

Rank-nine fusion rings of multiplicity two

Theoretical obstructions, coherent constructions and exact certification

FusionFrontier research prepared for Sébastien Palcoux
AI-generated research; independent human verification and formalization pending

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Abstract

We study the 2,083 supplied rank-nine based fusion rings of exact multiplicity two. The current certified partition consists of 2,046 general complex exclusions, 36 rings with unitary models and one unresolved ring. The proofs combine formal-codegree and subring obstructions, signed sphericalization, central induction, modular trace constraints, Morita theory and explicit categorical constructions. A Galois restriction theorem simultaneously excludes six rings by half-integral restriction multiplicities. A forced central $\text{Rep}(S_3)$ gives a further structural exclusion. Rank-one module categories and the dimension-twenty Hopf classification settle three additional rings, while explicit relative cocycles construct three integral models. Every decision is bound to an original tensor, a proof dependency and exact replay data. We classify existence of a categorification of a based ring, without classifying all its associators.

Verification scope. These are AI-generated mathematical claims. Separate mathematical audits and exact computational replays are provided; independent human verification and Lean formalization are pending.

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1 The census, identifiers and accepted partition

A fusion ring has a distinguished basis $1 = X_0, X_1, \dots, X_8$, nonnegative integer constants $X_i X_j = \sum_k N_{ij}^k X_k$, an involution $i \mapsto i^*$, and the usual unit and Frobenius

reciprocity identities. Multiplicity means $\max_{i,j,k} N_{ij}^k$. The source file through multiplicity two contains 2,225 tensors: 142 of multiplicity one and 2,083 of exact multiplicity two. Its complete producer run finished all five duality strata. Enumeration completeness is producer-reported; tensor validity and exact uniqueness are separately checked. We do not repeat the entire enumeration here.

The label $R9_2_n$ denotes the n th *exact-multiplicity-two* tensor in source order, not necessarily physical line n of the mixed source file. Every ledger entry also supplies its physical line and SHA-256 of compact JSON, without a trailing newline. The 142 multiplicity-one labels retain explicit basis transports to the multiplicity-one classification; fourteen of these transports are nonidentity. No literal unchanged-basis assertion is needed. The original tensor, rather than a dimension profile, determines case identity.

Theorem 1.1 (Certified partition of the supplied exact-M2 census). *The accepted ledger partitions the 2,083 source rings into 2,046 rings excluded from fusion categorification over $\mathbb{C}$, 36 admitting a unitary fusion categorification, and one unresolved ring. Every exclusion has a scope-valid theoretical argument and an exact certificate or an explicit transfer from an excluded tensor. Every realization has categorical existence data and a complete equality of its 9^3 structure constants with the source tensor under the recorded based permutation.*

Proof. The finite partition is the exact union of the source-bound exclusion and model records described below. The verifier checks all 2,083 tensor hashes, the 568 explicit negative-subring transfers, the subsequent theoretical certificates, and all 26,244 coefficients of the 36 positive tensors. It rejects any overlap between exclusion and realization records. Each family-specific proof supplies the mathematical implication used by its finite receipt. The model and exclusion arguments are audited separately from these bookkeeping checks. Complete data and replay commands appear in Section 20. $\square$

Subring-proof discipline. A closed based subring transfers an exclusion only when that subring has an actual negative proof. Absence from a positive model catalog has no logical force. In particular the rank-three half- E_6 ring

$$f^2 = 1, \quad fx = xf = x, \quad x^2 = 1 + f + 2x$$

has a unitary categorification [5, 6]. It is never used as a negative subring. The audit withdraws 26 claims based on that omission, constructs models for seven affected ambient rings, and replaces the proofs for $R9_2_97, 98, 1489$ by their different $\text{NG}(C_2^2, 2)$ subrings. For those subrings the established near-group restriction $0 < \kappa < |G| - 1$ gives the exclusion. The corrected code contains an explicit negative registry with tensor identities and proof dependencies.

2 Uniform criteria and finite certificates

2.1 Formal codegrees, cyclotomicity and subrings

For an irreducible representation E of a based algebra let $e_E = \dim E$ and f_E be its formal codegree. The fusion Casimir acts by $e_E f_E$; the degree factor must be retained in a noncommutative algebra. Formal codegrees of a fusion category are d -numbers. If the minimal polynomial of an algebraic integer a is $t^n + a_1 t^{n-1} + \cdots + a_n$, the exact divisibilities $a_n^i \mid a_i^n$ express the d -number test. Failure supplies a finite

integer witness. Character cyclicity is used with its proper scalar character hypotheses; a nonabelian Galois-group witness cannot be replaced by an approximate eigenvalue calculation [1, 2].

For an integral ring a hypothetical categorification is pseudo-unitary. Distinguished central dimensions D/f_E restrict to sums of integral source dimensions, hence are integers. In the noncommutative calculation the central Casimir eigenvalue is divided by e_E before imposing this condition. Near-group and Tambara-Yamagami restrictions use their stated group and coefficient hypotheses. Tensor closure, dual closure and a full based permutation are checked before any proper-subring transfer.

The baseline dependency audit reviews all 588 original structural uses. After removing the 26 invalid half- E_6 uses, the other 562 have direct proofs or explicit lower negative bindings. The 454 rank-six through rank-eight uses match excluded lower tensors exactly; 166 stored direct witnesses replay; 24 small-rank uses are supplied with fresh direct certificates. These counts describe overlapping audit classes, not a sum of distinct excluded ambient rings.

2.2 Central induction with representation degrees

Let $\mathcal{C}$ be pseudo-unitary, $D = \text{FPdim}(\mathcal{C})$ and $d_i = \text{FPdim}(X_i)$. Write $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ for forgetting the half-braiding and I for its right adjoint. The distinguished simple $A_E \subset I(1)$ has

$$[I(1) : A_E] = e_E, \quad \dim A_E = D/f_E.$$

For every central simple Z , its restriction column $v_Z = ([F(Z) : X_i])_i$ is nonnegative integral, and

$$[I(X_i) : Z] = v_{Z,i}, \quad \dim Z = \sum_i v_{Z,i} d_i.$$

The full noncommutative induction identity is

$$FI(X) = \bigoplus_{Y \in \text{Irr}(\mathcal{C})} Y \otimes X \otimes Y^*.$$

Consequently the columns satisfy the exact Gram identity

$$\sum_Z v_{Z,i} v_{Z,j} = [FI(X_i) : X_j].$$

In a commutative ring this is the usual Casimir Gram matrix; the simplification is not used when the multiplication is noncommutative.

Every summand of $I(1)$ has twist one. The first and second twist traces in $I(X_i)$, and the traces of cyclic rotations in tensor powers, give further necessary equalities. Arithmetic and Galois conditions on central dimensions and the integrality of modular character values restrict the finite column domains. Retaining a larger domain only weakens an exclusion test.

Proposition 2.1 (Exact separating certificate). *Suppose a finite rational system $Au = b$, $u \geq 0$, is proved necessary for a categorification. If an integer vector y satisfies $A^t y \geq 0$ and $b^t y < 0$, there is no such categorification.*

Proof. A solution would give $b^t y = u^t A^t y \geq 0$, a contradiction. Every entry of the matrix and ray is checked exactly, after clearing denominators. A failed numerical optimization or an unsuccessful search for y has no negative mathematical implication. $\square$

2.3 Signed reduction, neutral components and Morita theory

A criterion formulated for pseudo-unitary categories is transferred to arbitrary complex categories only through an established existence reduction. Weak integrality supplies pseudo-unitarity directly. In the indicated irrational cases, exact signed-sphericalization and character certificates show that a hypothetical categorification has a pseudo-unitary Galois conjugate of the same based ring. This is an equivalence of existence questions, not an assertion that every original category is pseudo-unitary. The signed audit checks the complete finite sign-character domains and nonzero-product constraints. A signed gate alone never excludes a ring.

The graded and Morita arguments use actual invertible module categories. For matrix conventions where source indices act on rows, module matrices represent the opposite multiplication: $M_i M_j = \sum_k N_{ji}^k M_k$. The linking-category Gram argument is applied with this convention, including noncommutative cases. A feasible Gram relaxation does not construct an invertible bimodule.

The accepted finite families, their whole-domain reductions and exact rays are retained in the replay archive. The 36 intermediate exclusions are grouped in Table 1. All their proofs and scope reductions passed a separate audit. The folded $\text{SO}(3)$ argument for $R9_2_920$ includes the odd-boundary step: split a boundary strand by the normalized trivalent vertex and retract by the bigon identity. Thus spanning and relations extend from even boundaries without an unsupported bending assertion.

Table 1: The 36 family exclusions. Listed integers are suffixes of $R9_2_n$.

Family	Cases
Signed nonzero products	686,690,726,728,736,1187,1241,1377,1424
Joint integral center	301,352,366,385,725,727,1028,1134,1238,1448, 1465,1501,1712,1778,1989
Prime descent	907
Solvable rank-two module	1071,1724
Joint quadratic center	1569,1886
Folded fermionic $\text{SO}(3)$	920
Solvable central invertibles	1356,1933
Quadratic neutral block	500,1235
Fixed-block Morita obstruction	396,1265

3 A Galois restriction theorem

The following criterion reconstructs an entire central restriction from the source algebra. It does not enumerate the rest of the center.

Theorem 3.1 (Galois-unit restriction). *Let R be a commutative fusion ring whose FP dimensions generate a real quadratic field K . Suppose every formal codegree belongs to K , and write $\sigma \neq 1$ for its automorphism. Let C_R be the regular Casimir matrix. Its distinct irreducible minimal factors have degree one or two. Define $p \in \mathbb{Q}[t]$ modulo their product by*

$$p(t) \equiv 1 \pmod{t - a}, \quad p(t) \equiv t^2/n \pmod{t^2 - at + n}.$$

If R has a pseudo-unitary categorification, the unit row of $p(C_R)$ is a nonnegative integer vector with unit entry one.

Proof. Let χ_h run through all scalar irreducible characters, with formal codegree f_h , and let $A_h \subset I(1)$ be their distinguished central simples. Put $D = \text{FPdim}(\mathcal{C})$ and let B correspond to $\sigma \circ \text{FPdim}$. Then $\dim A_h = D/f_h$, $\dim B = D/\sigma(D)$, and $[F(B) : 1] = 1$. The central-character identity and symmetry of the unnormalized modular S -matrix [3] give

$$\begin{aligned} (D/f_h)\chi_h(FB) &= S_{A_h, B} = S_{B, A_h} \\ &= \frac{D}{\sigma(D)}(\sigma \circ \text{FPdim})(FA_h) = \frac{D}{\sigma(D)}\sigma(D/f_h) = \frac{D}{\sigma(f_h)}. \end{aligned}$$

Restriction multiplicities are integers, which justifies commuting σ with the dimension of FA_h . Hence

$$\chi_h(FB) = f_h/\sigma(f_h).$$

For rational f_h this is one; for a quadratic root with norm n it is f_h^2/n . Rational Chinese remaindering therefore constructs the displayed p . All characters agree on FB and $p(C_R)$, and the semisimple commutative regular representation is faithful. Thus $p(C_R)$ is exactly multiplication by FB , proving its required integrality and unit entry. The characters themselves need not take their values in K ; only their formal codegrees do. $\square$

Corollary 3.2. *The rings $R9_2_114, 128, 1487, 1635, 1636, 1959$ have no complex fusion categorification.*

Proof. Their FP field is $\mathbb{Q}(\sqrt{3})$, with $D = 96 + 48\sqrt{3}$. The forced restrictions, in source order, are

$$\begin{array}{l|l} 114 & (1, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, 1, 0, 0) \\ 128 & (1, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0) \\ 1487 & (1, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}) \\ 1635 & (1, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 1) \\ 1636 & (1, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}) \\ 1959 & (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0). \end{array}$$

Each contains nonintegral multiplicities. The exact independent replay constructs the polynomial by a rational remainder linear system, checks its values in every character component, verifies the full multiplication matrix identity, and checks dimension $D/\sigma(D) = 7 + 4\sqrt{3}$. The separately audited signed gates reduce every general complex existence question here to pseudo-unitary existence. The contradiction therefore has general scope. $\square$

Remark 3.3 (Duality variant and noncommutative extension). A dual-invariant scalar character labels a self-dual distinguished central simple: $S_{A^*, Z} = S_{A, Z^*}$ and the character formula identify their S -rows. In cases 1487 and 1959 a character of codegree 24 forces dimension $4 + 2\sqrt{3}$. Its restriction must contain the unit and exactly one of a pair of dual simples of dimension $2 + 2\sqrt{3}$, which is impossible. For a noncommutative irreducible E of degree e_E , the central eigenvalue is χ_E/e_E . The same proof gives $f_E/\sigma(f_E)$, while the Casimir eigenvalue is $e_E f_E$. The integer e_E cancels in the Galois ratio. The extension yields integral nonnegative vectors for 1848 and 1975 and makes no exclusion for them. All eleven applicable commutative quadratic positive controls pass the criterion.

4 A forced central symmetric category excludes 516

Proposition 4.1. *The ring $R9_2_516$ has no fusion categorification over $\mathbb{C}$.*

Proof. Use its audited general-to-pseudo-unitary gate. The exact codegree data force precisely one distinguished object of dimension one and one, A , of dimension two in $I(1)$; every other such dimension exceeds four. Its restriction is $F(A) = 1 + g$, with $g^2 = 1$. Uniqueness makes A self-dual. The unit multiplicity of $F(A)^2$ is two. Since all distinguished multiplicities in $I(1)$ are one, A^2 has exactly one other distinguished summand in addition to the unit. The dimension bound forces that summand to be A , leaving an invertible z with $F(z) = g$. Thus

$$A^2 = 1 + z + A, \quad zA = A, \quad z^2 = 1.$$

The verified character of A has value one at g . The character formula and balancing give $\theta_z = 1$; all three objects have trivial double braiding on their closed fusion subcategory. Positive dimensions identify this symmetric category as Tannakian $\text{Rep}(S_3)$.

Condense the central regular algebra $1 + z$. Its image $1 + g$ is connected, so its module category $\mathcal{E}$ is a fusion category. The spherical trace descends after division by two, giving its positive spherical structure. Free and fixed source orbits under g give simple dimensions

$$1^3, \quad d^3, \quad e^3, \quad d = 1 + \sqrt{3}, \quad e = 2 + \sqrt{3}.$$

The local $1 + z$ -modules in the central $\text{Rep}(S_3)$ form $\text{Rep}(C_3)$. Under center descent [4, Theorem 3.20], its two nontrivial invertibles forget separately to the unit of $\mathcal{E}$: their sum is the free module on $F(A) = 1 + g$, namely two units. Distinct central structures on the unit give a faithful C_3 of tensor automorphisms of the identity. Their eigencharacters define a faithful dual C_3 grading, whose components have equal FP dimension [1]. Equating the rational and $\sqrt{3}$ coefficients, a component contains either three dimension- d simples or one each of dimensions $1, d, e$. The neutral component contains a unit, so it has dimensions $1, d, e$.

Its rank-three fusion rules are forced by dimension and duality:

$$X^2 = 1 + X + Y, \quad XY = X + 2Y, \quad Y^2 = 1 + 2X + 2Y.$$

Its three distinguished restriction columns are $(1, 0, 0), (1, 1, 1), (1, 2, 2)$, with dimensions $1, 4 + 2\sqrt{3}, 7 + 4\sqrt{3}$. Their twist-one contribution in $I(X)$ is $M = 18 + 10\sqrt{3}$. The total dimension of $I(X)$ is $T = 30 + 18\sqrt{3}$. Its first twist trace is zero, since $X \neq 1$. All other twists have modulus one, requiring $M \leq T - M$. But $2M - T = 6 + 2\sqrt{3} > 0$, the contradiction. An independently reconstructed labelled- S ray against 14,128 columns supplies an additional exact certificate, without being needed for this structural proof. $\square$

5 The dimension-forty rank-one extension family

5.1 A fiber functor gives two exclusions

Proposition 5.1. *The rings $R9_2_1225$ and $R9_2_1673$ have no complex fusion categorification.*

Proof. In either ring the eight labels other than 5 form a fusion subring $\mathcal{D}$ of dimensions $1^4, 2^4$ and FP dimension 20. The other simple m satisfies

$$Xm = mX = d_X m, \quad m^2 = \sum_X d_X X.$$

The additive category generated by m is a rank-one left $\mathcal{D}$ -module category, equivalent to Vec . Its action gives a strong exact faithful tensor functor $\mathcal{D} \rightarrow \text{End}(\text{Vec}) = \text{Vec}$. Ordinary Tannaka reconstruction gives $\mathcal{D} \simeq \text{Rep}(H)$ for a semisimple Hopf algebra of dimension 20. This uses neither a pivotal structure nor Hopf self-duality.

The complete dimension-twenty semisimple list consists of group algebras, their duals, and the two exceptional Gelaki algebras A_0, A_1 [7, 8, 9]. Function Hopf algebras are pointed; group representation rings are commutative. The two exceptional character rings are also commutative, as the direct calculation below proves. Both source neutral rings are noncommutative, witnessed respectively by the products of labels $(1, 2)$ and $(1, 3)$. Thus none of the possibilities can occur. $\square$

5.2 Explicit Hopf data and a unitary model

For an odd prime p and $t = 0, 1$, write $B(a, e, b) = x^a \delta_e g^b$, with $a \in \mathbb{Z}/p$ and $e, b \in \mathbb{Z}/2$. The Andruskiewitsch-Natale construction [9] has

$$\begin{aligned} B(a, e, b)B(c, f, d) &= \delta_{e,f}(-1)^{tebd}B(a + (-1)^b c, e, b + d), \\ \Delta B(a, e, b) &= \sum_{s=0,1} B(a, s, b) \otimes B((-1)^s a, e - s, b), \\ \epsilon B(a, e, b) &= \delta_{e,0}. \end{aligned}$$

Its four one-dimensional simples $L(e, s)$ have $x = 1$ and $g = (-1)^s i^{te}$. The remaining $p - 1$ simples $V(e, j)$, $1 \leq j \leq (p - 1)/2$, have dimension two and x -eigenvalues $\zeta_p^{\pm j}$. Their characters are invariant under $a \mapsto -a$, so inversion in the coproduct is invisible in every character convolution. This proves commutativity of both exceptional character rings. Explicitly,

$$\begin{aligned} L(e, s)L(f, r) &= L(e + f, s + r + tef), \\ L(e, s)V(f, j) &= V(e + f, j), \\ V(e, j)V(f, k) &= V(e + f, j + k) + V(e + f, j - k), \end{aligned}$$

where $V(e, -j) = V(e, j)$ and $V(e, 0) = L(e, 0) + L(e, 1)$.

Put $\zeta = \zeta_p$. An explicit symmetric nondegenerate ordinary Hopf pairing and its inverse copairing are

$$\begin{aligned} P(B(a, e, b), B(c, f, d)) &= \delta_{e,d} \delta_{f,b} \zeta^{ac} i^{tbd}, \\ \gamma &= \frac{1}{p} \sum_{a,c,e,b} \zeta^{-ac} i^{-teb} B(a, e, b) \otimes B(c, b, e). \end{aligned}$$

The positive Haar functional is $h(B(a, e, b)) = \delta_{a,0} \delta_{b,0}/2$. Set $\lambda = \sqrt{4p} h$. Directly,

$$\begin{aligned} (\Delta \otimes 1)\gamma &= \gamma_{13}\gamma_{23}, & (1 \otimes \Delta)\gamma &= \gamma_{12}\gamma_{13}, \\ (S \otimes 1)\gamma &= (1 \otimes S)\gamma, & (\lambda \otimes \lambda)(1 \otimes S)\gamma &= 1. \end{aligned}$$

The copairing is symmetric, $S^2 = 1$, and the distinguished grouplike is one. These are precisely the conditions of [10, Corollary 3.19]; rigidity is supplied by their Section 3.6. Hence there is a coherent fusion category $\text{Rep}(A_t) + \text{Vec}$ with the displayed odd rules. In their dual-Hopf convention the self-duality map is ST_0 , but their formula for the associator inserts S^{-1} . Thus the actual associator copairing is the γ displayed above, without another antipode.

The compact involution is

$$B(a, e, b)^* = (-1)^{teb} B(-(-1)^b a, e, b).$$

The Haar inner product makes these basis vectors orthogonal of squared norm $1/2$. Each block of γ is a Fourier bicharacter element times a unitary and a phase. The odd Fourier associator is

$$\phi(B(a, e, b)) = \frac{i^{-teb}}{\sqrt{p}} \sum_u \zeta^{-(-1)^{eua}} B(u, b, e),$$

a permutation of normalized unitary Fourier blocks. The remaining nonidentity associators are representations of γ , the Hopf transport maps $h \otimes v \mapsto h_{(1)} \otimes h_{(2)} v$, and their inverses. Haar invariance makes these transports unitary. In the odd-even-odd case right multiplication composed with the antipode is a star representation, with the corresponding opposite algebra understood. All associators are therefore unitary.

For $p = 5$, $t = 1$ gives $R9_2_965$, with source-to-model map $(0, 2, 3, 6, 7, 8, 5, 4, 1)$. The $t = 0$ construction recovers $R9_2_21$ as a positive control. The independent checker verifies both copairing identities, normalization, all character convolutions, and all 729 coefficients.

6 Three relative-cocycle models

6.1 A construction with its twisted character formula

Let G be a finite group, $H < G$, $\omega \in Z^3(G, U(1))$, and let ψ be normalized with $\delta\psi = \omega^{-1}|_H$ in the convention

$$\psi(a, b)\psi(ab, c) = \omega(a, b, c)\psi(b, c)\psi(a, bc).$$

The subgroup algebra $A = \bigoplus_{h \in H} h$ in Vec_G^ω is connected and separable. Its bimodules form the unitary group-theoretical category $\mathcal{C}(G, \omega, H, \psi)$ [1, 11]. For a double-coset representative u put $K_u = H \cap uHu^{-1}$. On its degree- u fiber, the stabilizer representation has multiplier

$$\alpha_u(a, c) = \frac{\psi(a, c)\omega(a, u, u^{-1}cu)}{\omega(a, c, u)\omega(u, u^{-1}au, u^{-1}cu)\psi(u^{-1}au, u^{-1}cu)}.$$

This follows by writing the left and right module actions:

$$L_a L_c = \psi(a, c)\omega(a, c, v)^{-1} L_{ac}, \quad R_c R_a = \omega(v, a, c)\psi(a, c) R_{ac}, \quad R_c L_a = \omega(a, v, c) L_a R_c.$$

One defines $\rho_u(a) = R_{u^{-1}au}^{-1} L_a$ and substitutes these identities. For the relative tensor product the balancing relation is

$$R_h(m_u) \otimes n_v = \omega(u, h, v) m_u \otimes L_h(n_v).$$

In a stabilizer trace at degree uv , a fixed coset contributes

$$\frac{\omega(u, h, v)}{\omega(a, u, v)\omega(u, v, b)} \chi_u^M(a) \chi_v^N(h), \quad h = u^{-1}au, \quad b = (uv)^{-1}a(uv).$$

Projective character inner products recover fusion coefficients exactly. The transport phase between homogeneous fibers is derived from the same three module identities and recorded with a separate derivation in the supplement. An untwisted convolution would be incorrect here.

6.2 The model of dimension 120

For $G = S_5$, $H = S_3 \times C_2$, let ε be parity and α the supplied normalized $\mathbb{F}_2$ two-cocycle, defined by a fixed section of the Schur double cover. Put

$$\omega(g, h, k) = (-1)^{\varepsilon(g)\alpha(h,k)}.$$

The explicit unit-modulus cochain ψ trivializes its restriction to H . The independent replay reconstructs S_5 from permutations and checks all 120^3 two-cocycle equations and all 12^3 algebra equations. The three stabilizer orders are 12, 2, 4, with projective irreducible sizes $(1, 1, 1, 1, 2, 2)$, $(1, 1)$, (2) . The nondegenerate multiplier on the last V_4 gives $M_2(\mathbb{C})$. Simple dimensions are $1^4, 2^2, 6^3$, and the complete tensor is $R9_2_92$ under $(0, 3, 1, 5, 7, 6, 4, 2, 8)$.

6.3 Two models on the same group and subgroup

Take $G = S_4 \times C_2$, write e for the external bit and s for parity, and set

$$x = ((12), 1), \quad y = ((34), 0), \quad H = \langle x, y \rangle \simeq V_4.$$

Let X, Y be its two coordinates and pull the explicit Schur cocycle α back from S_4 . The first associator and trivialization are

$$\omega_+ = (-1)^{e \smile (\alpha + e^2)}, \quad \psi_+(a, b) = i^{X(a)Y(b)}.$$

They construct $R9_2_392$ under $(0, 1, 2, 8, 7, 3, 5, 6, 4)$.

For the second model let Ω be the copied eighth-root-valued S_4 cocycle from the lower-rank project. Its full table and original source hash are supplied; all 24^4 cocycle identities and every transported table entry are independently checked. Set

$$\omega_- = \Omega(-1)^{e \smile \alpha + s \smile e^2}.$$

In the order $(1, y, x, xy)$ of H , a trivialization $\psi_-(a, b) = \zeta_8^{q(a,b)}$ is given by

$$(q(a, b)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 7 \\ 0 & 4 & 0 & 0 \\ 0 & 1 & 6 & 7 \end{pmatrix}.$$

The resulting tensor is $R9_2_1246$ under $(0, 8, 7, 5, 6, 3, 1, 2, 4)$. For each construction the double-coset sizes are 4, 4, 16, 16, 8, with stabilizer orders 4, 4, 1, 1, 2 and projective dimensions $(1, 1, 1, 1)$, (2) , (1) , (1) , $(1, 1)$. This gives type $1^4, 2^3, 4^2$. The independent checker verifies all 48^4 cocycle equations, the subgroup algebra and all tensor entries over the exact cyclotomic field. Neither construction relies on an unproved classification of dimension 24.

7 The half- E_6 family and the model inventory

7.1 Products and associative zesting

Let $\mathcal{E}$ be the unitary half- E_6 category and f its specified central fermion [6]. The products with Ising, $\text{Rep}(S_3)$, $\text{PSU}(2)_5$ and $\text{Vec}(C_3)$ give respectively $R9_2_71, 79, 499, 2020$. Their tensors are direct Kronecker products. For $R9_2_1218$ use the Ising parity on $\mathcal{E} \boxtimes \text{Ising}$ and zest the $(1, 1)$ product by f . The associator cochain $\nu(a, b, c) = i^{abc}$ has coboundary $(-1)^{abcd}$ on parity bits, exactly canceling the central fermion crossing. All sixteen coherence equations are checked. The cochains and the initial categories are unitary.

7.2 A central-fermion shear with two coherent actions

On $\mathcal{D} = \mathcal{E} \boxtimes \text{Vec}(C_2)$ define $T(Z, h) = (f^h Z, h)$. Its tensorator moves the inserted f^k left past Z using the inverse half-braiding, and multiplies the adjacent powers of f . The half-braiding identities prove its pentagon. Composition gives T^2 the identity tensorator multiplied by $(-1)^{hk}$, because $c_f = -1$. For either $u = i$ or $u = -i$, the components $\eta_{(Z, h)} = u^h$ are a monoidal isomorphism $T^2 \rightarrow \text{Id}$:

$$u^{h+k \bmod 2} (-1)^{hk} = u^h u^k.$$

The action equation $T\eta = \eta T$ holds because T preserves h . These are two coherent unitary C_2 actions.

The action fixes $(1, 0), (f, 0), (x, 0), (x, 1)$ and exchanges $(1, 1), (f, 1)$. Its equivariantization has four invertibles, four dimension- $(1 + \sqrt{3})$ simples and one dimension-two object Y . Write ϵ for the acting sign and choose lifts a, b over x, xh . A normalized $t : fx \rightarrow x$ and $r = tc_x^{-1} : xf \rightarrow x$ satisfy

$$r(t \otimes 1_f) = -t(1_f \otimes r).$$

This is naturality and the half-braiding hexagon with $c_f = -1$; it makes f fix the two lifts of x and exchange the two lifts of xh . The rules include

$$a^2 = 1 + f + 2a, \quad Y^2 = 1 + f + \epsilon + f\epsilon,$$

$$aY = b + \epsilon b, \quad bY = a + \epsilon a, \quad ab = Y + b + \epsilon b.$$

The sign of η changes the unit-pairing eigenvalue on the odd fixed simple. Thus one choice has $b^2 = 1 + f\epsilon + a + \epsilon a$, and the other $b^2 = \epsilon + f + a + \epsilon a$. These give *R9_2_96* and *R9_2_1266*, including their distinct dualities. All coefficients, action coherence and forgetful identities are independently reconstructed.

7.3 All 36 positive tensors

Table 2 lists the accepted models. For group-theoretical and equivariantization entries, the archive supplies the actual cocycles, action data and projective stabilizer calculations, rather than an action on the Grothendieck ring alone. The exceptional AH3 entry is matched to its published categorical tensor. The WZW, subfactor, product and known near-group entries use their specified unitary realizations. The word “model” asserts existence for a based ring and makes no claim of novelty up to tensor equivalence.

Table 2: All unitary models. The final column is the source-to-model permutation; indices are zero-based.

Case	Construction	Basis map
R9_2_21	Parity-twisted V4 equivariantization of TY(C5)	013576428
R9_2_71	half-E6 x Ising	043276518
R9_2_79	half-E6 x Rep(S3)	043576218
R9_2_92	C(S5, sign cup Schur, S3 x C2, explicit trivialization)	031576428
R9_2_96	C2 equivariantization of the central-fermion shear on half-E6 x Vec(C2)	023567418
R9_2_369	Rep(PGL(2, 7))	018245367
R9_2_389	Inversion orbifold of cyclic generalized Moore-Read cat-egory, dimension16	056241873

Case	Construction	Basis map
R9_2_392	$C(S4 \times C2, e \text{ cup } (\text{Schur} + e^2), V4 \text{ graph subgroup, explicit trivialization})$	012873564
R9_2_499	half-E6 x PSU(2)_5	035412678
R9_2_515	Diagonal-fermion quotient (Ising box PSU(2)_10)_1+psi box V10	054321687
R9_2_746	Inversion equivariantization of cyclic-C9 generalized Tambara-Yamagami category obtained by explicit central C3 condensation	016725384
R9_2_748	TY(C3)^S3 by inversion, parity coherence 0	016724583
R9_2_812	Rep(V4 semidirect Dih(C9))	016725384
R9_2_813	Rep(V4 semidirect Dih(C3 x C3))	016724583
R9_2_936	Rep(PSL(2,8))	085167243
R9_2_965	Davydov-Runkel rank-one extension of Gelaki A1, dim(A1)=20	023678541
R9_2_1016	$C(S4, 1, V4 \text{nonnormal}, 1)$	067854123
R9_2_1209	Inversion equivariantization of cyclic-C9 generalized Tambara-Yamagami category obtained by explicit central C3 condensation	067253814
R9_2_1211	TY(C3)^S3 by inversion, parity coherence 1	067245813
R9_2_1218	Fermionic associative C2 zest of half-E6 x Ising	025768143
R9_2_1246	$C(S4 \times C2, \text{exotic } S4 \text{ associator times } e \text{ cup Schur plus sign cup } e^2, V4 \text{ graph subgroup, explicit trivialization})$	087563124
R9_2_1254	Mixed order-four cyclic-cover functor descended to a coherent C2 action on generalized Moore-Read category, then equivariantized	087561243
R9_2_1266	C2 equivariantization of the central-fermion shear on half-E6 x Vec(C2)	067832154
R9_2_1372	Asaeda-Haagerup category AH3	034128576
R9_2_1373	$(\text{Fib box Fib})^{\wedge} S3$, sign quotient acts by swapping factors	045138762
R9_2_1374	D16 even category	087246135
R9_2_1396	Pulled-back LPR-24 action along Dih(C9)->S3	067253814
R9_2_1397	Pulled-back LPR-24 action along Dih(C3 x C3)->S3	067245813
R9_2_1557	$C(GL(3,2), 1, S4, 1)$	067124385
R9_2_1848	Izumi isotropic C3 orbifold of unitary NG(C18,18)	015426873
R9_2_1857	Spin(8)_2 WZW category equivariantized by order-three Dynkin triality	087213564
R9_2_1979	Izumi near-group (Q8,2)	013467582
R9_2_1983	Ordinary-swap equivariantization of the equal half-E6 diagonal-fermion product	013587642
R9_2_2020	half-E6 x Vec(C3)	045871236
R9_2_2029	Triality equivariantization of the Tannakian-zested Spin(8)_2 category	087352164
R9_2_2083	Rep(C19 semidirect C3, u=7)	012378564

8 Cyclic internal-end algebras and multiplicity

Date: 24 October 2026. Scope: arbitrary fusion categories over $\mathbb{C}$. No pivotal, spherical, pseudo-unitary, unitary, or braided hypothesis occurs.

8.1 Theorem

Let D be a fusion category, M a finite semisimple indecomposable left D -module category, and x a simple object of M . Suppose its internal end algebra has underlying object

$$A = [x, x]_D = \text{direct_sum}_{g \in H} g,$$

where H is a finite cyclic group of invertible objects, each with multiplicity one. Then $y \otimes x$ is multiplicity-free for every simple object y of D .

8.1.1 Proof

For each g in H , the nonzero morphism $g \otimes x \rightarrow x$ is an isomorphism. Consequently every component $g \otimes h \rightarrow gh$ of the multiplication of A is nonzero: it is induced by composition of these isomorphisms. This is not an assumption about an arbitrarily chosen algebra structure.

The internal-Hom equivalence of Ostrik, Theorem 1, identifies M with the category of right A -modules in D , sends x to A , and sends $y \otimes x$ to the free module $y \otimes A$. Therefore their endomorphism algebras agree. By free-module adjunction,

$$\begin{aligned} \text{End}_A(y \otimes A) &= \text{Hom}_D(y, y \otimes A) \\ &= \text{direct_sum}_{g \in K} \text{Hom}_D(y, y \otimes g), \end{aligned}$$

as vector spaces, where $K = \{g \in H : y \otimes g \text{ is isomorphic to } y\}$. K is a subgroup of H , hence cyclic. Every displayed summand is one-dimensional. Choose nonzero isomorphisms $u_g: y \rightarrow y \otimes g$. The associated free-module endomorphism is the composite

$$\begin{aligned} y \otimes A &\xrightarrow{u_g \otimes \text{id}} (y \otimes g) \otimes A \\ &\xrightarrow{\text{associator}} y \otimes (g \otimes A) \\ &\xrightarrow{\text{id} \otimes \text{multiplication}} y \otimes A. \end{aligned}$$

Call this endomorphism e_g . The multiplication of A and the u_g are isomorphisms on the relevant simple summands, so composition satisfies $e_g e_h = c(g, h) e_{gh}$ (or the reversed product according to convention), with $c(g, h)$ nonzero. In particular each e_g is invertible. If t generates K of order n , the powers of e_t span all these homogeneous summands and $e_t^n = c \text{id}$ for c nonzero. Thus

$$\text{End}_A(y \otimes A) \text{ is isomorphic to } C[T]/(T^n - c).$$

This algebra is commutative. In a semisimple C -linear category the endomorphism algebra of $\text{direct_sum}_i m_i z_i$, for distinct simples z_i , is the product of $\text{Mat}_{m_i}(C)$; it is commutative precisely when every m_i is at most one. This proves the theorem. The case $K=\{1\}$ is included.

The proof does not invoke $H^2(H, C^*)=0$, a fiber functor on D , a classification of Tambara-Yamagami categories, or a normalization of their associators. The cyclic hypothesis is essential: for V_4 a twisted group algebra can be $\text{Mat}_2(C)$.

8.2 Four rank-nine applications

All labels are zero-based source-tensor labels. In every row, the invertible group is $G = \{0,1,2,8\}$, generated by label 1 with powers $(0,1,8,2)$. In particular it is C4, not V4.

Ring	D simples	y	x	x'
R9_2_1025	0,1,2,6,8	6	4	5
R9_2_1709	0,1,2,7,8	7	5	6
R9_2_1759	0,1,2,6,8	6	3	4
R9_2_1988	0,1,2,7,8	7	5	6

For each tensor the listed D is a fusion subring with rules $g y = y g = y$ and $y^2 = \sum_{g \in G} g$. Its hypothetical categorical realization is a fusion subcategory of any hypothetical categorification of the full ring. The additive category M spanned by x, x' is a D-module subcategory: each g fixes both objects and

$$y \text{ tensor } x = 2 x', \quad y \text{ tensor } x' = 2 x.$$

It is indecomposable since y connects its two simples. The internal end of x in D is exactly $\sum G$: the four invertibles each fix x , and y has no diagonal action on x . The theorem forces $y \text{ tensor } x$ to be multiplicity-free, contrary to $2 x'$. Equivalently its endomorphism algebra would have to be both commutative and $\text{Mat}_2(\mathbb{C})$.

Conclusion: all four rings have no categorification over $\mathbb{C}$.

8.3 Exact certificate and provenance

`cyclic_tensors.json` contains the four full 9-by-9-by-9 tensors copied from the accepted phase-three corpus, without alteration. `verify_cyclic_obstruction.py` checks their SHA-256 values, all fusion ring axioms, C4 multiplication, dual closure and multiplication closure of D, the complete action of D on M, the internal-end support, and the off-diagonal multiplicity two. The output is `cyclic_obstruction_certificate.json`. These finite checks verify the hypotheses of the theorem; they do not replace the categorical proof.

8.4 Primary source

V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups 8 (2003), 177–206, arXiv:math/0111139, Section 3.2, Lemma 5 and Section 3.3, Theorem 1. The proof explicitly uses the internal end of a chosen object and the free-module adjunction, and imposes no unitarity assumption.

<https://arxiv.org/pdf/math/0111139>

The cyclic obstruction above is a direct algebraic consequence proved here; no claim of priority or literature novelty is made.

9 A nonsolvable dimension-sixty neutral component

Date: 24 October 2026. General-complex scope, with no braiding or unitarity assumptions on the hypothetical categorification.

9.1 Theorem

Let D be a fusion category with simple FP dimensions $1, 1, 2, 2, 5, 5$. Assume:

1. D equals its adjoint subcategory (its universal grading is trivial).
2. The nontrivial invertible g fixes the two dimension-two simples and exchanges the two dimension-five simples.

Then every faithful finite-group extension of D is group-theoretical, and in particular is integral.

9.1.1 Step 1: D cannot be solvable

Suppose D were solvable. By ENO Proposition 4.4 it admits a construction from Vec by successive extensions and equivariantizations by cyclic groups of prime order. The last step cannot be an extension, since D has no nontrivial faithful grading. Thus $D = E^{\{C_p\}}$ for a fusion category E , and the canonical central $\text{Rep}(C_p)$ embeds into D under forgetting its central structure. D has exactly two invertibles, so $p=2$, and the nontrivial character has underlying object g .

De-equivariantize by this central $\text{Rep}(C_2)$, or equivalently take modules over its regular central algebra $A=1+g$. This is a fusion category of dimension 30. Its simple dimensions can be determined without knowing any associator of D . The free orbit $\{1, g\}$ gives one invertible; the free orbit consisting of the two dimension-five objects gives one simple of dimension five. Each fixed dimension-two object gives two distinct simple A -modules of dimension one. Indeed, the endomorphism algebra of its free module has dimension two and is semisimple, hence is $C+C$; the two simple summands each have underlying object equal to the original dimension-two object. FP dimension after passage to A -modules is the underlying FP dimension divided by $\text{FPdim}(A)=2$. These six objects exhaust the simples since every simple A -module is a summand of a free module. Therefore the de-equivariantization has dimensions $1^5, 5$.

Its invertibles form the cyclic group C_5 . Writing z for its unique noninvertible simple, rigidity and dimensions force

$$hz = z = zh, \quad z^2 = \sum_{h \in C_5} h + 4z.$$

But Etingof-Gelaki-Ostrik, Corollary 7.4, applies to arbitrary complex fusion categories with n cyclic invertibles and one noninvertible of dimension n . It forces $n+1$ to be a prime power. Here $n=5$ and 6 is not a prime power. This contradiction proves that D is not solvable.

9.1.2 Step 2: D is Morita equivalent to $\text{Vec}_{\{A_5\}^\omega}$

ENO Theorem 9.16 says that every fusion category of dimension 60 is weakly group-theoretical. Hence D is categorically Morita equivalent to a nilpotent fusion category N of the same dimension. Choose a tower of faithful group extensions constructing N from Vec . The product of the orders of the grading groups is 60.

If all grading groups are solvable, their normal series refine this tower into cyclic-prime extensions; N and hence D are solvable, contrary to Step 1. Thus at least one grading group is nonsolvable. Every proper divisor of 60 is the order of only solvable finite groups; the unique nonsolvable group of order 60 is A_5 . These standard finite-group facts can also be expressed using the smallest nonabelian simple group A_5 , of order 60, and a composition series. The nonsolvable grading

group therefore has order 60, and every other extension step is trivial. Consequently N is a pointed category $\text{Vec}_{A_5}^{\omega}$ for some 3-cocycle ω . No value or trivialization of ω is assumed.

9.1.3 Step 3: all extensions of $\text{Vec}_{A_5}^{\omega}$ are pointed

Let $P = \text{Vec}_{A_5}^{\omega}$, and let B be an invertible P -bimodule category. As a left P -module category it has pointed dual (indeed the right action identifies its dual with P , up to the harmless opposite convention). By the classification of pointed-dual module categories, Naidu-Nikshych Example 2.2, such a module category is described by a normal abelian subgroup H of A_5 and compatible cochain data. Since A_5 has no nontrivial normal abelian subgroup, $H = \{1\}$. Therefore B is the regular left P -module category and has 60 simple objects, transitively permuted by the 60 invertibles of P .

Each homogeneous component of a faithful group extension of P is an invertible P -bimodule category. Consequently each component has these 60 simple objects, all with the same FP dimension d , because the invertibles of P permute them transitively. A faithful homogeneous component has total FP dimension $\text{FPdim}(P) = 60$. Hence $60 \cdot d^2 = 60$ and $d = 1$. Every simple object of the extension is invertible: the extension is pointed. This argument applies to every ω and every coherent extension, not merely to untwisted or preselected models.

9.1.4 Step 4: transport the extension through Morita equivalence

If C is any faithful finite-group extension of D , ENO Lemma 3.4 constructs an extension of P which is Morita equivalent to C . It does this by replacing D with its category of bimodules over an indecomposable algebra and applying the same construction to C ; the grading survives because the algebra lies in the neutral component. Step 3 makes the resulting category pointed. Thus C is group-theoretical. Group-theoretical fusion categories are integral: the standard double-coset description of simples gives dimensions

$$[H : H \cap tHt^{-1}] \cdot \dim(\text{projective irreducible representation}),$$

which are integers. This proves the theorem.

9.2 Application to two remaining rank-nine rings

Both tensors have global FP dimension 120. Their integral objects form a dimension-60 fusion subring D and the other three objects each have dimension $2\sqrt{5}$. The tensor verifies a faithful C_2 grading with D as neutral component. In the notation $(1, g, a, b, u, v)$ of dimensions $(1, 1, 2, 2, 5, 5)$, the neutral labels are:

Ring	Neutral labels $(1, g, a, b, u, v)$	Odd labels
R_{9_2-585}	$(0, 1, 7, 8, 3, 4)$	$2, 5, 6$
R_{9_2-1303}	$(0, 4, 5, 6, 7, 8)$	$1, 2, 3$

The neutral rules satisfy the theorem's two conditions. In particular the adjoint closure is all six labels, while g fixes a, b and exchanges u, v . A hypothetical categorification would therefore be an integral extension by the theorem. Its three dimensions $2\sqrt{5}$ contradict integrality.

Conclusion: R9_2_585 and R9_2_1303 have no complex categorification.

This proof does not assume that a fusion-ring match to a known dimension-60 model classifies its categorifications. Step 2 establishes the necessary Morita form for every hypothetical categorification.

9.3 Exact certificate

d120_tensors.json contains the two unchanged full source tensors. verify_dimension120_obstruction.py checks all fusion-ring axioms, all FP dimension equations exactly in $\mathbb{Q}(\sqrt{5})$, the neutral subcategory, faithful grading, its adjoint closure, and the complete permutation of simples by g . Its output is dimension_120_obstruction_certificate.json.

9.4 Primary references and theorem scopes

* P. Etingof, D. Nikshych and V. Ostrik, *Weakly group-theoretical and solvable fusion categories*, Adv. Math. 226 (2011), 176–205, arXiv:0809.3031v2. Definition 1.1; Proposition 4.4; Proposition 4.5; Theorem 9.16 (dimension 60); Lemma 3.4 (Morita transport of extensions). <https://arxiv.org/pdf/0809.3031> * P. Etingof, S. Gelaki and V. Ostrik, *Classification of fusion categories of dimension pq *, IMRN 2004, no. 57, 3041–3056, arXiv:math/0304194v2, Corollary 7.4. The statement concerns all complex fusion categories and is not a unitary-only classification. <https://arxiv.org/pdf/math/0304194> * D. Naidu and D. Nikshych, *Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups*, Commun. Math. Phys. 279 (2008), 845–872, arXiv:0705.0665v2, Example 2.2 and Section 4.2. The normal-abelian-subgroup condition covers arbitrary 3-cocycles ω . <https://arxiv.org/pdf/0705.0665>

The theorem is proved here by combining the stated results. No claim of priority or exhaustive literature novelty is made.

10 Cyclic generalized Tambara-Yamagami constructions

10.1 Result

The rings R9_2_746 and R9_2_1209 have explicit unitary categorifications. They are the two parity-coherence inversion equivariantizations of a 12-simple-object generalized Tambara-Yamagami category with invertible group C_9 and three simple objects of dimension $\sqrt{3}$. This category is constructed below by an explicit central C_3 condensation, rather than inferred from vanishing of a group cohomology obstruction.

All half-braidings in this note use the orientation $c_z(X): z \otimes X \rightarrow X \otimes z$. This is inverse to the orientation used for central structures in GNN Section 2.2; keeping this convention explicit avoids a sign reversal in the bicharacter.

10.2 Starting unitary categories and central object

Let $\omega = \exp(2\pi i/3)$, $\zeta = \exp(2\pi i/9)$. Take the unitary category $T = TY(C_3, \chi, 1/\sqrt{3})$, $\chi(a, b) = \omega^{ab}$, with the skeletal associators of GNN Section 4.1. Its objects are a in C_3 and m , with $m^2 = \sum_a a$. The central lift z of the invertible $a=1$ has right half-braiding

$$c_z(b)=\omega^b, \quad c_z(m)=\omega.$$

The equations are $c_z(m)^2=\chi(1,1)^{-1}$ and $c_z(z \otimes z')(m)=c_z(m)c_z'(m)\chi(a,a')^{-1}$. Thus z has order 3 as a central object, and $c_z(z)=\omega$. These formulas follow by substitution in the displayed TY associators; they are also the inverse-half-braiding form of the central invertibles in GNN Proposition 4.1. No braiding on T itself is asserted.

Let $P=\text{Vec}(C_9, \Omega)$, with representatives $0, \dots, 8$ and

$$\Omega(a, b, c) = \zeta^{(3a \bmod (b+c)/9)}.$$

This is a normalized unitary 3-cocycle. The underlying invertible $h=3$ has right half-braiding $\beta(x)=\zeta^{(2x)}$. Its central square has underlying object 6 and half-braiding $\zeta^{(4x)}$. Its central cube has half-braiding

$$\zeta^{(6x)} \Omega(x, 6, 3) = \zeta^{(9x)} = 1.$$

In the product category $B = T \boxtimes P$, the central object

$$g = (z, 3)$$

has order 3. Its self-braiding is $\omega * \zeta^6 = 1$. The associator on its three powers is identically 1, since $\Omega(3a, 3b, 3c) = 1$. Their braiding is also identically 1, because the self-braiding of a generator is 1 and this is a pointed category with trivial associator. They therefore give an explicit central Tannakian $\text{Rep}(C_3)$. Set $A = 1 + g + g^2$, with the ordinary group-algebra multiplication (all nonzero multiplication scalars 1). It is a commutative separable central algebra with the usual unitary normalization. The right A -module category $D = B_A$ is a unitary fusion category of dimension $(6 \cdot 9)/3 = 18$.

The action of g on every simple of B is free: its C_9 charge shifts by 3. Hence every free simple A -module is simple, and these are indexed by the 12 orbits. Writing the nine invertibles as u^a , a in C_9 , and the three other simples as X_r , r in C_3 , gives

$$\begin{aligned} u^a u^b &= u^{(a+b)}, \\ u^a X_r &= X_r u^a = X_{(r+a)}, \\ X_r X_s &= \sum_{h=0,1,2} u^{(r+s-3h)}. \end{aligned}$$

For example, use orbit representatives $(0, a)$ and (m, r) ; the relation $(1, 3) = 1$ in the quotient identifies $(h, r+s)$ with $(0, r+s-3h)$. The dimensions are $1^9, (\sqrt{3})^3$.

10.3 A coherent involution preserving the algebra

On T use the canonical orthogonal inversion autoequivalence $F(a) = -a, F(m) = m$. The tensorator is the canonical identity map, including the permutation of summands of m^2 . This is an actual involutive tensor autoequivalence, as in GNN Proposition 4.8.

On P use $F(x) = -x$ with tensorator

$$J(a, b) = \zeta^{\mu(a, b)}, \quad \mu(a, b) = -3a \cdot 1_{(b>0)} \bmod 9.$$

Writing w for the exponent of Ω , direct substitution gives

$$\begin{aligned} \mu(a, b) + \mu(a+b, c) - \mu(b, c) - \mu(a, b+c) \\ = w(-a, -b, -c) - w(a, b, c) \bmod 9, \\ \mu(a, b) + \mu(-a, -b) = 0 \bmod 9. \end{aligned}$$

Consequently $F^2 = \text{Id}$ as a tensor functor with identity square coherence. On T it sends the central z to z^2 . On P the transported half-braiding of $h=3$ at the new argument x is

$$\text{beta}(-x) J(3, -x)/J(-x, 3) = \text{zeta}^{(4x)},$$

the half-braiding of the inverse central object h^2 . Thus $F(g)=g^2$ as central objects. On powers of g the tensorator is identically 1 because $\mu(3a, 3b)=0$. The permutation $g^i \rightarrow g^{-i}$ therefore defines an actual algebra isomorphism $F(A) \rightarrow A$ whose square is identity, respecting both multiplication and the central structure. Hence F descends to a coherent unitary $C2$ action on D . It acts by

$$u^a \rightarrow u^{(-a)}, \quad X_r \rightarrow X_{(-r)}.$$

There is a second coherent action with the same tensor functor: multiply the square coherence by the parity natural automorphism, $+1$ on pointed simples and -1 on X_r . Parity is monoidal, invariant under F , and trivial on A ; hence it obeys the $C2$ action coherence before and after descent. Both actions consist entirely of unitary structure maps.

10.4 Exact equivariant fusion tensors

Let E_0 and E_1 be the equivariantizations for the identity and negative odd square coherence, respectively. They are unitary categories of dimension 36. Inversion fixes the unit and X_0 , and has four two-element pointed orbits and one two-element nonpointed orbit. Thus each E has simple dimensions $1^2, 2^4, (\sqrt{3})^2, 2\sqrt{3}$ and rank 9.

The two fixed objects give two equivariant lifts each. Every free orbit gives one simple. The fusion multiplicity for a fixed output is $(\dim + \text{trace})/2$; for a free output orbit it is the multiplicity at either representative. Only fixed input objects contribute to the trace. The only nontrivial fixed-input fusion line is $X_0 X_0 \rightarrow 1$. This line is inherited from $(m, 0)(m, 0) \rightarrow (0, 0)$, where the tensorator acts identically. In E_0 its trace is the product of the equivariant signs. In E_1 each fixed odd equivariant structure squares to -1 , so their product contributes the extra factor -1 . This determines every tensor coefficient without choosing unknown F -symbols on D .

The standalone generator `cyclic_ty_models.py` constructs both tensors from these orbit/trace formulas, verifies the unit, duality, Frobenius reciprocity and all associativity equations, and matches every one of the 729 original census coefficients. The source-to-model permutations are

Source	Coherence	Basis permutation
R9_2_746	identity	[0,1,6,7,2,5,3,8,4]
R9_2_1209	negative odd square	[0,6,7,2,5,3,8,1,4]

`cyclic_ty_models.json` contains both full tensors, the orbit labels, the intermediate rank 12 tensor, and the complete exact parameter checks. The finite checks corroborate explicit categorical data; associativity of the fusion tensor alone is not used as an existence argument.

10.5 References

- D. Tambara and S. Yamagami, Tensor categories with fusion rules of self-duality for finite abelian groups, J. Algebra 209(1998), 692–707.
- S. Gelaki, D. Naidu and D. Nikshych, Centers of graded fusion categories, Algebra & Number Theory 3(2009), 959–990; arXiv:0905.3117. Sections 2.4–2.5 give equivariantization/de-equivariantization; Section 4.1 gives the explicit TY associators; Proposition 4.1 describes central invertibles and Proposition 4.8 gives the orthogonal action. Primary source checked: <https://arxiv.org/pdf/0905.3117>.

This settles existence only; no classification or novelty claim about all tensor-equivalence classes with these fusion rings is made.

10.6 Uniform family

The same construction works with 3 replaced by any odd $n > 1$. Put $\zeta = \exp(2\pi i/n^2)$, $\chi(a,b) = \zeta^{(nab)}$ on C_n , choose $c_z(m) = \chi(1,1)^{((n-1)/2)m}$ (whose square is $\chi(1,1)^{(-1)m}$), and use

$\Omega(a,b,c) = \zeta^{(n \lfloor a \rfloor ((b+c)/n^2))}$,
 $h=n$, $\beta(x) = \zeta^{((n-1)x)}$,
 $J(a,b) = \zeta^{(-n \lfloor a \rfloor (b>0))}$.

The n -th central power of h has half-braiding $\beta(x)^n \zeta^{(nx)} = 1$. The diagonal generator's self-braiding is 1. Both the associator and inversion tensorator restrict trivially to its powers. Inversion sends its central lift to its inverse because $-(2n-1) = (n-1)^2 \pmod{n^2}$. This gives a unitary category of dimension $2n^2$ with invertibles $C_{(n^2)}$, n simples of dimension $\sqrt{n}$, and the same cyclic generalized-TY formulas. The two equivariantizations have dimension $4n^2$ and $\text{rank}(n^2+n+6)/2$. The two rank-nine models above are the $n=3$ members. This uniform construction does not classify all generalized Tambara-Yamagami categories or their actions.

11 An order-four cover and a coherent descended action

11.1 Result and outline

The ring $R_{9_2,1254}$ is realized by a coherent C_2 equivariantization of a unitary generalized Moore-Read category of dimension 16. The new action comes from a mixed order-four tensor functor on the dimension 32 covering category. Its square becomes tensor-isomorphic to the identity only after central boson condensation. Requiring the cover functor to have square identity would incorrectly discard this construction.

The certificate is `mr_order4_descent_2_3_1_4.json`: it gives every root-of-unity tensorator scalar and every component of the square isomorphism. `mr_order4_descent.py` checks all required equations. The categorical argument below explains why those equations suffice.

11.2 The existing unitary cover and its central algebra

Use a unitary Ising category I with simples $1, \psi, \sigma$, and $P = \text{Vec}(C_{16}, \Omega)$,

$\Omega(a,b,c)=\text{zeta}\backslash^{(2a \text{ floor}((b+c)/16))}$, $\text{zeta}=\exp(2 \pi i/16)$.

Let B be the parity subcategory of I boxtimes P : its simples are (i,a) , with $i=0,1,2$ for $1,\psi,\sigma$, and a even for $i=0,1$ and odd for $i=2$. There are 24 simples and $\text{FPdim}(B)=32$. The invertible

$g=(\psi,8)$

has right half-braiding $g \text{ tensor}(i,a) \rightarrow (i,a) \text{ tensor } g$ of scalar

$\beta(i,a)=\text{zeta}\backslash^{(7a+e\backslash_i)}$, $(e\backslash_0,e\backslash_1,e\backslash_2)=(0,8,12)$.

It has order_2 in $Z(B)$, self-braiding 1, and trivial associator on its powers. Indeed the pointed square correction is $\text{zeta}\backslash^{(2a)}$, giving $\beta_{\text{pointed}}(a)^2 \cdot \text{zeta}\backslash^{(2a)}=1$; the Ising central fermion squares to 1. The regular algebra $A=1+g$ is commutative, separable and central. Its multiplication $g \text{ tensor } g \rightarrow 1$ is the identity in the chosen skeletal basis. Thus $D=B_A$ is a unitary fusion category of dimension 16.

These are precisely the explicitly verified cover and central algebra in phase2 /weak_integral/mr_orbifold.md, with the right-half-braiding orientation made explicit.

11.3 The order-four mixed tensor functor

On simple objects define

$F(i,a)=(i \text{ xor } (a/2 \bmod 2), 3a \bmod 16)$, $i=0,1$;
 $F(2,a)=(2, 3a \bmod 16)$.

This preserves every fusion coefficient and the unit. The file lists the scalar $J_F(a,b;c)$ in μ_{16} on every multiplicity-one fusion channel, with tensorator orientation $F(a) \text{ tensor } F(b) \rightarrow F(a \text{ tensor } b)$. F is linear on each chosen elementary morphism and permutes channel labels in the obvious way. Its tensor-functor equations are verified entry by entry against the cover's actual F -symbols, not only the fusion rules. All nonzero Ising F -symbol magnitudes are preserved: the only non-trivial $\text{magnitude}/\sqrt{2}$ occurs for three σ inputs, and F fixes the Ising type of those inputs. Consequently checking the phase of every nonzero entry checks the entire tensor-functor diagram.

The actual F -symbol exponent is the standard Ising sign times 8, plus $2 \text{ a floor}((b+c)/16)$ from the charge factor. The script checks all 18432 nonzero associator entries and the unit constraints. Since J is unitary and the object map is a permutation, F is a unitary tensor autoequivalence.

$F(g)=g$. The certificate additionally verifies every central transport identity

$J\backslash_F(g,a;ga)/J\backslash_F(a,g;ga)=\beta(Fa)/\beta(a)$.

Choose an algebra isomorphism $\phi:F(A) \rightarrow A$ which is identity on 1 and has scalar $c=\text{zeta}32^2$ on g (equivalently $c=\text{zeta}16$). Here the saved value is $J_F(g,g;1)=\text{zeta}16^2$, so $c^2=J_F(g,g;1)$. This is exactly the algebra multiplication compatibility. The central transport equations make ϕ an isomorphism of central algebras. Therefore F descends to a unitary tensor autoequivalence f of D .

11.4 Squaring before and after condensation

On B , F^2 is not the identity object map. It is

$$K(a) = g^p(a) \text{ tensor } a,$$

where $p(a)=0$ for an even Ising object and 1 for a sigma object. Equip K with its canonical tensorator: move the central g factors to the left using the right half-braiding, and multiply $g^2=1$. More explicitly, reassociate

$$\begin{aligned} (g^p a)(g^q b) &\rightarrow g^p(a(g^q b)) \\ &\rightarrow g^p((a g^q)b) \rightarrow g^p((g^q a)b) \\ &\rightarrow g^p(g^q(ab)) \rightarrow (g^p g^q)(ab) \rightarrow g^{p+q}(ab). \end{aligned}$$

The middle interchange contributes $\beta(a)^{-q}$. All other contributions are specified cover associators. The script independently verifies the resulting J_K against all 18432 tensor-functor equations.

The certificate supplies a natural isomorphism $d:F^2 \rightarrow K$ on the common underlying simple objects. Its phases satisfy for every channel $a \rightarrow b \rightarrow c$

$$\begin{aligned} d(a)d(b)/d(c) \\ = J_F(a,b;c) J_F(Fa,Fb;Fc)/J_K(a,b;c). \end{aligned}$$

This is exactly monoidality of d . It also satisfies

$$d(g) = J_F(g,g;1) = c^2.$$

Thus d intertwines the algebra map $\phi F(\phi):F^2(A) \rightarrow A$ with the identity algebra map $K(A) \rightarrow A$. It descends to the module categories.

On D , K is canonically tensor-isomorphic to the identity. On a free module $q(a)$, use $q(g)^p \text{ tensor } q(a) \rightarrow q(a)$, where $q(g) \rightarrow 1_D$ is supplied by the regular algebra multiplication. The tensorator of K was defined by precisely the central movements appearing in this map. Its monoidality uses $\beta(g)=1$ and the group-algebra relation $g^2=1$. Hence f^2 is tensor-isomorphic to Id_D .

11.5 Coherent C2 action, with no unresolved obstruction

For completeness there are two ways to finish coherence.

First, the canonical universal grading of D is C_8 : writing its simples as $u^a(a \text{ in } C_8)$ and $X_r(r \text{ in } C_4)$, their degrees are $2a$ and $2r+1$ modulo 8. Its adjoint subcategory has basis $1, u^4$, as follows from $X_r X_r^* = 1 + u^4$. Therefore $\text{Aut}_{\text{tensor}}(\text{Id}_D) = \text{Hom}(C_8, C^*) = C_8$. The functor f acts on this group by multiplication 3. The obstruction to upgrading an order-two tensor-equivalence class to a C_2 action lies in

$$H^3(C_2, C_8 \setminus 3) = \ker(1+3)/(3-1)C_8 = 2C_8/2C_8 = 0.$$

Thus the action coherence can always be corrected. This is the standard monoidal-action obstruction (a direct cyclic coherence calculation), not an assertion that H_4 alone supplies an extension.

Second, for the displayed certificate the correction is explicit. If phases are written as exponents mod 16, replace $d(a)$ by

$$d_{\text{new}}(i,a) = d(i,a) - 4a \text{ modulo } 16.$$

This multiplies d by the universal-grading character with parameter-2. Before correction the C2 coherence defect is the parity character; after correction it is identically 1. In the saved bases the defect, written as an exponent mod 32, is

$$2d(\text{Fa}) - 2d(a) - p(a)(J_F(g, g; 1) - 2J_F(g, a; ga)).$$

The script `mr_1254_model.py` checks that it is 16 on odd objects and 0 on even objects before correction, and 0 everywhere afterwards; the final phases still satisfy all monoidality and algebra compatibility equations. Every correction is unitary. This produces an actual coherent unitary C2 action on D .

11.6 Fusion tensor and census identification

In D use representatives $(1, 2a)$ for u^a , and $(\sigma, 1+2r)$ for X_r . Then

$$u^a u^b = u^{(a+b)}, \quad u^a X_r = X_r u^a = X_{(r+a)}, \\ X_r X_s = u^{(r+s+1)} + u^{(r+s+5)}.$$

The descended involution acts by

$$u^a \rightarrow u^{(-a)}, \quad X_r \rightarrow X_{(3r+1)}.$$

It fixes $1, u^4$; it has three two-element pointed orbits, and two two-element odd orbits. The equivariantization $E = D^{C2}$ therefore has rank 9 and dimensions $1^4, 2^3, (2\sqrt{2})^2$. The odd orbits are exchanged by duality, giving the feature missing from the earlier model 389.

For the fixed object u^4 , the certificate has $d_{\text{new}}(u^4) = 14$ and $J_F(u^4, u^4; 1) = 14$ modulo 16. Thus its equivariant lifts have square 1, and the invertible group is V_4 . A later action-coherence correction by a universal-grading character cannot change this, because $\deg(u^4) = 0$.

The full tensor is computed by the invariant-subspace formula. For a fixed output the multiplicity is $(\dim + \text{trace})/2$; only fixed input objects contribute to the trace. These inputs are 1 and u^4 , whose representatives are literally fixed in the cover, so the relevant phases are exactly the saved J_F and d_{new} entries. For a free output orbit use the dimension at one representative. This determines every fusion coefficient.

`mr_1254_model.py` verifies all ring axioms and matches all 729 coefficients of $R_{9,2,1254}$ with source-to-model basis permutation

$$[0, 8, 7, 5, 6, 1, 2, 4, 3].$$

`mr_1254_model.json` includes the full rank 9 tensor, rank 12 quotient tensor, action, corrected coherence phases and equivariant orbit labels. The model is unitary by construction from unitary categories, a central unitary separable algebra, and a coherent unitary finite-group action.

11.7 Scope and references

The proof establishes categorical existence for this fusion ring. It does not classify all its categorifications or assert tensor-equivalence novelty of the resulting category.

- S. Gelaki, D. Naidu, D. Nikshych, Centers of graded fusion categories, arXiv:0905.3117, Sections 2.4–2.5, for actions and central condensation.

- J.Dong, S.Natale, H.Sun, A class of prime fusion categories of dimension 2^N , New York J.Math.27(2021),141–163, <https://nyjm.albany.edu/j/2021/27-5p.pdf>, for the generalized Moore–Read setting. Existence and the required action here are specified directly.
- P.Etingof, D.Nikshych, V.Ostrik, Fusion categories and homotopy theory, arXiv:0909.3140, for the obstruction-theoretic formulation of tensor actions; the relevant cyclic H3 calculation is supplied above.

12 Complete group-theoretical obstructions in dimensions 120 and 168

Scope. This argument excludes group-theoretical categorifications of $R9_2_12$, $R9_2_959$, and $R9_2_1965$. Together with ENO Proposition 9.11 for the simple exotic ring $R9_2_941$, all four integral residual rings are outside the group-theoretical class. This does not exclude arbitrary fusion categorifications. In particular, no theorem identifying all integral categories of dimensions 120 or 168 as group-theoretical is used.

The finite computations below are exact. A separate independent audit passed: `phase4/audit_integral_gt/AUDIT.md` and `audit_verdict.json`. The accepted conclusion has group-theoretical scope only.

12.1 An exhaustive projective-character profile reduction

Every group-theoretical category of dimension D has a presentation $C(G, \omega, H, \psi)$ with $|G|=D$, $d \psi = \omega |H|$. Its simples over a double coset HgH are indexed by projective irreducible representations of $K_g = H \cap gHg^{-1}$, of dimensions

$$[H:K_g] \dim(\pi).$$

The identity double coset supplies the ordinary category $\text{Rep}(H)$ as a fusion subcategory. A projective irreducible degree is at most the square root of the order of its group, so all simple dimensions are at most $|H|$. In particular, $|H| < 6$ cannot supply a dimension-six simple.

`all_twist_skeleton.g` enumerates every group of orders 120 and 168 and every conjugacy class of subgroups. It first checks that the ordinary character degrees of H are a submultiset of the target type. For every remaining double-coset intersection K , it constructs a Schur cover and partitions the irreducible characters of the cover by their central character on the kernel. These are precisely all projective degree profiles, for every class in $H^2(K, C^*)$. It independently chooses any of these profiles at every double coset, retaining those whose union is the target type. Independence is a relaxation of global compatibility; it cannot omit any global three-cocycle or relative two-cochain.

The complete output is:

D	Groups	Subgroup classes	Surviving pairs
120	47	1431	(S5, S3xC2) and (A5xC2, S3xC2)
168	57	1751	the two conjugacy classes (PSL(2, 7), S4)

Both dimension-120 pairs have intersections of orders 12, 2, 4, with the order-four intersection $V4$. Their required projective degree packets are $(1, 1, 1, 1, 2, 2)$, $(6, 6)$, and (6) respectively. Thus the $V4$ intersection must carry its nontrivial

Schur class. Both dimension-168 pairs have intersections S4 and V4; the required packets are (1,1,2,3,3) and (6,6,6,6).

The scan uses GAP 4.12.1 with SmallGrp and the exact character/Schur-cover algorithms. It is finite group classification plus finite exact arithmetic, not a restricted three-cocycle ansatz. The saved .json normalizes the GAP output; the original .jsonl contains line-wrapped JSON objects.

12.2 Immediate consequence for R9_2_959

For both surviving dimension-120 pairs, H is $S_3 \times C_2$, whose representation category has invertible group V4. It embeds as the identity double-coset subcategory. But the invertible group of R9_2_959 is C_4 . A based inclusion cannot embed V4 into C_4 . Thus no group-theoretical categorification of R9_2_959 exists, independently of omega and psi.

12.3 All relative three-cocycles for the S5 pair

Let $G=S_5$ and $H=S_3 \times C_2$. Every admissible class a in $H^3(G, C^*)$ restricts to zero on H. Restriction followed by corestriction is multiplication by $[G:H]=10$, hence $10a=0$.

The five-primary part of $H^3(S_5, C^*) = H^4(S_5, Z)$ is zero: restriction to a Sylow C_5 is injective on this part, and the image must be fixed by the normalizer $C_5:C_4$. An order-four automorphism of C_5 acts on H^4 by the square of a primitive automorphism, namely -1 modulo five, whose fixed space is zero. Therefore every admissible class has order at most two.

Every two-torsion class in $H^3(G, C^*)$ lifts to $H^3(G, F_2)$, by the long exact sequence of $1 \rightarrow \mu_2 \rightarrow C^* \xrightarrow{\text{square}} C^* \rightarrow 1$. Let P be a Sylow D8 of S_5 . Its index is fifteen, hence odd. If a is a two-torsion global class and b is any F_2 lift of its restriction to P, then the image of $\text{cor}_P^G(b)$ in $H^3(G, C^*)$ is $\text{cor}(\text{res}(a))=15a=a$. Consequently corestricting every F_2 third cohomology class of P exhausts all admissible global associator classes.

The normalized bar complex computes $\dim H^3(D_8, F_2)=4$: 343 normalized three-cochain variables, cocycle dimension 45, boundary dimension 41. The program enumerates all sixteen classes. For right coset representatives t of $P \backslash G$, define $h(t, g)$ and $t.g$ by

$$t.g = h(t, g) (t.g).$$

The explicit corestricted sign cocycle is

$$w(g_1, g_2, g_3) = \sum_t b(h(t, g_1), h(t.g_1, g_2), h(t.g_1g_2, g_3)) \mod 2.$$

This is the usual inhomogeneous cochain transfer, so it takes exact cocycles to exact cocycles. The program solves $d \psi = (-1)^w |H$ exactly in eighth roots of unity. Completeness of this finite primitive test is explained below.

Eight of the sixteen representatives admit a relative trivialization. Four yield rank twelve. The other four yield rank nine and match ****all** 729 coefficients of R9_2_92**, with explicit based isomorphisms; none matches R9_2_12. Reusing the independently audited twisted bimodule character formula is legitimate here: only its normalized cocycle and relative cochain inputs change. Every returned tensor is separately checked for the unit and every associativity equation: 6,561 at rank nine and 20,736 at rank twelve.

There is no missing choice of psi. $H^2(H, C^*)=C_2$, and the nontrivial class extends from the Schur cocycle of S_5 : lifted disjoint transpositions have nontrivial

commutator. Multiplying ψ by that class is induced by an identity-on-objects tensor autoequivalence of Vec_G^ω , so it does not change the based ring. This extension was independently checked in phase three's `S5_RELATIVE_TWIST.md` and audit.

Thus the entire S5 pair, with all relative cocycles, gives no model for $R9_2_12$.

12.3.1 Exact primitive solver and its bound

Since $H^2(H, C^*)$ has exponent two, a sign-valued three-cocycle on H that is a C^* -coboundary has a fourth-root-valued primitive. Indeed, if $d\psi = w$, then ψ^2 is a two-cocycle. Hence $\psi^4 = d f$ for a one-cochain f . Choose a fourth root b of f , and replace ψ by $\psi/(d b)$; its fourth power is one. Thus eighth roots are more than sufficient.

The congruence solver lifts successively modulo two, four and eight. At each lower stage it enumerates all possible $H^2(H, F2)$ classes, whose basis is x^2, xy, y^2 , for the two abelianization characters of $H = S3 \times C2$. This follows also from the normal odd-order $C3$ subgroup: $H^*(H, F2) = H^*(C2 \times C2, F2)$. Two solutions differing by an $F2$ two-coboundary have the same liftability, since that change can be lifted to an integral one-coboundary adjustment. Exact Gaussian elimination handles the inhomogeneous linear equations and tests every consistency row. The final primitive is checked against every normalized triple of H . Thus a failed test is an exact no-primitive conclusion, not an optimizer timeout.

12.4 The $A5 \times C2$ pair, including the odd-primary part

Again the global restriction kernel is killed by ten, so an admissible class is a sum of a two-torsion and a five-torsion class. We handle the two-primary part by the same exhaustive transfer construction, using a Sylow group $P = C2^3$ of index fifteen. The normalized bar complex has cocycle dimension 49 and boundary dimension 39, giving ten $F2$ third cohomology generators.

For efficiency, let n be an order-three element of the normalizer of P . Corestriction is unchanged in cohomology by conjugating a seed by n . Thus corestricting b is cohomologous to corestricting $b + n*b + n^2*b$, since three is one modulo two. The span of these norm classes has dimension four. Exact reduction modulo the bar boundaries therefore gives sixteen representatives covering the image of **all 1,024** original $F2$ cohomology classes. No global class is discarded.

Only two of these sixteen representatives admit a primitive on H . For each, both subgroup Schur choices ψ and $\psi*(-1)^{\{x \cup y\}}$ are explicitly tested. In all four cases the $V4$ intersection multiplier is trivial and the resulting category has rank twelve. Thus no two-primary relative class gives the required rank-nine profile.

The five-primary part cannot change this conclusion. Relative pairs (ω, ψ) form relative cohomology, and the multiplier on a double-coset intersection K is a homomorphism from that relative cohomology group to $H^2(K, C^*)$. The five-primary part of the relative group maps isomorphically to the five-primary global restriction kernel, because $H^2(H, C^*)$ has no five-torsion. Its image in the order-two Schur multiplier of $V4$ is therefore zero. Equivalently, multiplying the relative data by any order-five relative class does not change the $V4$ projective degree profile. The necessary rank-nine profile requires that multiplier to be nontrivial. Hence no admissible cocycles for $(A5 \times C2, S3 \times C2)$ realize $R9_2_12$.

Combining Sections 1, 3 and 4 proves the group-theoretical exclusion for $R9_2_12$.

12.5 The dimension-168 pair

The all-twist profile reduction leaves only $G = \text{PSL}(2, 7)$, $H = S_4$, for the two subgroup conjugacy classes. The following completes the old bounded argument by showing that these are now the *only* presentations that need to be considered.

Restriction $H^3(G, C^*) \rightarrow H^3(H, C^*)$ is injective. Its kernel is killed by index seven. But G has no seven-primary H^4 : restriction to its Sylow C_7 is injective on the seven-primary part, and the normalizer $C_7:C_3$ acts on $H^4(C_7, \mathbb{Z}) = C_7$ by multiplication by 2 and 4, with zero fixed part. Thus any admissible ω is globally trivializable.

After trivializing ω , $H^2(S_4, C^*) = C_2$, and both classes extend from $H^2(\text{PSL}(2, 7), C^*) = C_2$. Nontriviality of restriction is witnessed by a commuting pair in S_4 whose lifts to the Schur double cover fail to commute. The prior exact checker tests this for both subgroup classes: `phase3/group_models/psl168_relative_bound.g` and its saved log. Therefore ψ can also be removed by a global two-cocycle autoequivalence. Both presentations reduce to the untwisted category, whose full tensor is $R_9_2_1557$, not $R_9_2_1965$. The based nonisomorphism is checked again.

13 Fermion parity and solvable descent in dimension 120

13.1 A fermion-parity criterion

Proposition. Let C be an integral fusion category with commutative Grothendieck ring, trivial universal grading, and dimension N divisible by four. Write the induction of the unit into $Z(C)$ as a sum of distinct simple objects. Suppose exactly one nonunit summand has odd dimension, and that its dimension is a positive power of an odd prime. Then $Z(C)$ contains a nontrivial Tannakian subcategory.

Proof. Integrality gives a canonical positive spherical structure and makes $Z(C)$ an integral nondegenerate braided category. The odd-prime-power summand of $I(1)$ and ENO Corollary 7.2 produce a nontrivial symmetric fusion subcategory. If that symmetric category has dimension larger than two, Deligne's theorem gives a nontrivial Tannakian subcategory. A symmetric category of dimension two is either Tannakian or $s\text{Vec}$. It remains to handle $s\text{Vec}$, generated by a fermion χ in $Z(C)$.

Here is a short proof of the required dimension count. By Ostrik Proposition 2.8, for the unnormalized modular S -matrix,

$$S[I(1)] = N[I(1)].$$

The fermion χ has twist -1 , whereas every summand of $A = I(1)$ has twist $+1$ (Ostrik Theorem 2.5); hence χ is absent from A . The χ entry of this identity is zero. For each simple Z in A , monodromy with the order-two invertible χ is a scalar ϵ_Z in $\{+1, -1\}$, and $S_{\{\chi, Z\}} = \epsilon_Z \text{FPdim}(Z)$. Thus the total dimension of the constituents of A centralizing χ equals that of the other constituents, namely $N/2$. This uses no braiding on C .

For completeness the same dimension count has the following independent relative-center derivation. Let $F: Z(C) \rightarrow C$ be the forgetful functor and $g = F(\chi)$. Because the universal grading of C is trivial, the forgetful homomorphism on invertible objects is injective: its kernel is the group of tensor automorphisms of Id_C , namely the characters of the universal grading group. Thus g is a nontrivial order-two invertible, and $D =$ has dimension two.

Use the relative-center construction from the proof of ENO Lemma 9.17. The forgetful functor $Z(C) \rightarrow Z_D(C)$, which remembers the half-braiding only against D , is dominant. Its right adjoint on the unit gives a connected algebra B contained in A , of dimension $N/\text{FPdim}(D) = N/2$. This follows either from that lemma's dimension calculation or from the dominant-functor index formula and $\text{FPdim}(Z_D(C)) = \text{FPdim}(C) \text{FPdim}(D)$.

We claim that B is precisely the sum of those constituents of A that centralize χ . Here is the explicit morphism check, so this identification does not require a fixed-algebra theorem or a choice of associator. For a central simple Z occurring in A , take a nonzero map $f: F(Z) \rightarrow 1$. It is a morphism to the unit in the relative center precisely when

$$(\text{id}_{\backslash g} \otimes f) c_{\backslash \{Z, g\}} = f \otimes \text{id}_{\backslash g}.$$

The naturality of χ 's half-braiding gives

$$(f \otimes \text{id}_{\backslash g}) c_{\backslash \{g, Z\}} = \text{id}_{\backslash g} \otimes f.$$

Consequently the relative-center condition is equivalent to

$$(f \otimes \text{id}_{\backslash g}) (c_{\backslash \{g, Z\}} c_{\backslash \{Z, g\}}) = f \otimes \text{id}_{\backslash g}.$$

In $Z(C)$, $Z \otimes \chi$ is simple, so this monodromy is scalar. Since χ has order two, its monodromy eigenvalue is $+1$ or -1 . The displayed condition therefore holds exactly when Z centralizes χ . This proves the claim, including multiplicities (which are one because the based ring is commutative).

The dimension of B is $N/2$, which is even. B contains the unit, and every nonunit summand of A except the specified odd-prime-power object has even dimension. Parity therefore forces that odd object to occur in B . It lies in the centralizer $E = ' \text{ inside } Z(C)$. By nondegeneracy of $Z(C)$, the Muger center of E is exactly $= \text{sVec}$, so E is slightly degenerate. ENO Proposition 7.4 says that a slightly degenerate integral braided category containing a simple object of odd-prime-power dimension has a nontrivial Tannakian subcategory. This proves the proposition. QED.

The argument explicitly treats a possible central fermion. It does not infer a Tannakian category merely from an unspecified symmetric category.

13.2 Application to both dimension-120 tensors

Both input tensors have dimensions $(1^4, 2^2, 6^3)$, are commutative, and have trivial universal grading. Their formal codegrees, with multiplicity, are

120, 24, 20, 30, 30, 6, 6, 4, 4.

Ostrik's induction theorem (Theorem 2.13 of arXiv:1309.4822) gives a multiplicity-free decomposition of $A = I(1)$ with simple dimensions

1, 5, 6, 4, 4, 20, 20, 30, 30.

There is exactly one odd nonunit summand, of dimension five. The proposition therefore forces a nontrivial Tannakian subcategory of $Z(C)$.

Li Dai's Lemma 3.1 then implies that C is weakly group-theoretical. Its full hypothesis is exactly: an integral category of dimension 120 whose center contains a nontrivial Tannakian subcategory. For clarity, its reduction is as follows. If the Tannaka group's order is a power of two, it contains a quotient C_2 , and the

corresponding $\text{Rep}(C_2)$ in the center gives an extension or equivariantization of a dimension-60 category. All dimension-60 categories are weakly group-theoretical by ENO Theorem 9.16. If the Tannaka group's order is divisible by three or five, the neutral component of the de-equivariantized center has dimension $(120/|G|)^2$, with at most two prime divisors, and is solvable by ENO Theorem 1.6. The extension/equivariantization and center characterizations of weak group-theoreticality finish the reduction.

New conclusion: every hypothetical categorification of either $R_9_2_12$ or $R_9_2_959$ is weakly group-theoretical. This conclusion applies to arbitrary complex fusion categories and is not a model construction.

13.3 An auxiliary dimension-30 obstruction

Lemma. An integral fusion category of dimension 30 has no simple object of dimension three.

Proof. ENO Theorem 9.2 makes every integral fusion category of square-free dimension 30 group-theoretical. Every group G of order 30 has a normal cyclic subgroup P of order 15. One elementary proof is: if its Sylow-five subgroup were not normal, there would be 24 nonidentity order-five elements. Counting forces the Sylow-three subgroup to be normal. The inverse image of the normal Sylow-five subgroup in the order-ten quotient is then a normal subgroup of order 15, necessarily cyclic, making its Sylow-five subgroup normal after all. Thus Sylow five is normal. In the order-six quotient, the Sylow-three subgroup is normal; its inverse image is the desired cyclic subgroup of order 15.

In a group-theoretical presentation $C(G, \omega, H, \psi)$, simples over HgH are indexed by projective irreducible representations of $K = H \cap gHg^{-1}$, with dimension $[H:K] \dim(\pi)$. Every such K has square-free order, so $H^2(K, C^*) = 0$: restriction to a Sylow subgroup is injective on the corresponding primary part by restriction/corestriction, while every Sylow subgroup is cyclic and has trivial complex Schur multiplier. Thus all projective degrees coincide with ordinary character degrees.

Set $Q = H \cap P$. It is cyclic and, being a subgroup of the cyclic normal subgroup P , is normal in G . We have $[H:Q] \leq 2$, and Q lies in every intersection K . Hence $[H:K] \leq 2$. If $[H:K] = 2$, $K = Q$ is cyclic, so every projective irreducible of K has dimension one. If $[H:K] = 1$, $K = H$ has a cyclic normal subgroup of index at most two. Restriction of an ordinary irreducible representation to that cyclic subgroup is a sum of characters. Clifford theory gives irreducible dimensions at most two: an orbit of size two induces a two-dimensional representation, while an invariant character extends across the cyclic quotient and gives one-dimensional representations. Thus all resulting simple dimensions are one or two. In particular no dimension-three simple occurs. QED.

This argument allows arbitrary three-cocycles and subgroup cochains; an untwisted character-table search alone would not suffice. The elementary group-of-order-30 fact can equivalently be checked on the four groups of order 30. It is also a consequence of the classification of groups of square-free order at this particular order.

13.4 No solvable category has dimension type $(1^6, 3^6)$

Theorem. A fusion category D with six invertibles and six simple objects of dimension three is not solvable.

Proof. Its dimension is 60. For each dimension-three simple X , $G[X]$ has order dividing both $|G(D)|=6$ and $\text{FPdim}(X)^2=9$. In the decomposition of $X \otimes X^*$, all noninvertible summands have dimensions divisible by three, so $|G[X]|$ is divisible by three. Hence $|G[X]|=3$. The group of six invertibles has a unique subgroup H of order three, so this same H fixes all six noninvertible objects. It lies in D_{ad} . The remainder of $X \otimes X^*$ has dimension six, so D_{ad} is nonpointed.

Consider a nontrivial faithful grading with group order u . Each component has dimension $60/u$. The neutral component contains H and at least one three-dimensional object. A component with a invertibles and b three-dimensional simples has dimension $a+9b$. There are six invertibles and six three-dimensional simples in all. These constraints leave only $u=2$, with both components of type $(1^3, 3^3)$. This elementary enumeration is replayed exhaustively by the verifier. The neutral category would have dimension 30 and a three-dimensional simple, contrary to Section 3. Thus $U(D)=1$.

If D were solvable, the final step of a cyclic-prime extension/ equivariantization construction could not be an extension. It must be an equivariantization by C_p . Its central $\text{Rep}(C_p)$ embeds into D and supplies p invertibles, so $p=2$ or $p=3$.

For $p=2$, its nontrivial character acts freely on the six dimension-three objects: their stabilizers under $G(D)$ have order three. It also acts freely on all six invertibles. De-equivariantization has type $(1^3, 3^3)$ and dimension 30, again impossible.

For $p=3$, its character subgroup is H and fixes all six noninvertible objects. The six invertibles form two free orbits. De-equivariantization has dimension 20 and is pointed: each fixed three-dimensional simple splits into three invertibles, and each free orbit of invertibles gives one invertible. To justify the splitting, the endomorphism algebra of the free module of a fixed simple is a twisted group algebra of C_3 ; $H^2(C_3, C^*)=0$, so it has three one-dimensional simple modules. Each has underlying object X and normalized dimension $3/3=1$.

Write this pointed category as Vec_K^ω , $|K|=20$. The equivariantizing C_3 acts on K by group automorphisms. Its invariant invertibles each contribute exactly three invertibles in the equivariantization, again because $H^2(C_3, C^*)=0$. Six invertibles of D would require exactly two fixed elements of K . But K has a unique, hence characteristic, Sylow-five subgroup, and the C_3 action is trivial there because $\text{Aut}(C_5)=C_4$. There are at least five fixed elements, a contradiction. QED.

This theorem does not exclude categories of that dimension type in general: the group-theoretical category $C(A_5, 1, C_3, 1)$ supplies that type. The conclusion is nonsolvability, which is the needed restriction.

13.5 No solvable category has dimension type $(1^2, 2, 3^2, 6)$

Theorem. A fusion category D of this type is not solvable.

Proof. A nontrivial faithful grading of a dimension-60 fusion category has components of dimension at most 30. A simple object of dimension six alone contributes 36 to its component, which is impossible. Hence $U(D)=1$. If D were solvable, its last cyclic-prime step would be an C_2 -equivariantization, since it has exactly two invertibles. Let g be the nontrivial underlying character. It fixes the unique objects of dimensions two and six. It exchanges the two dimension-three simples: the stabilizer of a dimension-three simple has order dividing nine, so cannot contain an involution. De-equivariantization therefore has three invertibles (one from the pair of original invertibles and two from the fixed dimension-two object) and three dimension-three simples (one from their pair, two from the fixed dimension-six object). This is the forbidden dimension-30 type $(1^3, 3^3)$ of Section 3. QED.

13.6 Central involution descents of the two source rings

For a hypothetical central Tannakian $\text{Rep}(C2)$, the underlying involution must be one of the following source labels (zero-based):

Ring	Involution	Dimension type after de-equivariantization
R9_2_12	1 or 2	$(1^2, 2, 3^2, 6)$
R9_2_12	7	$(1^6, 3^6)$
R9_2_959	8	$(1^6, 3^6)$

The verifier checks every orbit of multiplication by each involution. A free pair of d -dimensional objects gives one d -dimensional object below; a fixed d -dimensional object splits into two objects of dimension $d/2$. This follows from the algebra $1+g$ and $H^2(C2, C^*)=0$, with all categorical premises stated. Sections 4 and 5 rule out both dimension types whenever the descending category is required to be solvable.

13.7 Nonsolvable weakly group-theoretical categories of dimension 120

Lemma. Every nonsolvable weakly group-theoretical fusion category of dimension 120 is group-theoretical.

Proof. By definition it is Morita equivalent to a nilpotent category N , built from Vec by successive faithful extensions by finite groups G_i . The product of the orders $|G_i|$ is 120. If every G_i is solvable, refine each extension by a composition series into cyclic-prime extensions. Then N , and hence the original category, is solvable, a contradiction.

Consequently one grading group is nonsolvable. A nonsolvable finite group whose order divides 120 has a nonabelian composition factor $A5$: the only possible proper nonsolvable order is 60, and its group is $A5$. If a grading group has order 120, every other stage is trivial and N is pointed. Otherwise an $A5$ extension uses a factor 60, leaving only a single factor two. Up to deleting trivial stages, there are two possible orders.

First suppose Vec is extended by $A5$ and then by $C2$. The intermediate category is $P = \text{Vec}_{\{A5\}}^{\omega}$. Every invertible P -bimodule category, as a left P -module category, has pointed dual. The classification of such module categories (Naidu-Nikshych, Example 2.2) requires a normal abelian subgroup H of $A5$. Its only possibility is $H=1$, so the left module is the regular P -module, with 60 simples permuted transitively by $A5$. Each component of any extension of P is such a bimodule, and has FP dimension 60. Its 60 equally dimensional simples therefore all have dimension one. The extension N is pointed, for every ω and all coherent extension data.

Second suppose Vec is extended by $C2$ and then by $A5$. The intermediate category is $P = \text{Vec}_{\{C2\}}^{\omega}$. Its center is pointed of rank four: its simples are labeled by the two elements of $C2$ and projective irreducible characters of $C2$, which are all one-dimensional since $H^2(C2, C^*)=0$. The group $\text{BrPic}(P)$ is the group of braided tensor autoequivalences of $Z(P)$, and embeds into the permutations of its three nonunit objects. For clarity, faithfulness follows from the classification of pointed braided categories by metric groups: a braided tensor autoequivalence fixing all simple objects is tensor isomorphic to the identity. Thus $\text{BrPic}(P)$ is a subgroup of $S3$ and is solvable. The homomorphism $A5 \rightarrow \text{BrPic}(P)$ of extension

components is trivial, since A_5 is simple nonabelian. Every component is the regular P -bimodule and has two simple objects. They have equal FP dimension and their squared dimensions sum to $\text{FPdim}(P)=2$; both are invertible. Again N is pointed.

In all cases N is pointed, so the original category is group-theoretical. QED. This is a dimension-120 statement with the nonsolvability hypothesis; it does not assert that all dimension-120 categories are group-theoretical.

13.8 General exclusion of R9_2_12 and R9_2_959

Theorem. Neither ring admits a complex fusion categorification.

Proof. Suppose C categorifies either ring. Section 2 proves that C is weakly group-theoretical. The independently audited phase-four theorem `phase4/integral/GROUP_THEORETICAL_OBSTRUCTIONS.md` excludes every group-theoretical categorification, including all associators and subgroup cochains. Section 7 consequently forces C to be solvable.

Its universal grading is trivial by the exact input tensor. The final step of a cyclic-prime construction of C must therefore be an equivariantization. Since C has four invertibles, the prime is two, and the canonical central $\text{Rep}(C_2)$ has one of the underlying involutions listed in Section 6. Its de-equivariantization D is solvable, being the preceding category in this construction. It has one of the two dimension types excluded for solvable categories by Sections 4 and 5. This is a contradiction. QED.

Every reduction concerns an arbitrary hypothetical categorification. The proof does not assume a fiber functor, braiding, unitarity, or an uncited classification of all dimension-120 categories. It uses the phase-four all-cocycle GT theorem only within its established scope.

13.9 References

- * Etingof-Nikshych-Ostrik, *Weakly group-theoretical and solvable fusion categories*, arXiv:0809.3031v2. Corollary 7.2; Proposition 7.4; Propositions 2.9, 2.10 and 4.4; Theorems 9.2 and 9.16; proof of Lemma 9.17 (relative-center algebra). <https://arxiv.org/pdf/0809.3031>
- * Ostrik, *Pivotal fusion categories of rank 3*, arXiv:1309.4822, Theorem 2.5, Proposition 2.8, and Theorem 2.13 (valid for arbitrary spherical fusion categories). <https://arxiv.org/pdf/1309.4822>
- * Dai, *Frobenius property for fusion categories of dimension 120*, Revista de la Union Matematica Argentina 68(2) (2025), 395–403, Lemma 3.1. DOI 10.33044/revuma.4390. <https://inmabb.criba.edu.ar/revuma/pdf/v68n2/v68n2a03.pdf>
- * Naidu-Nikshych, *Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups*, arXiv:0705.0665, Example 2.2 (pointed-dual module categories). <https://arxiv.org/pdf/0705.0665>
- * Etingof-Gelaki-Nikshych-Ostrik, *Tensor Categories*, Sections 8.4 (pointed braided categories and metric groups), specifically Theorem 8.4.12, and Section 8.27.2 (Brauer-Picard groups and centers). <https://math.mit.edu/~etingof/egnobookfinal.pdf>

The lemmas above are combinations of these primary results and explicit arguments; no claim of literature priority is made. Final status and any further phase-five inputs are recorded separately in REPORT.md.

14 Odd central dimensions and the dimension-504 ring

Theorem. The fusion ring $R_{9,2,941}$ admits no fusion categorification over C . This conclusion uses no unitarity assumption. The ring is exactly the rank-nine, dimension-504 ring in Alekseyev–Bruns–Dong–Palcoux, arXiv:2507.07023v2, §1.5, Question 1.14; the previously verified tensor isomorphism is recalled below.

14.1 Notation and categorical input

Use source labels $X_0=1$, $Y=X_2$, $B_1=X_1$, $B_2=X_3$, $B_3=X_4$, $B_4=X_5$, and $C_1=X_6$, $C_2=X_7$, $C_3=X_8$. Their dimensions are respectively 1, 8, 7, 7, 7, 7, 9, 9, 9, and total dimension is $D=504$. Suppose a categorification C exists. Since C is integral, it has the canonical positive spherical structure. Its center $Z(C)$ is an integral modular category of dimension D^2 . Write F for forgetful functor and I for its right adjoint. For a central simple Z write

$$F(Z) = t \cdot 1 + a \cdot Y + \sum_j b_j \cdot B_j + \sum_l c_l \cdot C_l,$$

and let $z = \dim(Z)$, $b = \sum_j b_j$, $c = \sum_l c_l$. Thus $z = t + 8a + 7b + 9c$.

The following standard facts will be used:

1. $z \mid D$, because modular square-divisibility makes D^2/z^2 an integer, and D, z are positive integers.
2. Since $K_0(C)$ is commutative, $I(1)$ is multiplicity-free. Its central simples A_{χ} are indexed by the nine scalar characters, with $\dim(A_{\chi}) = D/f_{\chi}$. Thus t is zero or one. The central dimensions in $I(1)$, with multiplicity, are 1, 63, 56, 56, 56, 72, 72, 72, 72.
3. If χ is a scalar character, the central character identity gives $S(A_{\chi}, Z) = \dim(A_{\chi}) \cdot \chi(F(Z))$. By symmetry of S and the Verlinde character formula, $\dim(A_{\chi}) \cdot \chi(F(Z))/z$ is an algebraic integer.
4. Set $G = \sum_i N_i N_i^T$. The induction Gram identity gives $G = \sum_Z v_Z v_Z^T$, where v_Z is the restriction column. Consequently $v_Z^T G^{-1} v_Z \leq 1$. Orthogonality of the real scalar characters gives
$$v_Z^T G^{-1} v_Z = \sum_{\chi} \chi(F(Z))^2 / f_{\chi}^2 \leq 1. \quad (1)$$
This is an ordinary positive semidefinite matrix identity and requires no C^* -structure on C .
5. Frobenius reciprocity and dimensions give
$$\sum_Z z \cdot [F(Z):X] = \dim(I(X)) = D \cdot \dim(X). \quad (2)$$

For the identity in item 3, see Deshpande, arXiv:1611.04486, Proposition 2.8 and §2.7, especially (36) and the description immediately preceding it. For item 2 see Ostrik, arXiv:1309.4822, Theorem 2.13.

14.2 Exact character data

The dimension character has formal codegree 504. One rational character of codegree 8 is

$$\chi_8(F(Z)) = t - b + c.$$

Four rational characters of codegree 9 are

$$\chi_{9,j}(F(Z)) = t + b - a - 3 \cdot b_j \quad (1 \leq j \leq 4).$$

The other three characters have codegree 7. Put $\alpha=2\cos(2\pi/7)$, $\beta=\alpha^2-2$, $\gamma=1-\alpha-\alpha^2$. These are the roots of x^3+x^2-2x-1 ; in a cyclic ordering the three characters are

$$\chi_7(F(Z)) = t+a+c_1\alpha+c_2\beta+c_3\gamma,$$

and its two Galois conjugates. In particular $\alpha+\beta+\gamma=-1$, $\alpha^2+\beta^2+\gamma^2=5$. The appropriate cyclic ordering is verified against every source product in the accompanying verifier; permuting the C labels has no effect on the argument.

We use the standard integral basis $0_-(Q(\alpha))=Z[\alpha]$, with basis $1, \alpha, \alpha^2$. For example see G. Eric Moorhouse, *Cyclotomic Fields with Applications*, Theorem 4.8. The accompanying verifier instead tests characteristic polynomials over Q , so its arithmetic check does not assume an integral basis theorem.

14.3 Only two restrictions can have central dimension 63

Lemma. If a central simple has dimension 63, its restriction is one of

$$U = Y+B_1+B_2+B_3+B_4+C_1+C_2+C_3,$$

$$V = 1+Y+2C_1+2C_2+2C_3.$$

Proof. Item 3 applied to the codegree-9 characters gives

$$t+b-a-3b_j \equiv 0 \pmod{9} \text{ for every } j. \quad (3)$$

Applied to any codegree-7 character it says $(8/7)*\chi_7(F(Z))$ is integral. In the integral basis above this forces

$$c_1 \equiv c_2 \equiv c_3 \equiv t+a \pmod{7}. \quad (4)$$

The dimension identity is

$$t+8a+7b+9c=63, \text{ with } t \text{ in } \{0,1\}. \quad (5)$$

For completeness, here is the entire short reduction of (3)-(5). Write r for the residue of $t+a$ in $\{0, \dots, 6\}$. Since each c_l is nonnegative and congruent to r , the inequality $t+8a+27r \leq 63$ leaves only the following aggregate possibilities:

t	a	b	c	Consequence
0	0	9	0	Every b_j is a multiple of 3
0	0	0	7	Exactly one c_l is 7
0	1	4	3	Every $b_j=1$ and every $c_l=1$: U
0	7	1	0	(3) requires every $b_j \geq 1$: impossible
1	0	5	3	(3) requires every $b_j \geq 2$: impossible
1	1	0	6	Every $b_j=0$ and every $c_l=2$: V
1	6	2	0	(3) requires every $b_j \geq 2$: impossible

In the first row $\chi_8(F(Z))=-9$, so its single contribution to (1) is $81/64 > 1$. In the second row the three codegree-7 contributions to (1) sum to $\alpha^2+\beta^2+\gamma^2=5 > 1$. Both are impossible. Only U and V remain. QED.

14.4 Two dimension parities force prime-power central simples

The only possible odd central dimensions are the odd divisors of 504: 1, 3, 7, 9, 21, 63. Dimension 3 is impossible. Dimension 1 is the unit. At dimensions 7, 9, 21, the restrictions are respectively a B_j , a C_l , or a sum of three B_j 's; indeed any occurrence of 1 would put the object in $I(1)$, whose dimension list has none of 7, 9, 21. At dimension 63 the preceding lemma applies.

Exactly one central simple has restriction V: it is the unique 63-dimensional summand of $I(1)$. Let m be the number of central simples with restriction U. Apply

(2) to Y and reduce modulo 2. Even-dimensional central objects contribute zero. Among odd-dimensional central objects, only the unique V and the m copies of U contain Y , each with multiplicity one. Therefore

$$0 == 504 \cdot 8 == 1 + m \pmod{2};$$

hence m is odd.

For each l let n_l be the number of central simples restricting to C_l . Apply (2) to C_l and again reduce modulo 2. The unique V contributes $63 \cdot 2$, the U columns contribute $63 \cdot m$, and the dimension-9 columns contribute $9 \cdot n_l$. All other contributions are even or zero. Thus

$$0 == 504 \cdot 9 == m + n_l \pmod{2}.$$

Consequently every n_l is odd. In particular, $Z(C)$ contains a simple of dimension $9 = 3^2$.

14.5 Contradiction and scope

The source ring is simple and has no braided categorification. The latter follows by the standard simple braided dichotomy: a nondegenerate realization would require $7^2 \mid 504$, while a symmetric realization would be a finite-group representation ring, which this exotic ring is not. These ring and braided facts were established in phase4 and in the published source identifying this ring.

Liu-Palcoux-Ren, arXiv:2102.01663v4, Theorem 5.1, states that the center of a nonpointed simple complex fusion category is simple exactly when the category does not admit a braiding. Hence $Z(C)$ is simple. But Etingof-Nikshych-Ostrik, arXiv:0809.3031, Corollary 7.2, states that an integral nondegenerate braided category containing a nontrivial prime-power-dimensional simple contains a nontrivial symmetric subcategory. Such a subcategory is proper in the nondegenerate $Z(C)$, contradicting simplicity. This proves the theorem.

The argument applies to arbitrary complex fusion categories because integrality supplies the positive spherical structure. It never replaces pseudo-unitarity by unitarity. No associator or indicator classification is assumed.

15 A forced Lagrangian subcategory in dimension 168

Theorem. The based ring $R_{9,2,1965}$ is not the Grothendieck ring of a fusion category over the complex numbers.

This is a general exclusion, not merely a braided, unitary, solvable, weakly group-theoretical, or group-theoretical exclusion. The new argument forces a hypothetical categorification to be group-theoretical and then applies the independently audited phase-4 exclusion of that class. In particular, it does not assert that every integral category of dimension 168 is group-theoretical.

15.1 Input and categorical tools

Use source labels 0 through 8. The positive dimension vector is

$$(1, 6, 6, 6, 6, 1, 2, 3, 3),$$

the dual permutation is $(0, 2, 1, 4, 3, 5, 6, 7, 8)$, and the global dimension is 168. Put $g=b_5$ and $u=b_6$. The invertible group is $\{1, g\} = C_2$. The based adjoint subring is the entire ring, so the universal grading group is trivial. These statements, all fusion axioms, and every finite identity below are checked by `verify_general_exclusion.py` against `source_tensor.json`. The tensor's canonical compact-JSON SHA256 is

07dac58d848bd451d8a3d9231bfd1c23c1b0117541e6de5055fa0ab90314c4f9.

Suppose that C categorifies this ring. Since its dimension is integral, C has its canonical positive spherical structure. Write $F: Z(C) \rightarrow C$ and I for its right adjoint. For any irreducible representation E of the complexified fusion ring, there is a distinguished simple A_E in $I(1)$ satisfying

$$[I(1) : A_E] = \dim E, \quad \dim A_E = \frac{\dim C}{f_E}, \quad \theta_{A_E} = 1.$$

For a scalar ring character χ , the associated central simple satisfies

$$\frac{S_{A_\chi, Z}}{\dim A_\chi} = \chi(F(Z)). \quad (1)$$

The induction matrix is a Gram matrix:

$$B_{ij} = [FI(b_j) : b_i] = \sum_Z [F(Z) : b_i][F(Z) : b_j], \quad FI(b_j) = \sum_t b_t b_j^*. \quad (2)$$

The order of multiplication in (2) matters because the source ring is noncommutative. Removing any specified central-simple summands from this Gram decomposition leaves a positive semidefinite matrix.

Primary references for these tools are [Ostrik, arXiv:1309.4822](#), Proposition 2.4, Theorems 2.5 and 2.13, and Example 2.9 with the discussion preceding Theorem 2.13; also [Deshpande, arXiv:1611.04486](#), Proposition 2.8 and Section 2.7. None requires the fusion ring to be commutative. Pseudo-unitarity and its canonical structure are [EGNO, *Tensor Categories*](#), Proposition 9.6.5 and Corollary 9.6.6.

15.2 A central Tannakian lift of $\mathfrak{g}$ is impossible

Assume there is a central Tannakian $\text{Rep}(C_2)$ whose nontrivial character forgets to $\mathfrak{g}$. The forgetful restriction to this subcategory is fully faithful. De-equivariantization by its regular algebra $1+\mathfrak{g}$ gives a fusion category D of dimension 84 and a dominant free-module tensor functor $Q: C \rightarrow D$ (EGNO Theorem 8.23.3).

Tensoring with $\mathfrak{g}$ has orbits

$\{0, 5\}$, $\{1, 3\}$, $\{2, 4\}$, $\{6\}$, $\{7, 8\}$.

By free-module adjunction, Q of each simple in a two-element orbit is simple, with the same dimension as the source. The fixed dimension-two simple u has $\dim \text{End}(Q(u))=2$. Hence it splits into two distinct simples. Their dimensions sum to two and each is at least one, so both are invertible. Thus D has precisely three invertibles $1, a, a^2$, and its remaining simples are Y, X, Z , of dimensions 3, 6, 6, respectively. Dominance ensures that this list is exhaustive. The source images are

Source Q -image	
b0, b5	1
b1, b3	X
b2, b4	Z
b6	$a+a^2$
b7, b8	Y

The order-three group of invertibles fixes Y, X and Z under both left and right multiplication: it acts on a one-element set of dimension-three simples and a two-

element set of dimension-six simples. Applying Q to the source products now forces all of the following:

$$\begin{aligned}
a^3 &= 1, \quad aY = Ya = Y, \quad aX = Xa = X, \quad aZ = Za = Z, \\
Y^2 &= 1 + a + a^2 + 2Y, \quad YX = XY = 3X, \quad YZ = ZY = 3Z, \\
X^2 &= 2X + 4Z, \quad Z^2 = 4X + 2Z, \\
XZ &= ZX = 1 + a + a^2 + 3Y + 2X + 2Z.
\end{aligned} \tag{3}$$

In particular D is integral. We next give a short theoretical obstruction to (3).

15.2.1 The dimension-84 induction-Gram obstruction

In basis $(1, a, a^2, Y, X, Z)$, (2) gives

$$B = \begin{pmatrix} 6 & 3 & 3 & 8 & 4 & 4 \\ 3 & 6 & 3 & 8 & 4 & 4 \\ 3 & 3 & 6 & 8 & 4 & 4 \\ 8 & 8 & 8 & 28 & 12 & 12 \\ 4 & 4 & 4 & 12 & 52 & 24 \\ 4 & 4 & 4 & 12 & 24 & 52 \end{pmatrix}.$$

Since (3) is commutative, all six irreducible ring representations are scalar. Its formal codegrees are $3, 3, 4, 28, 28, 84$, as verified by the characteristic polynomial of the Casimir/induction matrix:

$$\det(fI - B) = (f - 3)^2(f - 4)(f - 28)^2(f - 84).$$

Consequently $I_D(1)$ has six distinct simple constituents, of dimensions $1, 3, 3, 21, 28, 28$, each with multiplicity one. The dimension-one constituent is the central tensor unit. Let v range over the restriction vectors of the five other constituents. Each has $v_0=1$, and

$$\sum v = (5, 3, 3, 8, 4, 4). \tag{4}$$

The two dimension-three constituents each have $v_a + v_{a^2} = 2$, with no Y, X or Z coordinates. Thus only two of the total six a/a^2 coordinates in (4) remain. For each dimension-28 constituent, its dimension equation modulo three says $v_a + v_{a^2} \equiv 0 \pmod{3}$ after subtracting its unit coordinate. The remaining budget is at most two, so both dimension-28 constituents have zero a and a^2 coordinates. The dimension-21 constituent therefore has $v_a + v_{a^2} = 2$.

Consider the three pairs of a/a^2 coordinates belonging to dimensions $3, 3, 21$. Each pair sums to two and the three pairs have total $(3, 3)$. Either all are $(1, 1)$, or they are a permutation of $(2, 0), (1, 1), (0, 2)$. In the latter case, subtracting their outer products from B gives value $6 - 8 = -2$ on the vector supported as $(1, -1)$ in the a, a^2 positions. This violates positive semidefiniteness. Thus all three pairs are $(1, 1)$.

Let t be the Y -coordinate of the dimension-21 constituent, and let y_1, y_2 be those of the two dimension-28 constituents. Equation (4) gives $t + y_1 + y_2 = 8$. Remove from B the outer products of all six distinguished restriction vectors, and call the residual matrix M . Its a, Y principal entries satisfy

$$M_{aa} = 3, \quad M_{aY} = 8 - t, \quad M_{YY} = 28 - t^2 - y_1^2 - y_2^2 \leq 28 - t^2 - \frac{(8 - t)^2}{2}.$$

Therefore

$$\begin{aligned}\det M_{\{a,Y\}} &\leq 84 - 3t^2 - \frac{5}{2}(8-t)^2 \\ &= -\frac{(11t-40)^2}{22} - \frac{36}{11} < 0.\end{aligned}$$

This contradicts (2). Hence (3) has no fusion categorification, and the proposed central Tannakian lift of $\mathfrak{g}$ cannot exist.

For reference only, the full tensor (3) matches the already excluded lower-census ring R6_4_38, under permutation $(0, 1, 2, 5, 3, 4)$. The preceding proof is independent of its existing exclusion certificate.

15.3 Scalar codegrees force a nontrivial Tannakian subcategory

Set $z = -1 + \sqrt{-7}$ and $\bar{z} = -1 - \sqrt{-7}$. Direct substitution in all 81 multiplication equations gives two scalar ring characters

$$\chi_+ = (1, z, \bar{z}, z, \bar{z}, 1, 2, 3, 3), \quad \chi_- = (1, \bar{z}, z, \bar{z}, z, 1, 2, 3, 3).$$

Their formal codegrees are both 56. Thus there are two distinct distinguished central simples A_+, A_- of dimension three, each occurring once in $I(1)$, and each having twist one.

Their restrictions contain the unit once. The remaining dimension two can only be either $2g$ or u : no additional tensor unit is allowed, because its multiplicity is exactly one. The source induction matrix has $B_{\{0,5\}} = 1$. A restriction $1+2g$ would contribute two to this entry and is impossible. Therefore

$$F(A_+) = F(A_-) = 1 + u. \tag{5}$$

Using (1) and (5), for every $\sigma, \tau \in \{+, -\}$ we obtain

$$S_{A_\sigma, A_\tau} = 3\chi_\sigma(1 + u) = 9 = \dim A_\sigma \dim A_\tau. \tag{6}$$

We spell out why (6) gives actual centralization without any assumption of unitarity. The balancing formula, together with the trivial twists of A_σ and A_τ , says

$$9 = \sum_W N_{A_\sigma, A_\tau}^W \theta_W \dim W, \quad 9 = \sum_W N_{A_\sigma, A_\tau}^W \dim W.$$

All dimensions are positive, and twists are roots of unity. Equality in the triangle inequality forces $\theta_W = 1$ for every constituent W . The balancing identity for the double braiding consequently says that the double braiding is the identity. Thus A_+ and A_- centralize one another and themselves. The fusion subcategory $\mathcal{E}$ that they generate is symmetric. Its twist is identically one: this follows on all tensor words from balancing and then on their summands and duals. A positive spherical symmetric fusion category with trivial twist is Tannakian. Thus

$$\mathcal{E} \simeq \text{Rep}(G)$$

for a nontrivial finite group G . Here one can use EGNO's symmetric-category classification in Section 9.9; the nontrivial super case has a nontrivial canonical parity twist, which is absent here.

15.4 The Tannakian subcategory must be Lagrangian

The kernel of $\text{Inv } Z(C) \rightarrow \text{Inv } C$ consists of half-braidings on the tensor unit, equivalently tensor automorphisms of the identity functor. These are the characters of the universal grading group. The source adjoint closure is all of C , so this kernel is trivial.

It follows that $\text{Inv } E$ injects into $\{1, g\}$. If it were nontrivial, E would contain a Tannakian $\text{Rep}(C_2)$ whose nontrivial character forgets to g . Section 2 excluded precisely this situation. Therefore $\text{Inv } E$ is trivial: G has no nontrivial one-dimensional representations and is perfect.

Let E' be its centralizer in $Z(C)$. The centralizer dimension theorem and de-equivariantization give a fusion category $(E')_G$ of dimension

$$\text{FPdim}((E')_G) = \frac{168^2}{|G|^2}.$$

This rational number is an algebraic integer, so it is an integer. A rational number whose square is an integer is itself an integer; hence $|G|$ divides 168.

Every group of proper-divisor order of 168 is solvable. Indeed, the orders involving at most two primes are covered by Burnside's theorem. The only proper divisors involving all three primes 2,3,7 are 42 and 84; in either order the Sylow-seven number divides 6 or 12 and is congruent to one modulo seven, so it is one. The quotient by this normal C_7 has order 6 or 12 and is solvable. Since a nontrivial perfect group is not solvable, necessarily $|G|=168$.

Therefore $\text{FPdim } E=168$ and $\text{FPdim } Z(C)=168^2$. Since E is symmetric, the centralizer dimension formula now yields $E=E'$: it is a Lagrangian subcategory. EGNO Corollary 9.13.6 implies that C is group-theoretical.

15.5 Contradiction with the complete group-theoretical obstruction

Phase 4 already excluded every group-theoretical categorification of $R9_2_1965$. The proof is Section 5 of `phase4/integral/GROUP_THEORETICAL_OBSTRUCTIONS.md`, with independent replay in `phase4/audit_integral_gt/AUDIT.md`.

Briefly, the all-twists projective-character reduction exhausts all 57 groups of order 168 and all 1,751 subgroup conjugacy classes. Only $(\text{PSL}(2,7), S_4)$, for the two S_4 conjugacy classes, survives. Restriction on third cohomology is injective: its kernel is seven-primary, whereas the Sylow-seven normalizer acts without fixed points on $H^4(C_7, \mathbb{Z})$. Thus every admissible associator is globally trivializable. Both classes in $H^2(S_4, C^*)$ extend from $H^2(\text{PSL}(2,7), C^*)$ and can be removed by a global tensor autoequivalence. The resulting untwisted tensors are $R9_2_1557$, not $R9_2_1965$. Their duality types already differ (seven versus five self-dual simples).

The group-theoretical exclusion is incompatible with Section 4. This proves the theorem.

16 An isotropic orbifold model for R921848

$R9_2_1848$ is modelled. $R9_2_1975$ remains unresolved.

The new model is an application of an existing, general orbifold theorem to an existing unitary near-group category. It does not require a new associator ansatz or the unproved assertion that every spherical category is unitary.

16.1 Construction and complete fusion rules

Let N be Budinski's near-group category J_{18}^1 of type $C18+18$. Its simple objects consist of the group $G=C18$ and one object ρ with

$$\rho^2 = \sum_{g \in C18} g + 18 \rho, \\ \text{FPdim}(\rho) = 9 + 3 \sqrt{11}.$$

The source exists as a unitary category by the Cuntz algebra construction described in Budinski, Theorem 3.1.1 and Table 3.1 (the four rows for $n=18$). Table 3.1 displays rounded phase coordinates, but section 3.3.2 explicitly explains the exact algebraic certification of these solutions. We use the source's existence result as a literature dependency. This continuation does not claim to independently reconstruct or replay its complete associator.

For J_{18}^1 the nondegenerate symmetric pairing and quadratic refinement are

$$\langle x, y \rangle = \exp(2 \pi i xy/18), \\ a(x) = \exp(-\pi i x^2/18).$$

Take $H=\{0,6,12\}$. Then

$$H^\perp = \{0,3,6,9,12,15\}, \\ H \subset H^\perp, \quad a|_H=1, \\ G/H = C6, \quad H^\perp/H = C2, \quad G/H^\perp = C3.$$

Apply Izumi, Theorem 12.5, together with the irreducible decomposition and multiplication formulas immediately preceding it in section 12.1. This theorem covers isotropic H ; it does not require H to be Lagrangian. The resulting unitary orbifold has six distinct invertibles indexed by G/H , and three distinct noninvertibles indexed by $G/H^\perp$. With g a generator of $C6$ and X the distinguished component, its complete presentation is

$$g^6=1, \quad g^3 X=X, \quad Xg=g^{-1} X, \\ X^2=1+g^3+2(X+gX+g^2X).$$

Here the theorem's shift g_a may be chosen to be zero because $a|_H=1$. Its noninvertible coefficient is $|H^\perp/H|=2$. In particular, $\text{FPdim}(X)=3+\sqrt{11}$, all three noninvertibles are self-dual, and the category has rank nine and maximum fusion multiplicity two. The construction supplies associativity at the categorical level; checking the based tensor alone would not supply that missing premise.

The construction is a special case of the unified formula

$$\sigma^2 = \sum_{k \in H^\perp/H} \alpha_{(k-g_a)} \\ + |H^\perp/H| \sum_{q \in G/H^\perp} \alpha_q \sigma, \\ \alpha_g \sigma = \sigma \alpha_{-g}.$$

Thus the model uses a known theoretical construction and a known source category, rather than claiming a new family of associators.

16.2 Exact census binding

Order the model basis as

$$[1, g, g^2, g^3, g^4, g^5, X, gX, g^2X].$$

The map into the census tensor basis is

[0,1,4,8,3,2,5,7,6].

All 729 coefficients agree with R9_2_1848. The source tensor SHA256 is 42681407529a88aaaf34a909975776d8d1008ee8a99b6e231263dcb1aae9ff5c. The self-contained record 1848_model.json includes both tensors and the permutation. Run

```
python verify\_1848\_model.py
```

to verify the transport, the subgroup and root-of-unity exponent conditions, associativity, duality, and the dimension identities using exact integers. The receipt is 1848_model_verification.json. Its scope explicitly separates the new exact transport from the cited near-group existence theorem.

16.3 Primary sources and scope

1. P. G. Budinski, *Exotic Fusion Categories and Their Modular Data*, master's thesis, University of Alberta (2021), Theorem 3.1.1, Table 3.1 (n=18), section 3.3.2 (exactness, printed pp.46–47). DOI: <https://doi.org/10.7939/r3-7p7k-fg18>. Institutional PDF: <https://ualberta.scholaris.ca/bitstreams/3f73f320-a2a2-4184-ad81-d8708764f725/download>.
2. M. Izumi, *A Cuntz algebra approach to the classification of near-group categories*, arXiv:1512.04288v2, section 12.1, Theorem 12.5 and the preceding action formulas (printed pp.107–111): <https://arxiv.org/abs/1512.04288v2>.

Source URLs and hashes are in sources.json. Downloaded primary papers and extracted text are held in the local research cache; they are not included in the verified baseline archive or the final research package. In particular, Izumi's cached PDF and text under sources are URL/hash dependencies, not bundled replay inputs. The exact tensor replays do not read these cached papers. Later citations to the thesis in Yu's 2024 paper and Yu-Zhang's notes support that its solutions are used as existence results, but the proof dependency above is the primary thesis itself.

Izumi works in the unitary C* setting. That is sufficient here because the conclusion is the existence of one unitary model. No classification theorem for arbitrary original spherical categories is inferred.

17 Unified restrictions on the four quadratic cases

Throughout this section $d = 4 + 3\sqrt{2}$ and $D = 144 + 96\sqrt{2}$. The four quadratic rings have dimension type $(1^4, 2, d^4)$, and $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(D) = 2304 = 2^8 3^2$. All spherical and positive-center arguments use the already audited sphericalization and Galois-existence reduction; no claim is made that every original category is unitary or pseudo-unitary.

17.1 The integral part of the center is pointed

Proposition 17.1. *For a complex categorification of any of the four quadratic rings, the center contains no nontrivial Tannakian subcategory, and every integral-dimensional central simple is invertible. The integral center is Vec or sVec for the three commutative rings, and Vec for R9_2_1975.*

Proof. The established TY module reduction gives the nonalternating bicharacter on V_4 . In a commutative source, let h, h' stabilize its two distinct large-object orbits. Their product g is isotropic and acts freely. The adjoint subcategory is the full source, so the universal grading is trivial and forgetting central invertibles is injective. The TY central-invertible formulas give self-braiding $+1$ for a lift of g , and -1 for a lift of h or h' . Thus a lift of g would supply a central $\text{Rep}(C_2)$.

Central de-equivariantization would have four invertibles and two self-dual objects X, Y of dimension d . Its rules are

$$\begin{aligned} hX &= X, & hY &= Y, & uX &= Y, & uY &= X, \\ X^2 &= Y^2 = 1 + h + 4X + 4Y, & XY &= u + hu + 4X + 4Y. \end{aligned}$$

The alternatives $u^2 = 1$ and $u^2 = h$ are exactly the already excluded project tensors $R6_4_174$ and $R6_4_6$. All 216 coefficients of each identification agree. Their general exclusions are inherited lower-rank theorems, not deductions from failure to find a model. Hence g has no central lift. There is at most one fermionic lift of h, h' , since two would lift g . This proves the pointed-center bound.

For the noncommutative source, the corresponding hypothetical central quotients are the excluded $R6_4_1$ or $R6_4_155$ with a pointed permutation that cannot lift to a tensor autoequivalence. The latter obstruction follows from the distinct uniquely lifted pointed twists of $R6_4_155$. The exact source-to-quotient maps and both alternatives are checked in the proof record. The pointed center is therefore Vec .

If $\text{Rep}(H)$ were Tannakian in the center, FP-dimension divisibility and the norm of D would force the primes of $|H|$ to lie in $\{2, 3\}$. Burnside makes H solvable. A nontrivial solvable group has a nonunit one-dimensional representation, which would give a central boson, contradicting the preceding bounds.

Each central dimension is $a + bd$ with nonnegative integral a, b . Thus its integral-dimensional simples are exactly those forgetting into the integral TY subcategory. They form an integral braided fusion subcategory whose dimension has prime support in $\{2, 3\}$. The categorical Burnside theorem makes it solvable, hence weakly group-theoretical. With no Tannakian subcategory it is weakly anisotropic, so Natale's integral weakly-anisotropic theorem makes it pointed. This applies to the integral subcategory, not to the ambient nonintegral center. $\square$

The categorical inputs are the TY formulas of Gelaki-Naidu-Nikshych, Proposition 4.1 [13]; ENO FP-dimension divisibility; Etingof-Nikshych-Ostrik, *Weakly group-theoretical and solvable fusion categories*, Theorem 1.6 [14]; and Natale, *The core of a weakly group-theoretical braided fusion category*, Theorem 4.1 [15]. Their hypotheses and the exact lower-rank dependencies are recorded in the linked proof package. In particular no Morita representative has a nontrivial faithful finite-group grading, since such a grading would give central $\text{Rep}(H)$.

17.2 TY saturation and its sign

Let Y be the dimension-two TY object and ϵ the sign of its parameter in a positive spherical representative. Shimizu's formula [16] gives

$$(\nu_1(Y), \dots, \nu_8(Y)) = (0, \epsilon, 0, 0, 0, \epsilon, 0, 2).$$

The center induction formula, $\nu_8(Y) = \text{FPdim}(Y)$ and positivity force all constituents of $I(Y)$ to have eighth-root twists. Fourier inversion then fixes the weighted fibers.

Each primitive-eighth fiber has weight $D/4$; its only possible dimension pairs are

$$(2 + d, 2 + 7d) \quad \text{or} \quad (2 + 4d, 2 + 4d).$$

Combining this complete support condition with the integral-center obstruction and character/induction equations gives exact separating certificates for $\epsilon = +1$ in $R1607, R1975, R1976$. Both signs remain for $R137$. The negative-sign necessary systems remain feasible. An exact witness is evidence about those systems, not a category.

17.3 A general sign-twisted module-functor criterion

Proposition 17.2. *Let $\mathcal{D}$ be a fusion subcategory, $\mathcal{M}$ an indecomposable left $\mathcal{D}$ -component in a fusion category, and X a simple in it. Suppose h is an order-two invertible in $\mathcal{D}$, $\underline{\text{End}}_{\mathcal{D}}(X) = 1 + h$, $Xh \cong X$, and right tensoring by h moves another simple of $\mathcal{M}$. Then the normalized left and right stabilizer maps on X anticommute.*

Proof. The actual internal end $B = 1 + h$ has its subgroup-algebra structure and identifies $\mathcal{M}$ with right B -modules in $\mathcal{D}$, taking X to B . The functor $(-)_h$ is a left $\mathcal{D}$ -module functor. The module-functor correspondence [20] represents it by a B -bimodule P . Since $Xh \cong X$, its underlying right B -module is B . Its left action is an algebra automorphism of B : the unit is fixed and the h -summand maps to h or $-h$. These exhaust the possibilities because $\text{Hom}_B(hB, B)$ is one-dimensional and $h^2 = 1$. The positive action gives the regular bimodule and hence the identity module functor, impossible because another simple is moved. The action is therefore the negative one. Transporting it through $r : Xh \rightarrow X$ gives, with $l : hX \rightarrow X$,

$$r(l \otimes 1_h) = -l(1_h \otimes r),$$

with ambient associators understood. This is the required categorical anticommutator. It uses no central lift of the based-central h . $\square$

For each of the three commutative sources, the two pointed orbits have different stabilizers, so right h fixes the first and swaps the second. The criterion applies to every large simple. Exact source tensors give $\dim \text{Hom}(1, X^3) = 2$ in all twelve applications. The Clifford rotation theorem then forces

$$\nu_3(X) \in \{-1, -\zeta_3, -\zeta_3^2\}. \quad (1)$$

The normalized relative trace to $\mathbb{Q}(\sqrt{2})$ is consequently -1 or $1/2$. No reality assertion is inferred from self-duality. The noncommutative source has different left and right stabilizers; this source-local criterion is not applied to it.

17.4 The all-cochain Morita restriction

The rank-twelve Morita reductions have pointed group D_8 and four objects of dimension d . There are three types in their union: commutative sources allow duality shift zero or a reflection coset; $R1975$ allows reflection or rotation. Only the reflection type overlaps. The earlier nonidentity quotient action and extreme $R1975$ branch are excluded by their separate associator and pointed-automorphism obstructions.

The forced cyclic restrictions determine the pointed D_8 associator class: it restricts to class two on the rotation C_4 , trivially on the even V_4 , and to the TY sign

on the odd reflection groups. The cohomology calculation of Goff-Mason-Ng [17, Lemmas 4.4–4.5] shows that these cyclic restrictions detect the full class.

For a large simple, the pointed-bimodule stabilizer is $L = D_8 \times_{V_4} D_8 \cong D_8 \times C_2$. An explicit stabilizer cochain, checked against every coboundary equation, has mixed pairing -1 on its pure left and right central involutions h_L, h_R . This is independent of the chosen solution. Indeed any other solution differs by a 2-cocycle γ . Its commutator with the central h_R is a character of L , hence is trivial on $[L, L]$; but $h_L h_R \in [L, L]$ and its pairing with h_R equals that of h_L . Thus changing the cochain cannot change the sign.

The nondegenerate stabilizer pairing makes this four-object pointed bimodule invertible [18, Proposition 2.7]. The cubic Clifford theorem gives (1) for each of its four large objects. For pseudo-unitary Morita partners the weighted total indicator

$$v_3(\mathcal{C}) = \sum_T \text{FPdim}(T) \nu_3(T) = \frac{1}{D} \sum_{Z \in \text{Irr } \mathcal{Z}(\mathcal{C})} \text{FPdim}(Z)^2 \theta_Z^3$$

is Morita invariant by the center indicator formula [19]. The nonunit invertibles and TY object have third indicator zero. Subtracting the common unit and dividing by d therefore transfers the sum of four large-object indicators. Its projected value belongs to

$$\{-4, -5/2, -1, 1/2, 2\}.$$

This covers $R1975$ as well. Neither bimodule invertibility nor based centrality supplies a central half-braiding on the full category.

17.5 Exact surviving branches and exponent consequence

The full discrete projected choices were tested in the necessary center systems. Exact nonnegative rational witnesses survive as follows:

Source	TY sign	Four projected third indicators
$R137$	$-$	$(-1, -1, -1, -1)$
$R137$	$+$	$(1/2, 1/2, 1/2, 1/2)$
$R1607$	$-$	$(-1, -1, -1, -1)$
$R1976$	$-$	$(1/2, 1/2, 1/2, 1/2)$

The $R1975$ height-six system with the aggregate condition also has an exact witness at projected sum 2.

For positive-sign $R137$ and negative-sign $R1976$, all other discrete assignments are excluded: 25 exact separating vectors and five verified source relabellings cover all 15 alternatives in each case. The relabellings fix the distinguished pointed and TY objects and are used to relabel a hypothetical categorification; no tensor autoequivalence lifting them is assumed. Both saved matrices were independently reconstructed entrywise.

In these two conditional branches, every actual third indicator is $-\zeta_3$ or $-\zeta_3^2$, and 9 divides the FS exponent. Otherwise cubing every central twist would remove its 3-primary part. Since all central dimensions lie in $\mathbb{Q}(\sqrt{2})$, the induction formula would put $\nu_3(X)$ in a cyclotomic field of conductor coprime to 3, which cannot contain a nonrational cube root of unity. This contradicts the forced indicators.

The complete proofs, explicit cochains, bounded-system semantics and portable exact replays are curated in `evidence/unresolved_retry_20261024/README.md`. Feasibility of these relaxations does not establish associators or categorifiability. These restrictions alone do not decide the four source rings; the following section gives three general exclusions.

18 Three exclusions by Galois-unit pairing and parity

18.1 The projective-parity exclusion of $R9_2_1975$

We use the categorical TY/Morita reduction and the general-complex sphericalization and Galois-existence gate established above. The extreme Morita branch has already been excluded. In either uniform branch the rank-twelve dual has invertibles $G = D_8$, central subgroup $H = \langle z \rangle \cong C_2$, quotient $Q = G/H \cong V_4$, and four other simples X_q with

$$gX_q = X_qg = X_{q+gH}, \quad X_pX_q = \sum_{gH=p+q+\alpha} g + 2 \sum_{t \in Q} X_t, \quad X_q^* = X_{q+\alpha}.$$

Their common dimension is $d = 4 + 3\sqrt{2}$, with $d^2 = 2 + 8d$ and $D = 8 + 4d^2 = 144 + 96\sqrt{2}$. The shift α is a reflection or rotation coset. The forced pointed associator is the TY-origin class described in the preceding section. Its full class, not only its cyclic restrictions, is fixed by the detection argument cited there.

18.1.1 Duality and an arbitrary complex stabilizer cocycle

Let $K = G \times G$ act by $(g, k)X = gXk^{-1}$. The stabilizer of X_0 is $L = G \times_Q G \cong G \times C_2$, with $(g, e) \mapsto (g, gz^e)$. Use the coordinates

$$\begin{aligned} p &= (1, 0, 0), & s &= (0, 0, 1), & z &= (ps)^2 = (1, 1, 0), \\ P &= (p, 0), & S &= (s, 0), & Z &= (z, 0), & E &= (1, 1) \in L. \end{aligned}$$

An arbitrary pointed-bimodule cochain is $\beta' = \beta\gamma$, where β is a fixed cochain for the forced associator and γ is an arbitrary ordinary $\mathbb{C}^\times$ -valued two-cocycle on L . For commuting A, B write $C_\gamma(A, B) = \gamma(A, B)/\gamma(B, A)$. Let T exchange the two K factors and let ϕ_q be conjugation by $(q, 1)$ on the stabilizer. Categorical duality with shift qH requires

$$c'_q = \phi_q^*(\beta') \kappa_q T^*(\beta')$$

to be a coboundary. Here κ_q is the normalizer correction for the opposite-bimodule convention of Vainerman–Vallin, Sections 2.2.1, 2.2.2 and 3.1.1. No grading of the whole rank-twelve category is assumed. The explicit cochain and correction are recorded in `evidence/unresolved10_20261027/r9_noncommutative/REPORT.md`; the short cochain verifier checks every coboundary equation and the identities below.

For $q = (x, y, n)$ their commuting-pair identity is

$$C_{c'_q}(P, E)C_{c'_q}(S, E) = (-1)^{x+y+n}.$$

It excludes both reflection cosets for every complex γ . For either lift of the remaining rotation coset, represented by $q = ps = (0, 1, 1)$, direct substitution gives

$$C_{c_q}(P, E) = 1, \quad C_\beta(P, Z) = -1.$$

The change of the first expression on replacing β by $\beta\gamma$ is exactly $C_\gamma(P, Z)$. To see this without a root-of-unity bound, centrality of E makes $C_\gamma(-, E)$ a character; it kills $Z = [P, S]$ and has square one on P . For some $a \in \mathbb{F}_2$,

$$\phi_q(P) = P(ZE)^a, \quad \phi_q(E) = E, \quad TP = P, \quad TE = ZE.$$

Consequently

$$C_{\phi_q^* \gamma}(P, E) C_{T^* \gamma}(P, E) = C_\gamma(P, E)^2 C_\gamma(P, Z) = C_\gamma(P, Z).$$

A coboundary has trivial commutator on commuting pairs. The duality condition therefore forces $C_\gamma(P, Z) = 1$ and hence

$$C_{\beta'}(P, Z) = -1. \quad (2)$$

This holds for arbitrary complex cochains and gauges. The rotation-duality equation itself has an eighth-root trivialization; its consistency is not the contradiction, and a fourth-root-only search would be invalid.

18.1.2 Every central restriction has even multiplicity

The chosen pointed associator is identically one on $A = \langle p, z \rangle \cong V_4$. Diagonal conjugation by A therefore has its strict pointed convention, without a global normalization of the diagonal D_8 tensorator. For any $W \in Z(\mathcal{D})$, its half-braiding gives $F(W)$ an A -equivariant structure. Conjugation fixes every X_q , so $V = \text{Hom}(X_0, F(W))$ carries the projective stabilizer action from $\beta'|_A$, or its inverse. Either convention has the commutator in (2). Thus its two invertible operators obey $U_p U_z = -U_z U_p$. Taking determinants gives $(-1)^{\dim V} = 1$. We obtain

$$[F(W) : X_0] \text{ is even for every } W \in Z(\mathcal{D}). \quad (3)$$

This step uses neither indicator signs nor unitarity.

18.1.3 The distinguished Galois object has odd multiplicity

Pass to the pseudo-unitary Galois representative supplied by the signed existence gate, and take the actual Morita dual. Both categories are pseudo-unitary because their intrinsic global dimensions and FP dimensions are Morita invariant. This does not assert pseudo-unitarity of an arbitrary original category. Let $\sigma(\sqrt{2}) = -\sqrt{2}$ and $D' = \sigma(D)$. The uniform ring has eight scalar irreducibles and one irreducible of degree two. The two dimension characters have codegrees D, D' . For each nontrivial character ψ of Q , there are two representations $g \mapsto \psi(gH)$, $X_q \mapsto t\psi(q)$, $t^2 = 2\psi(\alpha)$; their codegrees are 16. The two-dimensional representation of D_8 extends by $X_q \mapsto 0$ and has codegree 4. Its Casimir eigenvalue is $2 \cdot 4 = 8$, not 4. The degree sum $8 + 2^2 = 12$ proves completeness.

The central element $v = 1 + \sum_q X_q$ acts on every irreducible E by $f_E/\sigma(f_E)$: it is one on the rational-codegree representations, and $1 + 4d = D/D'$ or $1 + 4\sigma(d) = D'/D$ on the dimension characters. Let $A_E \subset I(1)$ be the distinguished central simple of dimension D/f_E and let $B = A_{\sigma \text{FP}}$. With normalized central character $\lambda_E = \chi_E/\dim(E)$, symmetry of the unnormalized modular S matrix gives

$$\frac{D}{f_E} \lambda_E(F(B)) = S_{A_E, B} = S_{B, A_E} = \frac{D}{D'} \sigma(D/f_E) = \frac{D}{\sigma(f_E)}.$$

The degree normalization is retained. Semisimplicity of the based algebra therefore identifies $F(B)$ with v :

$$F(B) = 1 + X_0 + X_1 + X_2 + X_3.$$

This contradicts (3). Reflection, rotation and the previously eliminated extreme branch exhaust the possibilities, so $R9_2_1975$ has no fusion categorification over

©. The distinguished-object and central-character inputs are Ostrik, Theorem 2.13 and Section 2.3, and Deshpande, Proposition 2.8 and Section 2.7. The general-complex signed gate and both source tensor identities have separate exact completeness replays.

18.2 Two rank-nine exclusions from a symmetric Galois-unit pairing

18.2.1 Result and scope

R9_2_1607 and R9_2_1976 have no fusion categorification over the complex numbers. R9_2_137 remains unresolved; this argument forces all four of its large simple objects to have second Frobenius-Schur indicator +1 in the positive representative.

The argument combines the new reflection-duality obstruction in `./r9_noncommutative/REPORT.md`, the Galois-unit calculation in `./r9_noncommutative/GALOIS_PARITY_EXCLUSION.md`, and the symmetric pairing observation below. It uses the previously audited categorical TY/Morita reductions and the general-complex-to-pseudo-unitary existence gate. There is no unitarity assumption and no search over associators.

18.2.2 1. Exact source bindings

Inputs are the tensors in `data/current_remaining_cases.py` at the frozen source commit 6a426acf6465dd674351be92dcca41b72f371521.

Ring	TY object	Large objects	Selfdual large objects
<i>R9_2_137</i>	8	3,4,5,6	all four
<i>R9_2_1607</i>	3	1,2,7,8	7,8
<i>R9_2_1976</i>	5	1,2,3,4	none

The full tensor hashes are recorded in the source-bound accepted ledger.

Every source has four invertibles, one TY object Y of dimension 2, and four large objects of dimension

$$d = 4 + 3\sqrt{2}, \quad d^2 = 2 + 8d, \quad D = 144 + 96\sqrt{2} = 8 + 4d^2.$$

The short verifier `verify_source_checks.py` independently checks these dimensions, dualities, and hashes. Its receipt is `source_checks.json`.

The prior reductions establish a connected subgroup algebra and a rank-twelve Morita dual with invertible group $G = D_8$, central stabilizer $H = C_2$, quotient $Q = G/H = V_4$, and four other simples X_q . The earlier possibilities for these three sources had zero or reflection duality shift. The new all-complex commuting-pair obstruction excludes the reflection shifts. Thus a remaining categorification must have the uniform zero-shift rules

$$gX_q = X_qg = X_{q+gH}, \quad X_pX_q = \sum_{gH=p+q} g + 2 \sum_{t \in Q} X_t, \quad X_q^* = X_q.$$

The existing spherical/Galois existence gate allows us to work with a pseudo-unitary representative of this Morita pair. This does not assert that the original hypothetical category was unitary. Pseudo-unitarity is preserved under categorical Morita equivalence, so both sides have their canonical positive spherical structures.

18.2.3 2. A multiplicity-one Galois object is symmetrically selfdual

Let $\mathcal{D}$ be the hypothetical positive rank-twelve category, $F : Z(\mathcal{D}) \rightarrow \mathcal{D}$ the forgetful functor, and I its right adjoint. Its canonical algebra $L = I(\mathbf{1})$ is connected étale and has trivial twist. It has a nondegenerate Frobenius pairing. A commutative Frobenius algebra with trivial twist in a ribbon category is symmetric in the **pivotal** sense: see Fröhlich–Fuchs–Runkel–Schweigert, *Correspondences of ribbon categories*, Proposition 2.25(ii), <https://arxiv.org/pdf/math/0309465> or the author copy <https://www.math.uni-hamburg.de/home/runkel/PDF/corr.pdf>.

In explicit duality notation, its pairing gives an isomorphism $\phi_L : L \rightarrow L^*$ satisfying

$$\phi_L = \phi_L^* j_L,$$

where j is the pivotal structure. This is stronger than merely braided commutativity of the multiplication; it is the property used here.

Let $B = A_{\sigma_{\text{FP}}}$ be the distinguished summand of L for the conjugate dimension character, where $\sigma(\sqrt{2}) = -\sqrt{2}$. For the zero-shift uniform ring the complete list of irreducibles consists of eight one-dimensional representations and one two-dimensional representation. Their formal codegrees are

$$D, \quad D' = \sigma(D), \quad 16 \text{ (six times)}, \quad 4.$$

The distinguished central objects therefore have dimensions

$$1, \quad D/D', \quad D/16 \text{ (six times)}, \quad D/4.$$

Their multiplicities in L are respectively 1 for the scalar representations and 2 for the two-dimensional representation. In particular B occurs once and its dimension D/D' is unique among these constituents. Since $L \cong L^*$, duality must preserve this unique dimension and consequently $B^* \cong B$.

The pairing of L cannot pair this summand with any other simple summand. Its restriction to B is therefore nondegenerate and still pivotally symmetric. Thus B is symmetrically selfdual.

The normalized central-character calculation, with the factor $\dim(E)$ retained for the two-dimensional representation, gives

$$F(B) = \mathbf{1} + \sum_{q \in Q} X_q. \tag{1}$$

For a full proof see the sibling GALOIS_PARITY_EXCLUSION.md, §4. The same proof applies with zero shift: the rational codegrees remain 16 and 4. More explicitly, $1 + \sum X_q$ acts on every irreducible E by $f_E/\sigma(f_E)$, and the modular S -symmetry formula gives exactly the same scalar for $F(B)$. Equality follows from semisimplicity of the complexified based algebra.

The functor F is pivotal and strong monoidal. It therefore carries the nondegenerate symmetric pairing of B to one on $F(B)$. Every X_q is selfdual and appears once in (1). Restricting the pairing to each of these pairwise nonisomorphic simple summands is again nondegenerate and symmetric. Hence

$$\boxed{\nu_2^{\mathcal{D}}(X_q) = +1 \quad \text{for every } q \in Q.} \tag{2}$$

We are **not** claiming that numerical Frobenius–Schur indicators are preserved by arbitrary tensor functors. The transported object is a particular symmetric pairing, and multiplicity one is essential.

18.2.4 3. Exact Morita-invariant second-indicator comparison

For a pseudo-unitary fusion category define

$$T_2(\mathcal{C}) = \sum_{V \in \text{Irr}(\mathcal{C})} \dim(V) \nu_2(V).$$

The center-induction indicator formula gives

$$T_2(\mathcal{C}) = \frac{1}{\dim \mathcal{C}} \sum_{Z \in \text{Irr}(Z(\mathcal{C}))} \theta_Z^2 \dim(Z)^2. \quad (3)$$

Indeed, multiply $\nu_2(V) = \text{Tr}(\theta_{I(V)}^2) / \dim \mathcal{C}$ by $\dim V$, sum over V , and use Frobenius reciprocity: $\sum_V \dim(V) [I(V) : Z] = \dim F(Z) = \dim Z$. Formula (3) is invariant under Morita equivalence because the centers are braided equivalent and their canonical positive ribbon structures agree. The indicator formula is Ng-Schauenburg, *Frobenius-Schur indicators and exponents of spherical categories*, <https://arxiv.org/pdf/math/0601012>.

Write the TY sign in the source as $\varepsilon \in \{+1, -1\}$, so $\nu_2(Y) = \varepsilon$. Its four invertibles have trivial associator and second indicator 1. Thus its integral contribution is

$$T_{2,\text{integral}}(\mathcal{C}) = 4 + 2\varepsilon.$$

In the rank-twelve dual, the forced pointed associator is trivial on the even Klein four subgroup, while the two odd reflections have second indicator ε . The two order-four rotations are not selfdual and have second indicator zero. Thus its pointed contribution is also exactly

$$T_{2,\text{pointed}}(\mathcal{D}) = 4 + 2\varepsilon.$$

These signs are part of the earlier TY-origin associator calculation; Galois conjugation does not change them because they are $+1, -1$. Using (2) and cancelling these equal integral contributions in (3) gives

$$\boxed{\sum_{\text{four large } V \text{ in } \mathcal{C}} \nu_2(V) = 4.} \quad (4)$$

Now a nonselfdual simple has second indicator zero, whereas a selfdual simple has second indicator $+1$ or -1 :

- **R9_2_1607:** only two large simples are selfdual. Their total is at most 2, contradicting (4).
- **R9_2_1976:** none of the four large simples is selfdual. Their total is zero, contradicting (4).
- **R9_2_137:** all four are selfdual. Equation (4) forces all four indicators to be $+1$, but supplies no contradiction.

This completes the two exclusions over $\mathbb{C}$, using the stated general-complex existence gate. No numerical indicator optimization, bounded root search, or unitarity classification enters the new argument.

18.3 The remaining ring $R9_2_137$

In the remaining zero-shift Morita category, put $S = \sum_q X_q$ and let z be the central pointed involution. The pointed-subcategory canonical central algebra is $1 + B$ and its augmentation splits the forgetful algebra as $1 \times A$, with $[A] = 1 + S$. Thus A is forced in a hypothetical category. Its induction Gram gives eight modules underlying $g + S$ and four underlying X_q . Each of the latter has internal end $1 + z$. The module category is therefore the central-involution module already used, not a new construction. Adding all four positive second indicators to both TY-sign center systems leaves exact nonnegative rational witnesses, of support 102 and 97. Their feasibility is not a categorification. Complete matrices and exact solver-free certificates are in `evidence/unresolved7_20261027/audit_parity/R137_CONTINUATION.md`.

19 What remains unresolved

Type	Suffixes of unresolved $R9_2_n$
Quadratic	137

Only $R9_2_137$ remains unresolved, of dimension type $(1^4, 2, (4 + 3\sqrt{2})^4)$ and global dimension $144 + 96\sqrt{2}$. General, pivotal, spherical, pseudo-unitary and unitary existence remain undecided. The sphericalization and Galois existence gates equate general existence with pseudo-unitary existence, without asserting pseudo-unitarity of an arbitrary original category. Braided categorification is excluded. The two TY signs still pass the necessary systems with all four positive large second indicators. No compatible associator is constructed.

20 Exact evidence and reproduction

The public source is https://github.com/sebastienpalcoux/grothendieck-ring-classification/tree/main/evidence/rank9_m2_20261024. It exposes the accepted ledger, positive models, exact open tensors and readable proofs. Its checksummed replay archive retains every source cochain, certificate, constructor, independent audit and historical source context. The current mathematical partition is `evidence/frontier_20261027/rank9_m2_ledger.json`; `phase5/results/` is its immutable predecessor. Historical ledgers remain explicitly superseded; their unsupported half- E_6 claims are not used by the corrected verifier.

The baseline all-phase entry point, from a restored repository with its requirements installed, is

```
python3 evidence/correctness_20261025/replay_rank9.py
--output-dir /tmp/rank9-current-complete --frontier --jobs 2
```

The output directory must be new. Each phase starts from an independently hash-checked copy of the unchanged source archive. Two separately bound phase-three certificates omitted from that archive are supplied by the correction release. The command replays the accepted baseline and phases two-five, regenerates the model tensors and checks its frozen baseline partition. The archive's older root dispatcher checks phase five; its `--baseline` option adds phase four, not all preceding phases.

For R9_2_920, exact pseudo-Boolean associativity completeness and all algebraic branch checks are used. This encoding shares the discovery implementation; the memory-limited independent eight-bit run is not reported as successful. Stored exact group/cohomology narrowing records are checked by this command; fresh full regeneration is recorded in the separate correctness-audit evidence. The baseline optional frontier checks verify necessary restrictions. The three further exclusions and their fresh audits replay in their own frozen source context with python3 tools/replay_frontier_20261027.py. Each categorical argument retains its explicit external premises; independent human verification and formalization remain pending.

The fusion census repository publishes the same 2,225 supplied tensors through multiplicity two and the corresponding producer-run qualification. The public Atlas uses one rank/multiplicity table and one case directory for all ranks. Every M2 case has its source identity, current decision, proof links and a downloadable Python tensor. The remaining open case retains its separate categorical-scope statements.

References

- [1] P. Etingof, D. Nikshych, V. Ostrik, *On fusion categories*, Ann. Math. 162 (2005), 581–642; [arXiv:math/0203060](#).
- [2] V. Ostrik, *Pivotal fusion categories of rank 3*, Mosc. Math. J. 15 (2015), 373–396; [arXiv:1309.4822](#).
- [3] T. Deshpande, *On centers of bimodule categories and induction-restriction functors*, [arXiv:1611.04486](#), Proposition 2.8 and Section 2.7.
- [4] A. Davydov, M. Müger, D. Nikshych, V. Ostrik, *The Witt group of non-degenerate braided fusion categories*, J. Reine Angew. Math. 677 (2013), 135–177; [arXiv:1009.2117](#).
- [5] T. Hagge, S.-M. Hong, *Some non-braided fusion categories of rank three*, Commun. Contemp. Math. 11 (2009), 615–637; [arXiv:0704.0208](#).
- [6] D. Aasen, E. Lake, K. Walker, *Fermion condensation and super pivotal categories*, J. Math. Phys. 60 (2019), 121901; [arXiv:1709.01941](#), Section 7.
- [7] Y.-L. Cheng, S.-H. Ng, *On Hopf algebras of dimension $4p$* , J. Algebra 328 (2011), 399–419; [arXiv:1005.1267](#).
- [8] S. Natale, *On the semisolvability of semisimple Hopf algebras of low dimension*, [arXiv:math/0305038](#), Section 5.2.
- [9] N. Andruskiewitsch, S. Natale, *Examples of self-dual Hopf algebras*, J. Math. Sci. Univ. Tokyo 6 (1999), 181–215; <https://www.ms.u-tokyo.ac.jp/journal/pdf/jms060108.pdf>.
- [10] A. Davydov, I. Runkel, *$\mathbb{Z}/2\mathbb{Z}$ -extensions of Hopf algebra module categories by their base categories*, <https://www.math.uni-hamburg.de/home/runkel/PDF/hext.pdf>, Corollary 3.19 and Section 3.6.
- [11] C. Galindo, S.-M. Hong, E. C. Rowell, *Generalized and quasi-localizations of braid group representations*, [arXiv:1105.5048](#), Theorem 5.20.

- [12] M. Alekseyev, W. Bruns, J. Dong, S. Palcoux, *Classifying integral Grothendieck rings up to rank 5 and beyond*, [arXiv:2507.07023](#), Question 1.14.
- [13] S. Gelaki, D. Naidu, D. Nikshych, *Centers of graded fusion categories*, [arXiv:0905.3117](#), Proposition 4.1.
- [14] P. Etingof, D. Nikshych, V. Ostrik, *Weakly group-theoretical and solvable fusion categories*, Adv. Math. 226 (2011), 176–205; [arXiv:0809.3031](#), Theorem 1.6.
- [15] S. Natale, *The core of a weakly group-theoretical braided fusion category*, [arXiv:1704.03523](#), Theorem 4.1.
- [16] K. Shimizu, *Frobenius–Schur indicators in Tambara–Yamagami categories*, [arXiv:1005.4500](#), Theorem 3.4.
- [17] C. Goff, G. Mason, S.-H. Ng, *On the gauge equivalence of twisted quantum doubles of elementary abelian and extra-special 2-groups*, [arXiv:math/0603191](#), Lemmas 4.4–4.5.
- [18] L. Vainerman, J.-M. Vallin, *On $\mathbb{Z}/2\mathbb{Z}$ -extensions of pointed fusion categories*, [arXiv:1210.1732](#), Proposition 2.7.
- [19] S.-H. Ng, P. Schauenburg, *Frobenius–Schur indicators and exponents of spherical categories*, [arXiv:math/0601012](#), Theorem 4.1.
- [20] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor Categories*, AMS Mathematical Surveys and Monographs 205 (2015).