1^2, 2^2, 5^6)$ and trivial universal grading. Their nontrivial invertible g fixes the two dimension-two simples and exchanges the six dimension-five simples in pairs. ENO's categorical Burnside theorem makes a hypothetical category solvable. In a prime cyclic extension/equivariantization chain, the final operation cannot be an extension, because the universal grading is trivial. It must be a C_2 -equivariantization $\mathcal{C} = \mathcal{D}^{C_2}$.

The sign permutation forces $\mathcal{D}$ to have dimensions $(1^5, 5^3)$. Its invertibles form C_5 , act trivially on the three noninvertibles, and belong to $\mathcal{D}_{\text{ad}}$. The possible adjoint dimensions are $5 + 25k$ and must divide eighty; only five and eighty qualify. Dimension five cannot contain a dimension-five simple in any homogeneous component, so $\mathcal{D}_{\text{ad}} = \mathcal{D}$. A second application of the solvable last-step argument gives a central $\text{Rep}(C_5)$ whose image is exactly the five invertibles of $\mathcal{D}$.

An invertible object has at most one half-braiding when the universal grading is trivial: the ratio of two half-braidings is a tensor automorphism of the identity, hence a character of that grading. This central $\text{Rep}(C_5)$ is therefore stable under the C_2 action. Its equivariantization is a central Tannakian subcategory of $\mathcal{C}$ with dimensions $(1, 1, 2, 2)$, namely $\text{Rep}(D_{10})$. Direct de-equivariantization gives $\mathcal{C} = \mathcal{E}^{D_{10}}$ with $\text{FPdim } \mathcal{E} = 16$. The orbit-stabilizer dimension formula forces $\mathcal{E}$ to be pointed: every nonunit orbit contributes a dimension-five simple and hence has five invertible underlying objects. The order-five rotation acts freely on all fifteen nonidentity elements of its group. Characteristic-center and two-torsion arguments force that group to be $V = C_2^4$.

Identify V with $\mathbb{F}_{16}$. A rotation is multiplication by an order-five element a , and a reflection can be taken to be $b(x) = x^4$. The complete presentation of $H^3(V, \mathbb{C}^*)$ has fourteen binary generators. Their restrictions to cyclic subgroups are the squarefree monomials of degrees one through three on $V \setminus \{0\}$. They span the even-parity subspace of its fifteen functions. Rotation invariance leaves four classes, with constant values on the three length-five orbits and even total parity. Four explicit normal forms in the certificate cover these classes.

The rotation tensorator can be made strict because $H^1(C_5, H^2(V, \mathbb{C}^*)) = 0$. Relative to a reference reflection tensorator, the commutator of any other reflection tensorator is a rotation-invariant alternating form. Every such form is

$$B_c(x, y) = \text{Tr}_{16/2}(cxy^4), \quad c \in \mathbb{F}_4.$$

It vanishes on $\mathbb{F}_4 \times \mathbb{F}_4$, the reflection's fixed space. The reference tensorators are also symmetric there, as checked exactly. Consequently every allowed tensorator is symmetric on fixed pairs. This argument covers arbitrary complex actions, not only a bounded search of roots of unity.

For equivariant objects X, Y , the trace of an element h on their tensor product's fixed homogeneous degree z is

$$\sum_{\substack{x+y=z \\ hx=x, hy=y}} J_h(x, y)^{-1} \chi_X(h, x) \chi_Y(h, y).$$

The established symmetry makes this invariant under exchanging X, Y . Projective stabilizer characters determine equivariant objects, so the Grothendieck ring must be commutative. Both input tensors are explicitly noncommutative, proving the exclusions. The full proof and independent standard-library replay are in `work/audit_new_models/INT160_AUDIT.md` and `verify_int160_audit.py`.

9 The generalized Tambara–Yamagami family

Fourteen rank-ten inputs have profile $(1^8, 2^2)$. Liptrap’s classification of the corresponding fusion systems supplies necessary and sufficient associator data. Eleven have explicit unitary data in the package. The remaining three, $R10_1_118$, $R10_1_122$, and $R10_1_180$, admit no complex solution of those equations.

For a positive record the phases are powers of a sixteenth root of unity, with $\tau = 1/2$. The verifier checks the defining biderivation and coherence equations and all eighty-eight recorded nontrivial Fourier blocks for unitarity. Liptrap’s construction then supplies the entire fusion category. For a negative record, the stabilizer bicharacters are enumerated completely (sixteen pairs for V_4 , four for C_4). The remaining multiplicative equations have the form $\prod_j x_j^{A_{ij}} = i^{b_i}$ with arbitrary $x_j \in \mathbb{C}^*$. For every allowed bicharacter choice the receipt gives an integer row vector y with

$$yA = 0, \quad yb \not\equiv 0 \pmod{4}.$$

Multiplying the equations with these integer exponents would imply $1 = i^{yb} \neq 1$. The obstruction therefore applies over all of $\mathbb{C}^*$; failure of a finite-phase search is not being used as nonexistence evidence.

10 Complete near-group branch in dimension twenty-four

Consider a fusion category $\mathcal{D}$ of type $(1^8, 4)$ and a categorical C_2 action whose nontrivial element fixes precisely two invertible simples. Then its equivariantization has based ring $R9_1_12$ or $R9_1_108$. More precisely, the former occurs when the invertibles of $\mathcal{D}$ form D_8 , and the latter when they form Q_8 . In particular this branch cannot produce $R9_1_62$ or $R9_1_87$.

This is an exhaustive statement about complex fusion categories and actions. Indeed dimension $24 = 2^3 \cdot 3$ implies solvability by [8, Theorem 1.6], hence weak group-theoreticity and unitarizability by [12, Theorem 5.20]. Reutter’s Theorems 1–2 and Corollary 3 [16] allow tensor autoequivalences and their coherence to be transported to unitary ones. The near-group rules are $g\rho = \rho = \rho g$ and $\rho^2 = \sum_{g \in G} g + 2\rho$. Izumi’s classification [17, Theorems 1.1 and 6.1] shows $G = D_8$ or Q_8 , and supplies exactly three unitary categories for each group. The explicit admissible tuples below cover all six. Thus a unitary classification is being used only after unitarizability has been justified.

10.1 Normal form for all tensor autoequivalences

We use Izumi, *A Cuntz algebra approach to the classification of near-group categories*, arXiv:1512.04288v2, Remark 3.6, Definition 4.8, Theorem 4.9, and Lemma 6.13 [17]. Let π be the unique irreducible two-dimensional representation of G , on $K = \mathbb{C}^2$. Let j be an antiunitary commuting with $\pi(G)$ and satisfying $j^2 = \epsilon$, with $\epsilon = +1$ for D_8 and $\epsilon = -1$ for Q_8 . For any third root of unity ζ , the complete categorical data are

$$(K, V, U_K, j_1, j_2, \chi, l) = (\mathbb{C}^2, \pi, \pi, j, \epsilon\zeta j, \chi, 0), \quad \pi(g)\pi(h) = \chi_h(g)\pi(h)\pi(g).$$

The generating fusion isometries are $S_g \in \text{Hom}(g, \rho^2)$ and $T(v) \in \text{Hom}(\rho, \rho^2)$, $v \in K$. The left absorption $g\rho = \rho$ is strict in this presentation. All morphisms are generated by these isometries and their adjoints (Remark 3.6).

Every tensor autoequivalence is, up to tensor natural isomorphism, represented by a group automorphism ϕ and a unitary matrix W with

$$W\pi(g)W^* = \pi(\phi(g)), \quad Wj = jW. \tag{3}$$

The representative acts by

$$\gamma(S_g) = S_{\phi(g)}, \quad \gamma(T(v)) = T(Wv). \tag{4}$$

Here the normal-form assertion includes all tensorators, not just permutations of simples. To see this, normalize the one-dimensional left absorption maps $g\rho \rightarrow \rho$ under the functor. The tensor-functor coherence on (g, h, ρ) then normalizes the pointed multiplication maps as well. More explicitly, if λ_g is the tensorator on the strict left absorption $g\rho = \rho$, the natural-isomorphism component on g makes $\lambda_g = 1$, and the same coherence forces all pointed tensorators to one. A further scalar choice on ρ fixes S_1 . For a functor with tensorator L , its normalized trivalent images are $L_{\rho, \rho}^{-1}F(S_g)$ and $L_{\rho, \rho}^{-1}F(T)$. The following formulas refer to these normalized images. Since S_g is obtained by applying the left g action to S_1 , its image is now $S_{\phi(g)}$. The action on K is a unitary W . Covariance of the left G action and of the duality defined by S_1 gives exactly (3). Remark 3.6 then determines the functor on all morphisms. This is also the equivalence-of-tuples normal form in Definition 4.8 and the proof of Theorem 4.9.

Conversely (3) preserves χ and both j_1, j_2 , and $l = 0$ supplies no further constraint. The reconstruction formulas of Section 4 therefore give the tensor autoequivalence (4). Explicitly, the formula for $\rho(S_1)$ uses only $\sum_g S_g$ and $\sum_i T_i j(T_i)$, both covariant under γ ; $\rho(S_g) = U(g)\rho(S_1)U(g)^*$ is also covariant. Formula (3.12) for $\rho(T)$ is covariant because W intertwines π and commutes with j_1, j_2 . Thus this construction supplies genuine functors.

For each ϕ , Schur's lemma gives W uniquely up to a scalar. The condition $Wj = jW$ reduces that ambiguity to $\pm W$. Their quotient is the functor

$$\delta(S_g) = S_g, \quad \delta(T) = -T.$$

It is tensor-isomorphic to the identity: take $\eta_g = 1$ and $\eta_\rho = -1$, extend tensorially, and check naturality on S_g, T . For example $\eta_{\rho^2}\delta(T) = T\eta_\rho = -T$, while the S_g equation has both sides S_g . These generators suffice for all naturality equations. Consequently there is no additional invisible autoequivalence or free tensorator parameter at a fixed ϕ .

Finally $\text{Aut}_\otimes(\text{Id}_\mathcal{D}) = 1$. The relation $g\rho = \rho$ forces the scalar on every g to be one, and the occurrence of ρ in ρ^2 forces the scalar on ρ to be one. Thus a square isomorphism for an involutive tensor functor, when it exists, is unique, and automatically satisfies the C_2 action coherence. In particular there is no independent sign choice that could alter the duality calculation below.

10.2 The square of the action determines the large-object duality

If $\phi^2 = 1$, then W^2 commutes with $\pi(G)$ and is scalar. The antiunitary condition makes this scalar $+1$ or -1 . For D_8 choose the real matrices

$$\pi(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pi(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \text{complex conjugation}.$$

An automorphism with exactly two fixed group elements is $r \mapsto r^{-1}, s \mapsto r^k s$, $k = 1$ or 3 . Its intertwiner W is a real reflection, so $\text{tr } W = 0$ and $W^2 = 1$. For $k = 1$ one may take $W = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The action is strict and its square coherence has $\eta_\rho = +1$.

For Q_8 use

$$\pi(r) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \pi(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \pi(s) \circ \text{complex conjugation}.$$

The unitaries commuting with j form $SU(2)$. Every nontrivial involutive conjugation of $\pi(Q_8)$ lifts to a noncentral $W \in SU(2)$ with $W^2 = -1$ and trace zero. In particular this holds for all six involutions fixing precisely two group elements. Thus $\gamma^2 = \delta$, and the unique square coherence is $\eta_g = 1, \eta_\rho = -1$. It is invariant under γ , proving the action pentagon explicitly.

An equivariant structure on ρ is a scalar u with $u^2 = \eta_\rho$. Since $\gamma(S_1) = S_1$, the lift of ρ is self-dual exactly when $u^2 = 1$. Consequently the two lifts are both self-dual for D_8 , and form a dual pair for Q_8 . This conclusion is independent of the third-root parameter ζ .

10.3 Independent finite tensor replay

The pointed part is the strict equivariantization of Vec_G by ϕ . Its simple objects are the two characters over each fixed $g \in \{1, r^2\}$ and one object for each of the three free orbits. Products are obtained by counting ordered pairs (g, h) with prescribed product, and projecting the fixed output fibers onto the two C_2 characters. This computation allows noncommutative G throughout. Write P_+, P_- for the two lifts of ρ . A fixed invertible (g, t) multiplies P_u to P_{tu} , and each free-orbit simple multiplies it to $P_+ + P_-$. For $P_t P_u$, each free invertible orbit occurs once, and the fixed outputs are $(1, \epsilon tu)$ and $(r^2, \epsilon tu)$. Both P_+ and P_- occur once because the action on $\text{Hom}(\rho, \rho^2)$ is W with trace zero. These rules determine every coefficient.

The standard-library verifier `verify_ng_actions.py` enumerates all 8 automorphisms of D_8 and all 24 of Q_8 , selecting respectively the 2 and 6 required involutions. For every involution it supplies an exact matrix $W = A/\sqrt{2}$, with Gaussian-integer A , and checks unitarity, the intertwining equations, antiunitary compatibility, trace zero, and $W^2 = \epsilon$. It then constructs the complete equivariant fusion tensor and checks all associativity identities. Every D_8 result matches $R9_1_12$ and every Q_8 result matches $R9_1_108$, coefficient by coefficient, with an explicit model-to-input permutation. None matches $R9_1_62$ or $R9_1_87$. The complete receipt is `ng_actions.json`.

This argument resolves the entire $(1^8, 4)$ de-equivariantization branch; it does not by itself dispose of other possible de-equivariantization profiles of a dimension-forty-eight input.

11 The remaining integral dimension-forty-eight exclusions

Theorem 11.1. *The based rings $R9_1_62$ and $R9_1_87$ admit no fusion-category categorification over $\mathbb{C}$. The same argument distinguishes them from $R9_1_12$ and $R9_1_108$, for which unitary constructions are supplied separately.*

First reduction. All four rings have simple dimensions $(1^4, 2^3, 4^2)$, invertible group V_4 , and trivial universal grading group. The latter assertion is checked by generating the adjoint subring from all products XX^* . A categorification is solvable by the Etingof–Nikshych–Ostrik $p^a q^b$ theorem [8, Theorem 1.6]. In a sequence of cyclic prime-order extensions and equivariantizations constructing it ([8, Proposition 4.4]), the final step cannot be a nontrivial extension, since its universal grading is trivial. It must therefore be a C_p -equivariantization. Its canonical $\text{Rep}(C_p)$ embeds centrally, so p divides the order-four invertible group and $p = 2$. Write $\mathcal{C} = \mathcal{D}^{C_2}$, and let e be the corresponding sign object of $\mathcal{C}$.

The forgetful functor is completely constrained on simple-object orbits. A pair $\{X, eX\}$ forgets to one simple of dimension $\text{FPdim}(X)$; a simple fixed by tensoring with e forgets to two distinct simples, exchanged by the action, of dimension $\text{FPdim}(X)/2$. The projective stabilizers are cyclic and have trivial complex Schur multiplier, so no further projective-representation dimension occurs. The exact three possibilities are as follows; the entries give the zero-based input label of e .

input	$(1^8, 4)$	$(1^4, 2, 4)$	$(1^4, 2^5)$
$R9_1_12$	7	1	2
$R9_1_62$	5	6	7
$R9_1_87$	5	4	3
$R9_1_108$	5	7	6

The quotient has dimension 24 in each case. The integer verifier `verify_d48_exclusions.py` checks these orbits and every dimension identity directly against the input tensors.

The near-group branch. For profile $(1^8, 4)$, writing ρ for the unique noninvertible gives $g\rho = \rho g = \rho$ and $\rho^2 = \sum_{g \in G} g + 2\rho$. The acting involution fixes exactly two invertibles. The

complete near-group action theorem of Section 10 applies: its equivariantization is $R9_1_12$ for $G = D_8$, or $R9_1_108$ for $G = Q_8$. In particular this branch excludes the two rings in the theorem. This is an exhaustive categorical statement, including all six associator classes and all tensorators, not merely a test of strict group actions.

The $(1^4, 2^5)$ branch. The involution on the quotient fixes exactly two invertibles and one dimension-two simple m . It exchanges the remaining invertibles and pairs up the remaining four dimension-two simples. The fixed m is self-dual. The source products of either of its two equivariant lifts force

$$m^2 = \sum_{g \in \text{Inv}(\mathcal{D})} g.$$

The exact source products used here, in the input labeling, are $3 \cdot 3 = 0 + 7 + 8$ for input 12, $1 \cdot 2 = 0 + 5 + 8$ for input 62, $6 \cdot 6 = 0 + 5 + 7$ for input 87, and $1 \cdot 2 = 0 + 5 + 8$ for input 108. The last summand in each product forgets to the two nonfixed invertibles.

The quotient ring is necessarily multiplicity-free. For a nonfixed quotient simple Z , its coefficient in the forgetful image of a unique source simple is one. Thus the tensor-functor identity and the multiplicity-free source tensor bound every N_{XY}^Z by one. Invertible output coefficients are bounded by one by rigidity. The only remaining possible output is m . If an input is invertible the bound is immediate. Otherwise, if $Y \neq m$, Frobenius reciprocity gives $N_{XY}^m = N_{X*m}^Y \leq 1$, since Y is nonfixed. For $Y = m$, reciprocity reduces the coefficient to one in m^2 , which is already known. This proves the claim without assuming that a quotient of a multiplicity-free category is generally multiplicity-free.

A direct binary-tensor enumeration now completes the quotient ring, without consulting a catalog of quotient rings. Label the quotient so that its dimensions are $(1, 1, 1, 1, 2, 2, 2, 2, 2)$ and its involution is $(2\ 3)(5\ 6)(7\ 8)$. The source sign orbits determine the based restriction matrix F , up to this harmless choice of labels. The verifier `direct_quotient.py` enumerates every duality involution compatible with F , dimensions, and the action. It identifies tensor coefficients only under Frobenius reciprocity and the action, then exhausts all binary assignments satisfying unit, duality, dimension, and all identities

$$\sum_h N_{ij}^h(C) F_{hk} = \sum_{a,b} F_{ia} F_{jb} N_{ab}^k(\mathcal{D}).$$

For every variable, the positive linear equations also supply an explicit upper-bound witness at most one, independently confirming that no larger nonnegative integer coefficient is omitted. Every surviving assignment is checked against every associativity identity. There are respectively 8, 8, 4, 4 possible dualities for sources 12, 62, 87, 108; the numbers of linearly feasible completions are 36, 36, 4, 4. Exactly four labeled associative completions remain for each source. Only after this exhaustive completion does the verifier compare them with $R9_1_117$, and all sixteen comparisons give explicit based isomorphisms. Thus $R9_1_117$ is the only possible quotient, independently of completeness of any input census.

The complete triality-action classification, established in `work/resolve_integral/TRIALLI TY_ACTIONS.md`, now applies. Every categorification of $R9_1_117$ has neutral category $\text{Rep}(D_8)$ in its canonical C_3 grading, and every compatible C_2 action gives only $R9_1_12$ or $R9_1_108$. The argument includes the neutral-category restriction, all extension associator parameters, and the two possible coherence classes. Consequently this branch also excludes 62 and 87.

The rank-six branch. Here write G for the order-four invertible group and m, n for the unique simples of dimensions 2 and 4. No assumption that G is V_4 is needed at this stage. Dimension, rigidity, and Frobenius reciprocity force

$$gm = mg = m, \quad gn = ng = n, \quad m^2 = \sum_G g, \quad mn = nm = 2n, \quad n^2 = \sum_G g + 2m + 2n.$$

In particular this category, denoted $\mathcal{B}$, has trivial universal grading. Solvability once again makes it a C_2 -equivariantization $\mathcal{B} = \mathcal{E}^{C_2}$. The orbit calculation gives $\text{FPdim}(\mathcal{E}) = 12$, profile $(1^4, 2^2)$; the involution exchanges its two dimension-two simples and transposes two of its invertibles.

We give the necessary ring reduction explicitly. Let A be the order-four group of invertibles of $\mathcal{E}$, and let X, Y be its dimension-two simples. Duality shows that the left and right actions of A on $\{X, Y\}$ coincide. If this action were transitive, its stabilizer H would have order 2 and $XX^* = \sum_H h + Z$ for one dimension-two simple Z . This expression is self-dual, so Z is self-dual and hence both X, Y are self-dual. Choose $g \notin H$; then $Y = gX = Xg$ and $g^2 \in H$, whence $Y^2 = g^2 X^2 = X^2$. An automorphism exchanging X, Y would have to preserve the unique noninvertible summand Z of this common square, a contradiction. Thus A fixes both simples, and $XX^* = YY^* = \sum_A a$. If both were self-dual, write $XY = aX + bY$; the dimension identity gives $a + b = 2$, whereas $X(XY) = X^2 Y = 4Y$ forces $a = 0$, and $(XY)Y = XY^2 = 4X$ then gives an incompatible invertible summand. Therefore $X^* = Y$, and associativity, or Frobenius reciprocity, gives

$$XY = YX = \sum_A a, \quad X^2 = 2Y, \quad Y^2 = 2X.$$

This is the canonical faithful C_3 grading with neutral pointed category on A and one simple in each nonneutral component.

A rank-one homogeneous module forces the neutral associator to be cohomologically trivial, so the neutral category is Vec_A . The Brauer–Picard action of a generator of C_3 is an isometry $M \in O(A \oplus A^*, q)$, $q(a, \chi) = \chi(a)$, whose block $\beta : A^* \rightarrow A$ is invertible. These standard extension facts are also stated and proved in Jordan–Larson, [18, Lemma 2.6 and Proposition 3.1]. If $A = C_4$, its split quadratic group has exactly four automorphisms: $I, -I$, coordinate interchange, and its negative. None has order 3, so this case is impossible. The verifier exhausts all 256 candidate matrices modulo 4 to check this assertion.

For $A = V_4$, identify $A \oplus A^*$ with $\mathbb{F}_2^4$, with $q(a_1, a_2, \chi_1, \chi_2) = a_1 \chi_1 + a_2 \chi_2$ in exponent notation. The split orthogonal group has 72 elements. Exactly six of them have order 3 and invertible β . In every one of these six cases,

$$Mv = v, \quad q(v) = 0 \implies v = 0. \quad (5)$$

This is checked by exhaustive exact enumeration in the same verifier. Equivalently, the fixed space is zero or is an anisotropic plane. No choice of scalar associator or of a lift of the action changes this necessary fixed-vector statement. Jordan–Larson’s six-category classification [18, Theorem 1.4] provides an independent description of the same two matrix orbits and their three C_3 associator twists.

A characteristic central sign. Put $\mathcal{K} = \mathcal{E} \rtimes C_2$. It is Morita equivalent to $\mathcal{B}$ and is faithfully S_3 -graded with neutral category Vec_{V_4} , because the involution exchanges the two nonneutral C_3 components. Any central invertible of $\mathcal{K}$ has degree in $Z(S_3) = 1$; restriction of its half-braiding to the neutral category gives an M -fixed vector in $A \oplus A^*$. Its twist is precisely $(-1)^{q(v)}$ for that vector v . By (5), every central invertible of twist $+1$ has trivial restriction. The kernel of this restriction consists exactly of grading characters $\text{Hom}(S_3, \mathbb{C}^*) = C_2$. Indeed its underlying object is the unit, its half-braiding is trivial on the neutral component, and its scalar is constant on each indecomposable homogeneous bimodule component and multiplicative in the grading. Both kernel objects have twist $+1$. Hence $\mathcal{K}$, and therefore $\mathcal{B}$, has exactly one nontrivial central invertible of twist $+1$. This proof does not require that any nonzero invariant fermion lift to the center.

All categories under consideration are weakly integral and therefore have the canonical pseudo-unitary spherical structure. The braided equivalence of centers induced by Morita equivalence preserves its twist. Moreover $U(\mathcal{B}) = 1$, so the forgetful map $\text{Inv } Z(\mathcal{B}) \rightarrow \text{Inv}(\mathcal{B})$ is injective. The

unique nontrivial boson thus has a nontrivial underlying object and generates a central Tannakian $\text{Rep}(C_2)$. Every tensor autoequivalence preserves this characteristic central subcategory.

Return to the original equivariantization $\mathcal{C} = \mathcal{B}^{C_2}$. Equivariantizing this stable central $\text{Rep}(C_2)$ gives a central Tannakian subcategory of dimension 4 in $\mathcal{C}$: the equivariantization remains symmetric, and the composite forgetful functor to $\text{Rep}(C_2)$ and then to vector spaces is a symmetric fiber functor. It is $\text{Rep}(V_4)$ or $\text{Rep}(C_4)$. The latter is impossible since $\text{Inv}(\mathcal{C}) = V_4$. Thus all four invertible objects of $\mathcal{C}$ belong to central $\text{Rep}(V_4)$. We may therefore choose any of its three signs for de-equivariantization, in particular the sign in the $(1^8, 4)$ column of the displayed table. The already exhaustive near-group branch forces $R9_1_12$ or $R9_1_108$, excluding 62 and 87 in the last remaining branch. This completes the proof.

Replay and scope. The exact finite receipt is `work/resolve_rank9_integral/d48_exclusions_receipt.json`. Run `pythonwork/resolve_rank9_integral/verify_d48_exclusions.py` for the dimension identities, all source orbit profiles, the independent replay of all completed quotient tensors, the C_4 obstruction, and all six quadratic fixed-space checks. The detailed near-group theorem and its independent matrix/tensor replay are `work/resolve_rank9_neargroup/NG_ACTIONS.tex` and `verify_ng_actions.py`. The triality action classification is `work/resolve_integral/TRIALLITY_ACTIONS.md`. The exhaustive quotient completion and its receipt are `work/resolve_rank9_neargroup/direct_quotient.py` and `direct_quotient.json`. These categorical reductions and exact finite enumerations supply the complete exclusion; no infeasible numerical search or failure to find a model is used.

12 Unitary constructions

12.1 Established models and audited operations

The baseline contains 84 realized input rings. Seventy-nine have selected full tensor-model receipts; the Haagerup–Izumi C_5 model and four additional Liptrap-positive inputs complete this count. Standard sources include pointed and group-representation categories, $SU(2)$ and adjoint $PSU(2)$ categories, Tambara–Yamagami and metaplectic categories, Deligne products, equivariantizations, group-theoretical categories, and explicit lower-rank unitary models from Liu–Palcoux–Ren. Every selected receipt includes a unit-preserving basis permutation and a complete coefficient comparison.

The construction audit checks more than tensor matching for the less standard operations. It replays 27 associative C_2 zestings, eight central-extension quotients, six cyclic actions on Tambara–Yamagami categories, and the Ising-swap C_4 subextension. In a C_2 zesting, a neutral central involution supplies the two-cocycle, and the fourth-cohomology obstruction vanishes; the scalar coherence correction is included in the proof. Bases for which the required central or braided data have not been supplied are not used for this operation. The eight quotient constructions instead start from $SU(2)_4$, a specified central extension of V_4 , and a central boson; their fusion tensors follow from the actual free-orbit quotient.

12.2 Explicit coherent pointed actions

Six new small models are equivariantizations of pointed unitary categories:

Inputs	Construction
$R9_1_47$	$\text{Vec}(C_{12}, \omega)^{C_2}$, inversion action
$R9_1_98$	$\text{Vec}(C_3 \times V_4, \omega)^{C_2}$, inversion and swap
$R10_1_61, 84, 109, 158$	$\text{Vec}(\mathbb{F}_2^3, \omega)^{S_3}$, coordinate permutations

For each model the certificate gives all associator exponents w , forward tensorators J_g , and compositors $\mu_{g,h}$ modulo eight. The verifier checks the pointed pentagon and, for every tuple of indices,

$$\begin{aligned} J_g(a, b) + J_g(ab, c) + w(a, b, c) &= w(ga, gb, gc) + J_g(b, c) + J_g(a, bc), \\ \mu_{g,h}(ab) + J_g(ha, hb) + J_h(a, b) &= J_{gh}(a, b) + \mu_{g,h}(a) + \mu_{g,h}(b), \\ \mu_{g,h}(ka) + \mu_{gh,k}(a) &= \mu_{h,k}(a) + \mu_{g,hk}(a). \end{aligned}$$

These equalities define coherent categorical actions. Independent code then constructs all equivariant simple objects as projective matrix actions on their homogeneous fibers, verifies the action equations, and computes every fusion coefficient by exact Hom-space characters. Roots of unity of orders sixteen or forty-eight suffice for these matrix calculations. Thus neither the existence nor the fusion rules are inferred from a permutation action on a based ring alone.

The dimension-160 cases $R10_1_39$ and $R10_1_138$ are likewise realized as D_{10} -equivariantizations of $\text{Vec}(\mathbb{F}_{16}, \omega)$. One may take $R10_1_39 = \text{Rep}(\mathbb{F}_{16} \rtimes D_{10})$, with action $v \mapsto a^k v^{4^b}$. The second model uses a nontrivial associator and explicitly stored fourth-root tensorators. The replay checks all 65,536 pointed pentagons, all action diagrams, the projective stabilizer data, and the 1,000 fusion coefficients of each input.

12.3 A twisted subgroup-bimodule model

For $R9_1_92$, put $V = \{a \in \mathbb{F}_2^3 : \sum_i a_i = 0\}$ and $G = V \rtimes S_3 \cong S_4$. Restrict one of the coherent S_3 actions above to V . The crossed-product associator is

$$\Omega((a, g), (b, h), (c, k)) = w(a, gb, ghc) + J_g(b, hc) - \mu_{g,h}(c) \pmod{8}.$$

It is verified directly on every quadruple of the 24-element group. Its restriction to the chosen reflection subgroup $H = C_2$ is trivial. The category of bimodules over $\bigoplus_{h \in H} h$ in $\text{Vec}(G, \Omega)$ is therefore unitary. The supplied homogeneous bimodules are checked for left, right, and mixed associativity; the independent reconstruction forms their balanced tensor products, checks that both actions descend to the quotient, and identifies all 729 coefficients with $R9_1_92$.

12.4 The triality model of dimension twenty-four

Nikshych–Riepel, Example 9.3, constructs a nontrivial C_3 extension of $\text{Rep}(D_8)$, integral of dimension 24, with every noninvertible simple of dimension two. Their definition of nontrivial means that a homogeneous component is not equivalent to the regular category as a left module. An invertible simple in that component would give such an equivalence. Thus both nonneutral components have exactly two dimension-two simples.

Write the neutral simples as $G = V_4$ and m , with $m^2 = \sum G$. Every rank-two invertible $\text{Rep}(D_8)$ -module has transitive G action. Indeed the module-category classification gives reflection subgroups C_2 or the nontrivial projective class of D_8 ; both are transitive under the linear characters. The central C_2 alternative has a pointed dual and cannot be an invertible $\text{Rep}(D_8)$ -bimodule. The other subgroups have one, four, or five projective simples, so no case is omitted.

Label the nonneutral simples X_0, X_1 and Y_0, Y_1 with $X_i^* = Y_i$. Their left and right actions are nonzero homomorphisms $\phi, \psi : V_4 \rightarrow C_2$. Multiplication by m is the all-ones matrix. Frobenius reciprocity determines the mixed products:

$$X_i Y_j = \sum_{\phi(g)=i+j} g + m, \quad Y_i X_j = \sum_{\psi(g)=i+j} g + m.$$

The only unknowns are $X_i X_j = a_{ij} Y_0 + (2 - a_{ij}) Y_1$, with $a_{ij} \in \{0, 1, 2\}$; duality determines $Y_j Y_i$. The exact replay checks all $3 \cdot 3 \cdot 3^4 = 729$ choices, allowing multiplicity two. Precisely six are associative, all with distinct left and right kernels and $a_{ij} = 1$. Every one is based-isomorphic to $R9_1_117$. This identifies the existence theorem’s category with the supplied tensor. The category is nilpotent and hence unitarizable by Galindo–Hong–Rowell, Theorem 5.20.

12.5 An actual unitary model for $R10_1_21$

Let $\mathcal{P} = PSU(2)_6$ have simples $1, f, x, y = fx$, with $f^2 = 1, x^2 = y^2 = 1+x+y$, and $xy = f+x+y$. The invertible f is a transparent fermion and its pointed associator is trivial. All four simples are self-dual. Write $\mathcal{W} = (\mathcal{P} \boxtimes \mathcal{P})^{C_2}$ for the canonical permutation equivariantization, a unitary braided fusion category. Its simples are $d_i^\epsilon = (i \boxtimes i, \epsilon)$ and $p_{ij} = (i \boxtimes j) \oplus (j \boxtimes i)$ for $i < j$.

The object $h = d_f^-$ is an order-two transparent boson: the two fermionic self-braidings multiply to $+1$, its pointed associator is trivial, and its equivariance squares to $+1$. Consequently $A = 1 \oplus h$ is a connected central commutative separable algebra, and $\mathcal{C} = \mathcal{W}_A$ is an actual unitary fusion category. The free-module functor F is tensor. The h -orbits of size two give one simple each; a fixed simple splits into two simples, since its stabilizer is cyclic of order two. There are six free orbits and two fixed objects, namely p_{1f} and p_{xy} . Thus $\mathcal{C}$ has rank ten and dimensions $1^4, (2+2\sqrt{2})^2, (3+2\sqrt{2})^4$.

The fixed invertibles, including their tensor squares. Put $s = F(d_1^-)$, and let g, sg be the two summands of $F(p_{1f})$. The pointed subcategory of $\mathcal{P} \boxtimes \mathcal{P}$ can be made strict. For $X = p_{1f}$ write $e_1 = 1 \boxtimes f, e_2 = f \boxtimes 1$. The equivariant structure exchanges e_1, e_2 , whereas that of h has sign -1 . An equivariant right h -action on X is therefore

$$r(e_1 \otimes h) = ie_2, \quad r(e_2 \otimes h) = -ie_1;$$

it squares to the identity. Its negative is the other simple module. The half-braiding $c_{h,X}$ is -1 on both components, so the left h -action used in the balanced product is $-r$. Hence $g \otimes_A g$ is the cokernel of $r \otimes 1 + 1 \otimes r$ inside $X \otimes X$. This operator is Hermitian, so that cokernel identifies with its kernel. Its two homogeneous lines are

$$e_1 e_1 + e_2 e_2, \quad e_1 e_2 - e_2 e_1.$$

They have equivariance $+$ and $-$ respectively and underlie $1 \oplus h$. The remaining right action $1 \otimes r$ sends the first line to i times the second and the second to $-i$ times the first; rescaling the second line by i identifies this with the regular right A -module. Thus $g^2 = 1$. Multiplication by s exchanges the two module structures, so the four invertibles form $C_2 \times C_2$, with $s^2 = g^2 = 1$. This calculation is essential: the choice $h = d_f^+$ gives a different fixed-point resolution and does not give the multiplicity-one tensor here.

All fusion coefficients. Choose $a = F(p_{1x})$ and $b = F(d_x^+)$. Then

$$\begin{array}{c} \text{source} \\ F \end{array} \left| \begin{array}{cccccccccc} d_1^+ & d_1^- & d_f^+ & d_f^- & d_x^+ & d_x^- & d_y^+ & d_y^- & p_{1f} & p_{xy} \\ 1 & s & s & 1 & b & sb & sb & b & g+sg & gb+sgb \end{array} \right.$$

and $F(p_{1y}) = F(p_{fx}) = ga, F(p_{fy}) = a$. Indeed s interchanges the signs on the diagonal simples, and source fusion gives $(g+sg)b = F(p_{xy})$ and $(g+sg)a = 2F(p_{1y})$. Since $sb \neq b$ and $g^2 = 1$, these identify a free orbit of four b 's and an orbit of two a 's, with $sa = a$. Ordinary equivariant tensor products give

$$\begin{aligned} b^2 &= 1 + a + ga + b + sb + gb + sgb, \\ a^2 &= 1 + s + a + ga + b + sb, \\ ab &= a + b + sb + gb + sgb. \end{aligned}$$

For example $d_x^+ d_x^+ = d_1^+ + d_x^+ + d_y^+ + p_{1x} + p_{1y} + p_{xy}$. The three displayed rules, commutativity, and the invertible action therefore determine the entire tensor without choosing any unknown fusion coefficient. They also show all ten simples are self-dual.

The standalone constructor `model21.py` generates these rules in the displayed model basis. It checks all 10^4 associativity identities, and verifies all 10^3 coefficients against `R10_1_21` under the zero-based model-to-input basis map

$$(0, 9, 1, 2, 3, 8, 6, 7, 5, 4).$$

The resulting model receipt is `model21.json`. Its inverse input-to-model map is $(0, 2, 3, 4, 9, 8, 6, 7, 5, 1)$. The independent standard-library verifier `work/audit_new_models/verify_model21.py` replays the fixed-point balance, all 196 source products, the free-module map, and the complete original-tensor comparison. Its receipt is `work/audit_new_models/model21_audit.json`. The categorical inputs are the standard unitary $PSU(2)_6$ category, canonical permutation equivariantization, and central Tannakian condensation (EGNO, Theorems 8.4.9 and 8.23.3). No reconstruction from a ring-consistent induction matrix is being asserted.

12.6 Triality equivariantizations: `R9_1_12` and `R9_1_108`

Audit status. Both constructions below have passed independent categorical and coefficient audits. The second uses a direct inner modification of the first action and the vanishing of its action-coherence obstruction.

Put $D = \text{Rep}(D_8)$, where D_8 has order eight. We use its Tambara–Yamagami presentation $D = \text{TY}(V_4, \chi, +\frac{1}{2})$, with alternating bicharacter χ . The standard associators are invariant under every permutation of the three nonidentity elements of V_4 . Consequently these permutations, with identity tensorators, define a coherent unitary S_3 action. The tensor autoequivalence group is S_3 , while $\text{BrPic}(D) \cong S_4$; see Nikshych–Riepel, *Categorical Lagrangian Grassmannians and Brauer–Picard groups of pointed fusion categories*, [arXiv:1309.5026](#), Section 9.1, and Diatlyk–Luo–Wang–Weller, *Gauging Non-Invertible Symmetries: Topological Interfaces and Generalized Orbifold Groupoid in 2d QFT*, [arXiv:2311.17044](#), Section 3.5, Equation (3.30), for the explicit tensor triality.

A coherent Morita transport. The three nonregular invertible left D -module classes of rank two are the two reflection-subgroup modules and the projective D_8 -representation module for the nontrivial Schur class. Their simple objects have internal End objects $1 + h + m$, with the three distinct nontrivial choices of h . Tensor triality therefore permutes these three module classes transitively. Choose the projective module class and realize it by the Azumaya algebra $A = \text{End}(V)$ in the symmetric category D , where V is a real two-dimensional irreducible projective representation of D_8 arising from its central cover D_{16} . Thus $A = 1 + h + m$ and $A\text{-Bimod}(D) \simeq D$.

Choose the transposition represented by the actual group involution

$$a(r) = r^{-1}, \quad a(s) = sr, \quad r^4 = s^2 = 1, \quad srs = r^{-1}.$$

It fixes the projective module class. This invariance has a coherent lift: the nontrivial Schur cocycle of D_8 extends to $D_8 \rtimes_a C_2 \cong D_{16}$, because the cover $D_{32} \rightarrow D_{16}$ restricts to the nontrivial cover $D_{16} \rightarrow D_8$ on its normal D_8 subgroup. Equivalently, writing the projective representation as a rotation R through $\pi/4$ and a reflection S , the real reflection $T = SR^{1/2}$ satisfies

$$T^2 = 1, \quad TRT = R^{-1}, \quad TST = SR,$$

and conjugation by T makes A a coherent equivariant algebra.

There is also no ambiguity between this coherent group action and the strict TY transposition. A fixed transposition class has two coherent C_2 actions, distinguished by $H^2(C_2, \text{Aut}_\otimes(\text{Id}_D)) = H^2(C_2, C_2) = C_2$. For the strict action the two extensions of m have equivariant scalars $+1, -1$

and are self-dual; in the other class they have scalars $+i, -i$ and form a dual pair. The group-automorphism action has self-dual extensions in $\text{Rep}(D_{16})$, so it is coherently equivalent to the strict restriction.

Form the crossed product $B = D \rtimes S_3$ and its graded Morita dual $C = A\text{-Bimod}(B)$. The neutral component is again D . Its C_3 restriction E is nonregular, since triality moves the module class of A ; the preceding triality construction identifies its fusion ring with $R9_1_117$. Coherent C_2 equivariance of A supplies an invertible $t \in C$ with $t^2 \cong 1$ and trivial restricted associator. Thus conjugation by t is an actual coherent C_2 action on E , and $C = E \rtimes C_2$. On the neutral category its equivariantization is $\text{Rep}(D_{16})$. Indeed, if $q = st$ in $D_8 \rtimes_a C_2$, then $q^2 = r$ and $sqs = q^{-1}$.

Duality of the dimension-four objects. The subcategories of a faithfully graded fusion category supported on conjugate subgroups are Morita equivalent: the sum of components on the corresponding coset is the equivalence bimodule. Hence all three order-two subgroup subextensions of C are Morita equivalent to $D \rtimes C_2$, and therefore to $\text{Rep}(D_{16})$. The two nonregular ones have dimension profile $(1^4, 2^3)$. Every category in this Morita class is $\mathcal{C}(D_{16}, 1, H, \psi)$. This profile forces $H = D_{16}$ or a reflection C_2 . Cyclic normal subgroups give pointed duals; a Klein-four subgroup or an index-two D_8 subgroup already gives eight invertibles, for either available Schur class. For $H = D_{16}$ the fusion ring is the representation ring of D_{16} . For a reflection C_2 , the three non-normalizer double cosets are inverse-stable with trivial stabilizer, and the four invertibles form V_4 . In both permitted cases all simples are self-dual.

Write the nonneutral C_3 components of E as $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$, with $X_i^* = Y_i$. The self-duality just proved for $X_i t$ implies $t(X_i) = Y_i$. Thus both four-dimensional orbit objects in E^{C_2} are self-dual.

Exact tensor identification. Label E by $0, 1, 2, 3, m, X_0, X_1, Y_0, Y_1$, choosing left and right stabilizers $\{0, 1\}$ and $\{0, 2\}$ for the X component. The restriction of the seven neutral simples of $\text{Rep}(D_{16})$ is

$$0, 3, 0, 3, m, 1 + 2, m,$$

and the other two restrictions are $X_i + Y_i$. Let F be this restriction matrix. Induction is $F^\top$ by Frobenius reciprocity, and

$$\text{Ind}(X) \otimes W = \text{Ind}(X \otimes \text{Forget}(W)), \quad W \otimes \text{Ind}(X) = \text{Ind}(\text{Forget}(W) \otimes X).$$

These formulas determine every product outside the neutral subcategory; inside it we use the ordinary D_{16} representation ring. The resulting multiplicity-free tensor agrees with $R9_1_12$ under

$$(0, 3, 2, 4, 8, 7, 6, 1, 5).$$

The independent verifier `work/audit_new_models/verify_triality_equivariant.py` checks all 729 coefficients, the restriction homomorphism, and the D_{16} subgroup calculation. The certificate is `work/resolve_small_models/triality_equivariant_certificate.json`. The construction is unitary throughout; alternatively, unitarizability follows from Galindo–Hong–Rowell, [arXiv:1105.5048](https://arxiv.org/abs/1105.5048), Theorem 5.20.

The second coherent action and $R9_1_108$. The following direct construction has passed the independent categorical audit and supplies the second unitary realization. Let α denote the accepted action and choose $g \in V_4$ with $g\alpha(g) = h$, the nontrivial fixed invertible. Set $\beta = \text{Ad}_g \circ \alpha$. Then $\beta^2 \simeq \text{Ad}_h$. For the nonregular triality action, the fixed central invertible in $\text{Inv } Z(D)$ is $w = (z, h)$, where z is the central involution of D_8 . Its obstruction to a central lift on $h \in E$ lies in $H^2(C_3, \mathbb{C}^\times) = 0$; hence $\text{Ad}_h \simeq \text{Id}$. The obstruction to making β a coherent C_2 action lies in

$$H^3(C_2, \text{Aut}_\otimes(\text{Id}_E)) = H^3(C_2, C_3) = 0,$$

and the corresponding H^2 torsor is trivial. Any correction is a character of $U(E) = C_3$, so it is the identity on D . Relative to the symmetric central lift of h , the lift w differs by z , changing the neutral square coherence by -1 on m . The inner modification changes the orbit pairing to $X_i + Y_{i+1}$. These obstruction statements use Etingof–Nikshych–Ostrik, *Fusion categories and homotopy theory*, [arXiv:0909.3140](#), Corollary 7.9(i) [19], and Gelaki–Naidu–Nikshych, *Centers of graded fusion categories*, [arXiv:0905.3117](#), Theorem 3.5. The detailed central-lift and coherence comparison is in `work/resolve_integral/TRIALITY_ACTIONS.md`.

The neutral ring is therefore $\text{Rep}(SD_{16})$, since the corresponding extension has $t^2 = z = r^2$, and $q = st$ satisfies $q^8 = s^2 = 1$ and $sqs = q^3$. Replacing the neutral group ring accordingly and using this pairing in the same induction formulas gives exactly `R9_1_108`, with based permutation

$$(0, 4, 6, 8, 7, 1, 2, 3, 5).$$

Both this coefficient identification and the categorical coherence of the modification have passed independent audits. No classification of all possible involutions is required for these two positive witnesses.

13 A uniform Morita reduction at rank ten

Put $\delta = 1 + \sqrt{2}$. The five inputs

$$R10_1_114, \quad R10_1_115, \quad R10_1_152, \quad R10_1_153, \quad R10_1_179$$

have dimension type $(1^4, (2\delta)^2, (\delta^2)^4)$. We prove their noncategorifiability over $\mathbb{C}$. The finite computations in this section reconstruct necessary module actions and fusion tensors; they are not searches for numerical associators. The full source-bound derivation and receipts are in [the rank-ten continuation package](#).

13.1 The involution algebra and the forced dual

In each input there is a unique nontrivial invertible h fixing the two simples Y_0, Y_1 of dimension 2δ . The remaining invertibles exchange them, and h acts freely on the other six simples. Directly in all five tensors,

$$N_{Y_i, Y_j}^{Y_k} = 1 \quad (i, j, k \in \{0, 1\}). \quad (6)$$

If $\mathcal{C}$ categorifies the input, the rank-one action of $\langle h \rangle$ on Y_0 trivializes its pointed associator. Thus $A = 1 \oplus h$ is a connected separable algebra. Its simple right modules are four free modules from length-two h -orbits and two modules Y_i^+, Y_i^- over each fixed Y_i . Twisting the right action by the sign automorphism of A fixes the free modules and exchanges each split pair.

Let W be the anti-invariant two-space with basis $w_i = [Y_i^+] - [Y_i^-]$. Left multiplication induces a based-ring representation ρ on W . Left h acts by $\text{diag}(\epsilon_0, \epsilon_1)$, with $\epsilon_i = \pm 1$. An invertible exchanging the Y_i acts by an invertible signed swap. Commutation with h gives $\epsilon_0 = \epsilon_1$. If both signs were negative, $hY_i = Y_i$ would imply $\rho(Y_i) = 0$. This is impossible: each entry is $a - b$ with $a + b = 1$ by (6). Hence left h fixes every simple right module.

An A -bimodule is a right module with coherent left C_2 action. Every right-module simple is fixed and has two one-dimensional equivariant structures, since $H^2(C_2, \mathbb{C}^\times) = 0$. The dual therefore has sixteen simples of dimensions

$$1^4, \quad \delta^8, \quad (\delta^2)^4. \quad (7)$$

We use the endofunctor-composition convention $\mathcal{D} = \text{End}_{\mathcal{C}}(\mathcal{C}_A)$. The corresponding literal bimodule convention takes the opposite tensor product; all arguments below are invariant under this change.

Here the full eight-simple module action is forced, rather than just its dimension vector. Normalize the signed swap on W . Its $+1$ eigenspace has integer character values y on both Y_i and z on the four δ^2 -dimensional simples. The input tensor gives

$$y^2 = 2 + 2y + 2z, \quad yz = y + 4z.$$

Thus $(y-2)(y^2-4y-4) = 0$, so $y = 2, z = -1$. The odd entries of $\rho(Y_i)$ are consequently all $+1$. Forgetting module structures then determines the remaining entries; the fixed-sector matrices of each δ^2 -dimensional simple are the negatives of the corresponding identity or swap matrices. All module fusion and duality equations are checked exactly by `module_action.py`.

For clarity, the subsequent finite step has a proved bound. A simple dual action matrix B is nonnegative integral, commutes with the original module action, and has prescribed unit column $Be_0 = e_m$. Let w be the positive module dimension vector, with entries in $\{1, \delta, \delta^2\}$ and $w_0 = 1$. The sum of the original module matrices is symmetric and irreducible. Its unique Perron line gives both $Bw = \lambda w$ and $w^t B = \lambda w^t$, and the unit column gives $\lambda = w_m$. Consequently

$$0 \leq B_{ij} \leq \lambda w_i / w_j \leq \delta^4 = 17 + 12\sqrt{2} < 34.$$

The rational commutant has dimension twelve. Prescribing a unit column leaves four parameters, each explicitly an entry of B . Necessary linear inequalities narrow their integer ranges; exhaustive enumeration leaves exactly two matrices for each of the eight columns. Tensoring with the canonical sign bimodule τ exchanges these two matrices without fixed points. Therefore both candidates occur as the two actual equivariant structures; duplicate action matrices are not being assumed distinct.

Matrix products determine 240 of the 256 ordered dual products uniquely. The remaining sixteen are precisely products of the four δ^2 -dimensional simples. Each such simple is already a simple product of two δ -dimensional objects; associativity forces its remaining products. The receipts verify all 4096 associativity equations and all 256 matrix-action equations per dual tensor. This is a complete necessary reconstruction with no arbitrary cutoff. The realized ring $R10_1_21$ is included as an independent control.

13.2 Relative factorization with a common invertible

The adjoint subcategory $\mathcal{B} = \mathcal{D}_{\text{ad}}$ has rank eight. Its forced tensor has two rank-four fusion subcategories $\mathcal{P}, \mathcal{Q}$ whose intersection has simples $1, f$. The involution τ is outside $\mathcal{B}$, is distinct from f , and conjugation by τ exchanges the two subcategories. Every cross product of simples is simple; fixing either factor gives four distinct products. Both rank-four factors have trivial universal grading, and f acts freely on their simple bases.

For inputs 114 and 115 their based rings are the ordinary $\text{PSU}(2)_6$ ring P :

$$f^2 = 1, \quad fx = y, \quad x^2 = y^2 = 1 + x + y, \quad xy = f + x + y, \quad x^* = x, \quad y^* = y.$$

For inputs 152,153,179 they are the pseudo ring Q :

$$f^2 = 1, \quad fx = y, \quad x^2 = y^2 = f + x + y, \quad xy = 1 + x + y, \quad x^* = y.$$

All these identifications, the simple cross products, and the action of τ are checked coefficient by coefficient in the forced tensors.

Lemma 13.1. *In this situation multiplication extends to a dominant tensor functor $\mathcal{P} \boxtimes \mathcal{Q} \longrightarrow \mathcal{B}$. The unique central lifts of f in the two factors have self-braidings $q_{\mathcal{P}}, q_{\mathcal{Q}}$ satisfying $q_{\mathcal{Q}} = q_{\mathcal{P}}^{-1} = q_{\mathcal{P}}$.*

Proof. Choose a noninvertible $Y \in \mathcal{Q}$. The complement $\mathcal{M} = \mathcal{P}Y$ of $\mathcal{P}$ in $\mathcal{B}$ is a $\mathcal{P}$ -bimodule subcategory. The functor $X \mapsto XY$ is an equivalence of left module categories. Transporting the

right action therefore identifies $\mathcal{M}$ with a twisted regular bimodule $\mathcal{P}_\phi$. The tensor autoequivalence ϕ fixes simple classes because XY and YX are the same simple class.

Multiplication followed by projection to $\mathcal{M}$ is a bimodule functor $\mathcal{P}_{\phi^2} \rightarrow \mathcal{P}_\phi$. Its value at the unit is $V = 1 \oplus f$, since the $\mathcal{M}$ -part of Y^2 is $Y \oplus fY$. Its right-module constraint gives coherent isomorphisms $\phi^2(X)V \simeq V\phi(X)$, or, replacing X by $\phi^{-1}(X)$, $\phi(X)V \simeq VX$. On a simple X the two constituent classes X, fX are distinct. These maps are therefore block diagonal for $V = 1 \oplus f$. The unit block is a tensor isomorphism $\phi \simeq \text{Id}$. Thus Y has an ordinary relative half-braiding with $\mathcal{P}$.

Two relative half-braidings on a fixed simple $Z \in \mathcal{Q}$ differ by scalars λ_X , because XZ is simple. Tensor compatibility makes them a character of the universal grading of $\mathcal{P}$, which is trivial. The half-braiding is unique. Products of lifted $\mathcal{Q}$ -simples inherit half-braidings, and the latter preserve each simple summand: tensoring by any fixed simple of $\mathcal{P}$ sends the four $\mathcal{Q}$ -simples to distinct simples. Tensor powers and summands of Y generate $\mathcal{Q}$. These unique lifts are therefore tensor closed and give the required functor into the relative center, hence the dominant multiplication functor.

The relative crossing of the common f with itself gives its unique central lift in $\mathcal{P}$; reversing the crossings gives its lift in $\mathcal{Q}$. Hence the two self-braidings are inverse. Conjugation by τ identifies these central lifts, so they are also equal. Their squares lie over the tensor unit and are trivial because $\text{Aut}_\otimes(\text{Id})$ is trivial in each factor. Thus $q_{\mathcal{P}}^2 = 1$. $\square$

The case $q = +1$ is already impossible in either factor. It would give the central boson algebra $1 \oplus f$, whose fusion category of modules has two simples and rule $W^2 = 1 + 2W$, excluded by the rank-two classification [20]. The forced central lift is consequently fermionic.

13.3 The pseudo branch

Theorem 13.2. *The based rings $R10_1_152$, $R10_1_153$ and $R10_1_179$ have no complex fusion categorification.*

Proof. Their rank-four factor has based ring Q and, by Lemma 13.1, a central fermion f . The standard associated superfusion construction [23, Definition 4.9] has morphism spaces

$$\text{Hom}_{\widehat{Q}}(U, V)_{\bar{0}} = \text{Hom}_Q(U, V), \quad \text{Hom}_{\widehat{Q}}(U, V)_{\bar{1}} = \text{Hom}_Q(U, fV).$$

The central self-braiding -1 gives exactly the super interchange sign. Up to homogeneous isomorphism there are two simple representatives $I = 1, W = x$, both of m -type. The fusion spaces have dimensions

$$\begin{aligned} \dim \text{Hom}(W^2, I) &= (0 \mid 1), & \dim \text{Hom}(W^2, W) &= (1 \mid 1), \\ \dim \text{Hom}(IW, W) &= (1 \mid 0), & \dim \text{Hom}(WI, W) &= (1 \mid 0). \end{aligned}$$

Thus $W^2 = \Pi I + (1 + \Pi)W$. This is exactly the odd-evaluation family $\widehat{\mathcal{C}}_m^{(1,1)}$ excluded by the finite superpentagon calculation in [24, Section 3]. That calculation explicitly includes the pair $(1, 1)$ and admits only $(0, 0)$ among the tested odd-evaluation families. We use this finite result, not the subsequent conjecture about all rank-two superfusion categories.

Neither the associated-superfusion construction nor that finite polynomial-system exclusion assumes unitarity. In particular the cited paper retains odd $(0, 0)$ solutions even while discussing their failure of a proposed physical locality condition; locality is not used as an exclusion filter here. The final published Mathematica/Magma calculation is a literature input and was not independently rerun in this project. The precise rule and its applicability have been independently audited. $\square$

13.4 The ordinary branch

Theorem 13.3. *The based rings $R10_1_114$ and $R10_1_115$ have no complex fusion categorification.*

Proof. The earlier tensor-specific sphericalization and character receipts give a pseudo-unitary Galois conjugate of any hypothetical categorification. Pseudo-unitarity passes to its Morita dual: both global dimension and Frobenius–Perron dimension are preserved [1, Theorem 2.15, Corollary 8.14]; the positive spherical structure then exists by [1, Proposition 8.23] and restricts to the factors. Etingof–Ostrik’s classification [21, Theorem A.3] identifies each pivotal ordinary rank-four factor with $P = \text{Rep}(\text{SO}(3)_q)$, q a primitive sixteenth root. Conjugation by τ identifies the two factors monoidally.

Lemma 13.1 and central-algebra reconstruction [22, Corollary 5.6] identify $\mathcal{B}$ with $(P \boxtimes P)_a$, where $a = 1 \oplus (f, f)$. The induced central structure is unique because $U(P \boxtimes P) = 1$. The common f in each quantum factor is the central fermion, so (f, f) is a boson. The canonical inverse Morita algebra is $1 \oplus \tau$; its pointed associator is trivial. Hence $\mathcal{D}$ is the crossed product of $\mathcal{B}$ by the coherent order-two action $T = \text{Ad}_\tau$, and the original category is $\mathcal{B}^T$ (or its opposite under the opposite dual convention).

We must exhaust the categorical actions, not just their based permutations. Identify the second factor with the first using T . The lift of T to $P \boxtimes P$ is then ordinary factor interchange L . Uniqueness of the cross-half-braidings gives a tensor comparison $TF \simeq FL$, where $F : P \boxtimes P \rightarrow \mathcal{B}$ is induction. Indeed applying T to the crossing gives its inverse after exchanging the factors, and these relative half-braidings are unique. Passing to right adjoints gives a central algebra isomorphism $L(a) \rightarrow a$. Conversely that isomorphism descends L to a -modules: transport the module action along it. Every module is a summand of a free one, so the comparison recovers the action on every module and morphism.

The unital algebra automorphisms of a are exactly the two signs on (f, f) . Their squares are trivial. There is no extra action coherence, since $U(\mathcal{B}) = 1$ implies $\text{Aut}_\otimes(\text{Id}_{\mathcal{B}}) = 1$. Equivariantization commutes with this central condensation. The two possible categories therefore have the form

$$((P \boxtimes P)^{\text{swap}})_{1 \oplus (f, f)_\epsilon}, \quad \epsilon = \pm.$$

For $\epsilon = +$, write d_x^+ for the positive equivariant lift of (x, x) and p_{uv} for the off-diagonal orbit on $(u, v), (v, u)$. In the permutation equivariantization,

$$(d_x^+)^2 = d_1^+ + d_x^+ + d_y^+ + p_{1x} + p_{1y} + p_{xy}.$$

Induction identifies d_x^+, d_y^+ and their common image b is simple, so b^2 contains $2b$. This contradicts multiplicity one.

For $\epsilon = -$ it suffices to determine the invertibles. Let $s = F(d_1^-)$ and let g, sg be the two summands of $F(p_{1f})$. The pointed subcategory of $P \boxtimes P$ has trivial associator. With $e_1 = (1, f), e_2 = (f, 1)$, the equivariant structure exchanges the two lines and the right (f, f) -action can be written

$$r(e_1) = ie_2, \quad r(e_2) = -ie_1.$$

The half-braiding is -1 on both lines, so the left action used in the balanced product is $-r$. The quotient by the image of $r \otimes 1 + 1 \otimes r$ is represented by its zero eigenspace, with homogeneous lines

$$u = e_1e_1 + e_2e_2, \quad v = e_1e_2 - e_2e_1.$$

Their equivariance signs are $+, -$, respectively. The remaining right action sends $u \mapsto iv$, $v \mapsto -iu$. Rescaling v by i identifies this module with the regular algebra. Hence $g^2 = 1$; also $s^2 = 1$, and s exchanges the two module structures. The four invertibles form $C_2 \times C_2$, whereas both target rings have invertible group C_4 .

Every primitive sixteenth root is Galois conjugate to the standard one. The quantum category, central fermion, swap action, and algebra transport under that conjugation. Fusion multiplicities and the invertible group are unchanged. Thus the two contradictions apply to all forms of the ordinary factors. Taking opposites does not alter either target based ring, since both are commutative. This exhausts and excludes both inputs. $\square$

14 Structural constraints on the cyclic rank-ten ring

The ring $R10_1_189$ has normal form

$$g^5 = 1, \quad gx = xg, \quad x^* = x, \quad x^2 = 1 + (g + g^2 + g^3 + g^4)x, \quad (8)$$

with $d = \text{FPdim}(x) = 2 + \sqrt{5}$ and $D = 50 + 20\sqrt{5}$. The original basis map for $(1, g, g^2, g^3, g^4, x, gx, g^2x, g^3x, g^4x)$ is $(0, 1, 8, 7, 2, 9, 4, 6, 5, 3)$. We record structural results valid for every categorification; the next section supplies an actual unitary model.

Proposition 14.1. *Any complex categorification of (8) has pointed subcategory tensor equivalent to the untwisted Vec_{C_5} .*

Proof. The established Galois reduction allows a positive spherical structure. Vanishing of the pointed associator class is Galois invariant, so it suffices to work there. Put $X_i = g^i x$, with subscripts modulo five. The restriction Gram matrix has blocks

$$B_{GG} = 10I, \quad B_{GX} = 5(J - I), \quad B_{XX} = 15I + 15J.$$

Writing a central restriction as $(a_i; b_i)$, the Galois automorphism $\zeta_5 \mapsto \zeta_5^2$ gives the exact transform

$$a'_i = a_{3i}, \quad b'_i = \sum_j a_j - a_{3i} - b_{3i}.$$

These identities follow from the normalized center character formula; the complete character matrix and the identity $HT = \sigma(H)$ are verified exactly in the structural receipt. In particular $T^4 = 1$. Every constituent of $I(g^j)$ has positive Galois sign and restriction $g^j + \sum_{i \in S} X_i$, with $S \subset \{i : i \neq j\}$.

For completeness $I(g^j)$ is multiplicity free with ten constituents. For $j = 0$ this follows from commutativity of the based ring and induction. For $j \neq 0$, the fifth powers of their twists coincide [6, Proposition 3.8]. The indicator identities $\nu_m(g^j) = 0$ for $1 \leq m \leq 4$ distribute the total dimension equally among the five phases, each of weight $D/5 = 2 + 4d$. If a constituent restricts with multiplicity a to g^j and has dimension $a + bd$, it contributes $a(a + bd)$ to that weight. Comparing constant coefficients gives $\sum a^2 = 2$ in each phase. There are therefore two constituents, both of multiplicity one, in each phase.

Suppose the pointed associator class $k \in H^3(C_5, \mathbb{C}^\times)$ is nonzero. The cyclic indicator formula [25, Section 3.3] gives a primitive fifth root for $\nu_5(g)$. Equality in the trace triangle inequality forces every constituent of $I(g)$ to have primitive twenty-fifth-root twist. Choose a cyclotomic automorphism σ of exponent 2 modulo 25, with odd exponent on its 2-primary part. The normalized center has anomaly one, so modular Galois symmetry gives $\theta_{\hat{\sigma}Z} = \sigma^2(\theta_Z)$ [26, Theorem 2.9]. Its fourth iterate fixes each restriction but raises a primitive twenty-fifth-root twist to its sixth power, an operation of order five. Thus each restriction in $I(g)$ occurs a multiple of five times.

The ten restrictions are consequently five copies each of complementary subsets S, S^c of $\{0, 2, 3, 4\}$. Their four Galois transforms supply all forty nontrivial pointed-induced columns. The ten columns in $I(1)$ have d -coefficients $0, 4, 1, 1, 1, 1, 3, 3, 3, 3$, from the exact formal codegreens. Their singleton and complementary-triple counts coincide coordinatewise by $FI(1)$, and Galois

multiplication by two makes these four counts equal, hence all one. Their noninvertible Gram contribution has diagonal entries five and off-diagonal entries three away from coordinate zero.

Subtracting these known columns from $15I + 15J$ leaves diagonal ten. For the eight complementary patterns, an off-diagonal entry of the residual matrix is as follows; an entry exceeding ten contradicts a nonnegative principal 2×2 minor:

Representative S	Entry	Conclusion
$\emptyset$	–	five central lifts, treated below
$\{4\}, \{3\}, \{3, 4\}, \{2\}, \{2, 4\}, \{2, 3\}$	12	determinant -44
$\{2, 3, 4\}$	15	determinant -125

The exact verifier checks all patterns, and also all 35 possible singleton-count compositions before imposing their Galois equality. The empty/full pattern would give five distinct central invertibles forgetting to the same g . This is impossible: the kernel of forgetting central lifts is $\text{Aut}_{\otimes}(\text{Id})$, which is trivial here. Indeed scalars u on g and v on x satisfy $v^2 = 1$ and $v^2 = u^j v$ for $j = 1, 2, 3, 4$; hence $u = v = 1$. All nonzero pointed associator classes are excluded. $\square$

Proposition 14.2. *The regular-algebra Morita dual of any categorification of (8) has the same based ring.*

Proof. Proposition 14.1 supplies $A = \bigoplus_{i=0}^4 g^i$. Its right-module category has two simples, and its dual has dimensions $1^5, d^5$. This uses $H^2(C_5, \mathbb{C}^\times) = 0$ when splitting the left equivariant structures; no multiplicity-one assumption on the dual is needed. The invertibles act freely and transitively on its five noninvertibles. The commutation automorphism is either the identity or inversion. Inversion would force five equal noninvertible coefficients in a square, contradicting their total four. The ring is commutative. Since five is odd, choose a self-dual noninvertible generator z . Its square is

$$z^2 = 1 + \{a + b(g + g^{-1}) + c(g^2 + g^{-2})\}z, \quad a + 2b + 2c = 4.$$

Up to automorphism there are four types: $(a, b, c) = (4, 0, 0), (2, 1, 0), (0, 2, 0), (0, 1, 1)$. The first contains the excluded rank-two ring $z^2 = 1 + 4z$. For the middle two, a formal codegree divided by five has respective minimal polynomial

$$t^4 - 15t^3 + 60t^2 - 90t + 45, \quad t^4 - 20t^3 + 100t^2 - 160t + 80.$$

Both are Eisenstein at five and have degree four. But a Morita-equivalent center has all simple dimensions in $\mathbb{Q}(\sqrt{5})$, since forgetting to the original category writes them in the original dimension semiring. The central objects of dimensions D/f furnished by induction therefore exclude these quartic codegrees. The final type is exactly the original ring. These polynomial identities and the four-type exhaustion are checked with rational arithmetic in the Morita receipt. $\square$

The irrational braided generalized-near-group classification [27, Theorem 3.6] excludes a braided categorification with noninvertible dimension $2 + \sqrt{5}$. The two propositions above do not exclude a nonbraided categorification with untwisted pointed subcategory. Equality of the dual's based ring is not a tensor equivalence, and the rank-two module category is not a rank-two fusion quotient. These necessary structural constraints supplement the actual unitary construction in Section 15.

15 A unitary conformal-embedding model for R10_1_189

Let R be the based ring with basis

$$1, g, g^2, g^3, g^4, x, gx, g^2x, g^3x, g^4x$$

and relations

$$g^5 = 1, \quad gx = xg, \quad x^* = x, \quad x^2 = 1 + (g + g^2 + g^3 + g^4)x. \quad (9)$$

The involution sends g to g^{-1} . Put $d = 2 + \sqrt{5}$; then

$$d^2 = 1 + 4d, \quad \text{FPdim}(R) = 5(1 + d^2) = 50 + 20\sqrt{5}. \quad (10)$$

Section 17.1 identifies R exactly with R10_1_189.

Write $\mathcal{C} = \mathcal{C}(\mathfrak{sl}_4, 6)$ for the unitary WZW modular category. Its simple objects are denoted $\mathbf{abc}$, where (a, b, c) are Dynkin labels, $a, b, c \geq 0$ and $a + b + c \leq 6$. The conformal embedding induced by the symmetric-square representation of $\mathfrak{sl}_4$ is

$$\text{SU}(4)_6 \subset \text{SU}(10)_1. \quad (11)$$

This is a realized conformal inclusion, with vacuum branching object

$$B = 000 \oplus 060 \oplus 202 \oplus 222; \quad (12)$$

see [28, p. 29 of the arXiv version] and [29, §§1.4.3–1.4.5]. In particular, B carries an actual connected commutative separable algebra structure. Its category of local modules is braided equivalent to $\mathcal{C}(\mathfrak{sl}_{10}, 1)$. These assertions concern the conformal inclusion and its unitary extension, rather than the existence of an algebra inferred from a numerical modular invariant.

Let $\mathcal{D} = \mathcal{C}_B$ be the unitary fusion category of right B -modules, with relative tensor product. Write $F : \mathcal{C} \rightarrow \mathcal{D}$ for free induction and U for forgetting the module structure. The functor F is tensor and central [30, §§3.3–3.4, equation (17)]. Denote the invertible simple local modules by τ_j , $j \in \mathbb{Z}/10$, with $\tau_i \tau_j = \tau_{i+j}$. We suppress the symbol $\otimes_B$ for products in $\mathcal{D}$.

Theorem 15.1. *Put $\beta = F(010)$, $g = \tau_2$ and $x = \tau_5\beta$. The ten objects*

$$g^j, \quad g^j x \quad (0 \leq j < 5)$$

are pairwise nonisomorphic simple objects of $\mathcal{D}$ and are closed under tensor products and duality. Their full replete semisimple subcategory $\mathcal{K}$ is a unitary fusion category satisfying $K_0(\mathcal{K}) \cong R$ as based rings. Thus R10_1_189 is unitarily categorifiable.

The proof below needs only a small amount of ground-state branching; it does not require a completeness assertion about an affine branching table. The additional construction in Section 17 identifies $\mathcal{K}$ with the neutral component of $\mathcal{C}_B$ for the natural $\mathbb{Z}/4$ grading.

16 A minimal branching argument

Let $v = 010$, the exterior square of the defining four-dimensional representation. The exterior-Pieri rule gives

$$v^2 = 000 + 101 + 020, \quad (13)$$

$$v^3 = 3010 + 2111 + 030 + 200 + 002. \quad (14)$$

All dominant summands encountered have level at most three, so these classical products are also the level-six fusion products.

Lemma 16.1. For $j \in \mathbb{Z}/10$,

$$\dim \operatorname{Hom}_{\mathcal{C}}(v^2, U\tau_j) = \delta_{j,0}, \quad (15)$$

$$\dim \operatorname{Hom}_{\mathcal{C}}(v^3, U\tau_j) = \begin{cases} 1, & j \in \{1, 3, 7, 9\}, \\ 0, & \text{otherwise.} \end{cases} \quad (16)$$

Proof. Conformality of (11) means equality of the two Sugawara conformal vectors. The lowest conformal energy of the level-one $\mathfrak{sl}_{10}$ module indexed by j is

$$h_j = \frac{j(10-j)}{20}, \quad 0 \leq j \leq 9.$$

An affine $\mathfrak{sl}_4$ highest-weight module of highest weight λ occurring in it must satisfy

$$h_\lambda = h_j + n, \quad n \in \mathbb{Z}_{\geq 0}. \quad (17)$$

If $n = 0$, its multiplicity is the ordinary finite-dimensional restriction multiplicity in the ground-state representation $\Lambda^j \mathbb{C}^{10}$. With long roots of squared length two, $h_\lambda = (\lambda, \lambda + 2\rho)/20$. The needed values are

λ	000	101	020	010	111	200,002	030
h_λ	0	2/5	3/5	1/4	3/4	9/20	21/20.

The residues of the h_j are $0, 9/20, 4/5, 1/20, 1/5, 1/4$. Consequently 101 and 020 cannot occur in any $U\tau_j$, while 000 occurs only in $U\tau_0$, with multiplicity one. This proves (15).

For (14), the residue of h_{111} excludes that weight. The only possible index for 010 is $j = 5$, but $h_5 = 5/4 > h_{010} = 1/4$, contradicting (17). The weights 200,002 can occur only for $j = 1, 9$, at ground energy. The defining ten-dimensional representation restricts as $\operatorname{Sym}^2 V$, $V = \mathbb{C}^4$, or its dual according to orientation. Thus precisely one of these two weights occurs, once, at each of $j = 1, 9$.

Finally, 030 can occur only for $j = 3, 7$, again at ground energy. The ordinary Schur-functor identity

$$\Lambda^3(\operatorname{Sym}^2 V) = \mathbb{S}_{(4,1,1)} V \oplus \mathbb{S}_{(3,3)} V \quad (18)$$

has highest weights 301 and 030. The latter is self-dual and occurs once; dualizing gives the same multiplicity for $j = 7$. This proves (16). Identity (18) is also checked as a full weight-character identity in the accompanying certificate. $\square$

Proof of Theorem 15.1. The object v is self-dual, so β is self-dual. Rigidity, the tensor property of F , and its adjunction to U give

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{D}}(\beta, \tau_j \beta) &= \dim \operatorname{Hom}_{\mathcal{D}}(\beta^2, \tau_j) = \dim \operatorname{Hom}_{\mathcal{C}}(v^2, U\tau_j), \\ \dim \operatorname{Hom}_{\mathcal{D}}(\beta^2, \tau_j \beta) &= \dim \operatorname{Hom}_{\mathcal{D}}(\beta^3, \tau_j) = \dim \operatorname{Hom}_{\mathcal{C}}(v^3, U\tau_j). \end{aligned}$$

Lemma 16.1 proves that β is simple and its ten translates $\tau_j \beta$ are pairwise nonisomorphic. It also gives one copy of each of $\tau_1 \beta, \tau_3 \beta, \tau_7 \beta, \tau_9 \beta$ in β^2 . Self-duality gives one copy of the unit.

For $[r] = \sin(r\pi/10)/\sin(\pi/10)$ the quantum Weyl formula gives

$$\operatorname{FPdim}(\beta) = \operatorname{FPdim}(v) = \frac{[4][3]}{[2]} = 2 + \sqrt{5} = d.$$

The tensor functor F preserves Frobenius–Perron dimension. The known constituents of β^2 have total dimension $1 + 4d = d^2$; positivity therefore proves the exact decomposition

$$\beta^2 = \mathbf{1} + \tau_1 \beta + \tau_3 \beta + \tau_7 \beta + \tau_9 \beta. \quad (19)$$

Centrality of F supplies isomorphisms $\beta \tau_j \cong \tau_j \beta$. Hence $x = \tau_5 \beta$ is self-dual, $g = \tau_2$ has order five, and (19) becomes (9). The five invertibles and the five indicated translates of β are distinct, their dimensions being 1 and $d > 1$. Relations (9) show closure under products and duals. The full subcategory they span is consequently a unitary fusion subcategory of $\mathcal{D}$, as asserted. $\square$

17 An accompanying rank-fourteen modular category

The same conformal inclusion supplies a concrete modular category and an algebra which recover $\mathcal{K}$. This strengthens the existence statement beyond consistency of candidate modular data.

Proposition 17.1. *Let $J = 600$, let $L = \mathbf{1} \oplus J^2$, and put $\mathcal{Q} = \mathcal{C}_L^0$. The image of J in $\mathcal{Q}$ generates a semion subcategory $\mathcal{S}$. Its Müger centralizer $\mathcal{E} = \mathcal{S}' \subset \mathcal{Q}$ is a unitary modular category of rank fourteen and dimension $100d^2$. There is a connected étale algebra*

$$A = \mathbf{1} \oplus U_0 \in \mathcal{E}, \quad \text{FPdim}(U_0) = d^2,$$

obtained from the inclusion $L \subset B$, and a tensor equivalence

$$\mathcal{E}_A \simeq \mathcal{K} \simeq (\mathcal{C}_B)_0.$$

Here the subscript 0 denotes the neutral component of the $\mathbb{Z}/4$ grading by $a + 2b + 3c$ modulo four.

Proof. The simple current J has order four, with $J^2 = 060$, $\theta_J = i$ and $\theta_{J^2} = 1$. We first justify the algebra inclusion $L \subset B$. The regular right B -module is simple because B is connected. Tensoring it by the invertible J^2 again gives a simple right module. Multiplication induces a nonzero right-module map $J^2 \otimes B \rightarrow B$ and hence an isomorphism. Its restriction to the summand $J^2 \otimes J^2 = \mathbf{1}$ is nonzero and lands in the unique unit summand of B . Thus multiplication restricts to $\mathbf{1} \oplus J^2$. The restricted commutative algebra is the separable order-two group algebra object; in particular L is connected étale.

The image of J in $\mathcal{C}_L^0$ is invertible of order two and twist i . It generates a nondegenerate semion subcategory. Since $\mathcal{Q}$ is unitary modular, its centralizer $\mathcal{E}$ is unitary modular as well. To enumerate its simples, define the charge of $\mathbf{abc}$ by $t = a + 2b + 3c \pmod{4}$. All four summands of B have charge zero, so this grading descends to $\mathcal{C}_B$. Simple-current monodromy with J is $\exp(\pm 2\pi i t/4)$, where the sign depends on the braiding convention. Locality for L requires even charge; centralizing the image of J requires charge zero.

On weights of charge zero the action of J^2 is

$$(a, b, c) \longmapsto (c, 6 - a - b - c, a).$$

There are twenty-two such weights. The only fixed weights are 121 and 303. A nonfixed orbit gives one simple local L -module. Each fixed weight gives two: its stabilizer is cyclic of order two, and its twisted group algebra has two one-dimensional irreducible representations. The resulting rank is $(22 - 2)/2 + 2 \cdot 2 = 14$.

For completeness, Table 2 gives all dimensions and twists. They follow from the quantum Weyl formula

$$\dim(\mathbf{abc}) = \frac{[a+1][b+1][c+1][a+b+2][b+c+2][a+b+c+3]}{[2]^2[3]}$$

and $\theta_\lambda = \exp(2\pi i h_\lambda)$. For a nonfixed orbit the module dimension is the dimension of a representative; at a fixed weight each of the two resolved modules has half that dimension. Their squared dimensions sum to $100d^2$.

The algebra B descends along $L \subset B$ to a connected étale algebra A in $\mathcal{Q}$ by [30, Proposition 3.16]. Its underlying object is $\mathbf{1} \oplus U_0$, where U_0 is the module of the orbit $\{202, 222\}$. Both summands have charge zero, so A lies in $\mathcal{E}$ and has dimension $1 + d^2$.

We detail the module-in-stages identification to fix its scope. Giving an A -module in $\mathcal{C}_L^0$ amounts to giving a B -module whose restriction to L is local: the action maps are the same after balancing over L , and the relative tensor products agree. A B -module supported on charge-zero objects restricts to a local L -module and centralizes the residual semion. Conversely,

Table 2: Simple objects of $\mathcal{E}$, with $\zeta = \exp(2\pi i/5)$. A row with two representatives describes two distinct orbits; a fixed-weight row describes its two resolved modules.

Orbit representative(s)	Number	Dimension	Twist
000	1	1	1
004, 024	2	$1 + d$	ζ
303 (fixed)	2	$1 + d$	ζ^{-1}
012, 032	2	$1 + 3d$	ζ^{-1}
121 (fixed)	2	$1 + 3d$	ζ
020	1	$2d$	ζ^{-2}
101	1	$2d$	ζ^2
105	1	d	1
113	1	$5d$	-1
202	1	d^2	1

any A -module in $\mathcal{E}$ has underlying $\mathcal{C}$ -object of charge zero. Thus this tensor equivalence identifies $\mathcal{E}_A$ with precisely $(\mathcal{C}_B)_0$.

Under the symmetric-square embedding the central element iI_4 maps to $-I_{10}$. It therefore acts on the entire affine module τ_j by $(-1)^j$: its ground state has this character and the current modes preserve it. Thus $U\tau_j$ has charge $2j \pmod{4}$. Consequently τ_2 has charge zero, whereas τ_5 and β both have charge two. It follows that $\mathcal{K} \subset (\mathcal{C}_B)_0 \simeq \mathcal{E}_A$. Finally,

$$\text{FPdim}(\mathcal{E}_A) = \frac{100d^2}{1+d^2} = 5(1+d^2) = \text{FPdim}(\mathcal{K}),$$

using $(1+d^2)^2 = 20d^2$. The full fusion-subcategory inclusion is therefore an equivalence. $\square$

Remark 17.2. The proposition constructs both $\mathcal{E}$ and its algebra A inside known unitary categories. Table 2 records their dimensions and twists; it is not used to assert uniqueness of a modular category or to identify a full S -matrix from those invariants alone. The rank-ten existence proof in Theorem 15.1 is independent of this optional modular construction.

17.1 Exact identification with the census tensor

Let $(e_0, \dots, e_9)$ be the original, zero-indexed census basis of R10_1_189. The based-ring map sends the ordered basis in Section 1 to

$$(e_0, e_1, e_8, e_7, e_2, e_9, e_4, e_6, e_5, e_3). \quad (20)$$

In particular, $g \mapsto e_1$ and $x \mapsto e_9$. If $p = (0, 1, 8, 7, 2, 9, 4, 6, 5, 3)$, the accompanying verifier checks

$$N_{p(i), p(j)}^{\text{census}} p(k) = N_{i,j}^{R,k} \quad (0 \leq i, j, k < 10)$$

for all 1000 structure constants. The right-hand tensor is generated directly from (9). The source tensor is bound by SHA-256 of its compact JSON serialization, namely

c44e21d1844ded91dba30cd6c691180afa4d4f1be1c4f27806bbc12c8c8292f4.

This binds the certificate to the tensor itself rather than to a potentially changing display label.

The standard-library Python files accompanying this note have the following roles:

- `verify_conformal_model.py` checks the two Pieri products, the needed intersections with the published branching table, dimension exhaustion, and the exact census map (20); its receipt is `conformal_model_certificate.json`.

- `verify_minimal_branching.py` checks the conformal energies over the rationals and verifies (18) as an equality of four-variable weight characters by enumerating the exterior-cube and semistandard-tableau characters. Its receipt is `minimal_branching_certificate.json`.
- `verify_r189_rank14_wzw.py` checks the charge-zero orbits, fixed-point count, dimensions and twists in Table 2, using exact arithmetic in $\mathbb{Q}[t]/(t^4 - 5t^2 + 5)$ with $t = 2\cos(\pi/10)$. Its receipt is `r189_rank14_wzw_receipt.json`.

The finite checks supplement the categorical arguments. Existence of the associativity constraints and of the algebra multiplications comes from the realized conformal inclusion and the module constructions above.

18 Complete classification and its boundary

All 336 supplied multiplicity-one rings at ranks nine and ten now have an accepted general exclusion or a unitary model. The counts are 102 exclusions and 40 models at rank nine, and 136 exclusions and 58 models at rank ten. The last cyclic ring is realized in Section 15; its structural restrictions remain valid. This classifies Grothendieck rings with at least one model, not all inequivalent fusion categories. Independent human verification remains pending, and the enumeration qualification is unchanged.

19 Reproduction and evidence boundaries

The baseline package is `evidence/rank9_rank10_m1_20261023`; its `pythonverify.py` replays the preserved baseline. Those inherited verifiers require ordinary Python because some use assertions. The five later exclusions retain their separate proof commands in `evidence/rank10_m1_exclusions_20261023/README.md`.

The current ledger, cyclic unitary model and new exact proofs are curated in `evidence/unresolved_retry_20261024/README.md`. After extracting its archive, run

```
python3 unresolved11_retry_20261024/verify_retry.py
```

The thirteen new portable checks also pass with assertions disabled. Discovery dependencies are separate from the standard-library replays. Neither the old nor new arithmetic checks replace verification of the categorical premises stated in the proofs.

The two tensor digests are explicitly distinguished. One hashes compact JSON with separators (',', ':'); the other hashes the flattened tensor as unsigned bytes. The original compressed archives have separate file hashes. The ledger accounts for every one of the 336 physical input lines exactly once and asserts that no accepted positive and negative outcomes overlap.

The exact calculations check the finite claims in the accompanying proofs. They do not formally verify the cited category-theoretic theorems in a proof assistant. Standard categories are grounded in established constructions; new pointed-action models carry explicit unit-modulus cochains and complete coherence checks. The triality construction uses an existence theorem and an exhaustive identification of its possible fusion tensor. These distinct forms of evidence are identified in the model records.

A Complete input assignment

The following compact table is generated from the current `rank9_rank10_m1_ledger.json` in the theoretical continuation package. E means excluded over $\mathbb{C}$, U means a unitary realization exists, and ? means unresolved. The JSON and CSV files give the proof and replay links.

Rank 9

Case / status	Case / status	Case / status	Case / status
R9_1_1: U	R9_1_2: E	R9_1_3: E	R9_1_4: E
R9_1_5: E	R9_1_6: E	R9_1_7: E	R9_1_8: U
R9_1_9: U	R9_1_10: U	R9_1_11: E	R9_1_12: U
R9_1_13: U	R9_1_14: E	R9_1_15: E	R9_1_16: U
R9_1_17: E	R9_1_18: U	R9_1_19: E	R9_1_20: E
R9_1_21: E	R9_1_22: U	R9_1_23: E	R9_1_24: E
R9_1_25: E	R9_1_26: U	R9_1_27: E	R9_1_28: E
R9_1_29: E	R9_1_30: E	R9_1_31: E	R9_1_32: E
R9_1_33: E	R9_1_34: U	R9_1_35: U	R9_1_36: U
R9_1_37: E	R9_1_38: E	R9_1_39: E	R9_1_40: U
R9_1_41: E	R9_1_42: E	R9_1_43: E	R9_1_44: U
R9_1_45: U	R9_1_46: E	R9_1_47: U	R9_1_48: E
R9_1_49: E	R9_1_50: E	R9_1_51: E	R9_1_52: E
R9_1_53: E	R9_1_54: E	R9_1_55: E	R9_1_56: E
R9_1_57: U	R9_1_58: U	R9_1_59: E	R9_1_60: U
R9_1_61: E	R9_1_62: E	R9_1_63: U	R9_1_64: E
R9_1_65: E	R9_1_66: E	R9_1_67: U	R9_1_68: E
R9_1_69: E	R9_1_70: U	R9_1_71: E	R9_1_72: U
R9_1_73: E	R9_1_74: E	R9_1_75: E	R9_1_76: E
R9_1_77: E	R9_1_78: E	R9_1_79: E	R9_1_80: E
R9_1_81: E	R9_1_82: E	R9_1_83: E	R9_1_84: E
R9_1_85: E	R9_1_86: E	R9_1_87: E	R9_1_88: E
R9_1_89: E	R9_1_90: E	R9_1_91: E	R9_1_92: U
R9_1_93: E	R9_1_94: U	R9_1_95: E	R9_1_96: E
R9_1_97: E	R9_1_98: U	R9_1_99: E	R9_1_100: E
R9_1_101: E	R9_1_102: U	R9_1_103: U	R9_1_104: U
R9_1_105: E	R9_1_106: E	R9_1_107: U	R9_1_108: U
R9_1_109: E	R9_1_110: E	R9_1_111: U	R9_1_112: E
R9_1_113: E	R9_1_114: E	R9_1_115: E	R9_1_116: E
R9_1_117: U	R9_1_118: E	R9_1_119: E	R9_1_120: E
R9_1_121: E	R9_1_122: E	R9_1_123: E	R9_1_124: E
R9_1_125: E	R9_1_126: E	R9_1_127: E	R9_1_128: U
R9_1_129: E	R9_1_130: E	R9_1_131: U	R9_1_132: U
R9_1_133: E	R9_1_134: E	R9_1_135: E	R9_1_136: U
R9_1_137: E	R9_1_138: E	R9_1_139: E	R9_1_140: E
R9_1_141: U	R9_1_142: U		

Rank 10

Case / status	Case / status	Case / status	Case / status
R10_1_1: U	R10_1_2: E	R10_1_3: E	R10_1_4: U
R10_1_5: E	R10_1_6: E	R10_1_7: E	R10_1_8: U
R10_1_9: E	R10_1_10: U	R10_1_11: E	R10_1_12: E
R10_1_13: E	R10_1_14: U	R10_1_15: E	R10_1_16: E
R10_1_17: U	R10_1_18: E	R10_1_19: E	R10_1_20: E
R10_1_21: U	R10_1_22: U	R10_1_23: E	R10_1_24: U

Case / status	Case / status	Case / status	Case / status
R10_1_25: E	R10_1_26: E	R10_1_27: E	R10_1_28: E
R10_1_29: E	R10_1_30: E	R10_1_31: U	R10_1_32: E
R10_1_33: U	R10_1_34: E	R10_1_35: U	R10_1_36: U
R10_1_37: E	R10_1_38: E	R10_1_39: U	R10_1_40: U
R10_1_41: U	R10_1_42: E	R10_1_43: E	R10_1_44: E
R10_1_45: E	R10_1_46: E	R10_1_47: U	R10_1_48: E
R10_1_49: E	R10_1_50: E	R10_1_51: U	R10_1_52: U
R10_1_53: E	R10_1_54: U	R10_1_55: E	R10_1_56: E
R10_1_57: U	R10_1_58: E	R10_1_59: E	R10_1_60: U
R10_1_61: U	R10_1_62: E	R10_1_63: E	R10_1_64: E
R10_1_65: E	R10_1_66: E	R10_1_67: E	R10_1_68: U
R10_1_69: E	R10_1_70: E	R10_1_71: U	R10_1_72: E
R10_1_73: E	R10_1_74: E	R10_1_75: E	R10_1_76: U
R10_1_77: E	R10_1_78: E	R10_1_79: U	R10_1_80: U
R10_1_81: E	R10_1_82: U	R10_1_83: E	R10_1_84: U
R10_1_85: E	R10_1_86: E	R10_1_87: E	R10_1_88: E
R10_1_89: E	R10_1_90: E	R10_1_91: E	R10_1_92: E
R10_1_93: E	R10_1_94: E	R10_1_95: E	R10_1_96: E
R10_1_97: U	R10_1_98: E	R10_1_99: E	R10_1_100: U
R10_1_101: E	R10_1_102: E	R10_1_103: U	R10_1_104: E
R10_1_105: U	R10_1_106: U	R10_1_107: E	R10_1_108: E
R10_1_109: U	R10_1_110: E	R10_1_111: E	R10_1_112: E
R10_1_113: E	R10_1_114: E	R10_1_115: E	R10_1_116: U
R10_1_117: E	R10_1_118: E	R10_1_119: E	R10_1_120: E
R10_1_121: E	R10_1_122: E	R10_1_123: E	R10_1_124: U
R10_1_125: U	R10_1_126: U	R10_1_127: E	R10_1_128: E
R10_1_129: E	R10_1_130: E	R10_1_131: U	R10_1_132: U
R10_1_133: U	R10_1_134: E	R10_1_135: E	R10_1_136: E
R10_1_137: E	R10_1_138: U	R10_1_139: E	R10_1_140: U
R10_1_141: E	R10_1_142: E	R10_1_143: E	R10_1_144: E
R10_1_145: E	R10_1_146: E	R10_1_147: E	R10_1_148: U
R10_1_149: E	R10_1_150: U	R10_1_151: E	R10_1_152: E
R10_1_153: E	R10_1_154: U	R10_1_155: E	R10_1_156: E
R10_1_157: U	R10_1_158: U	R10_1_159: E	R10_1_160: E
R10_1_161: E	R10_1_162: E	R10_1_163: U	R10_1_164: E
R10_1_165: E	R10_1_166: E	R10_1_167: U	R10_1_168: E
R10_1_169: E	R10_1_170: E	R10_1_171: E	R10_1_172: E
R10_1_173: E	R10_1_174: E	R10_1_175: E	R10_1_176: E
R10_1_177: E	R10_1_178: E	R10_1_179: E	R10_1_180: E
R10_1_181: E	R10_1_182: E	R10_1_183: U	R10_1_184: E
R10_1_185: U	R10_1_186: U	R10_1_187: E	R10_1_188: E
R10_1_189: U	R10_1_190: E	R10_1_191: U	R10_1_192: E
R10_1_193: U	R10_1_194: E		

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