

RANK-EIGHT GROTHENDIECK RINGS: CATEGORIFICATION THROUGH MULTIPLICITY THREE

ABSTRACT. We investigate complex categorification of the supplied 4500 rank-eight fusion rings of multiplicity at most three. The exact-multiplicity-three continuation treats 3527 new tensors: 3507 general exclusions, 16 unitary models, and 4 unresolved cases. The complete classification through multiplicity two is preserved: 896 exclusions and 77 unitary models. Unified obstructions combine signed sphericalization, full character-labeled central induction, modular symmetry and rotations, integer central restriction and cubic fusion constraints, and module-category and Morita arguments. Positive constructions include group representations, near-group equivariantizations, exceptional subfactor categories, a twisted group-theoretical category and Morita duals. An inherited generalized-Haagerup Q-system realizes `R8_3_721`, using a categorical vertex obstruction, exact coefficient bounds and a direct exhaustive fusion completion. The triality-zero component of the known unitary E_{21} category for $SU(3)_{21} \subset (E_7)_1$ realizes `R8_3_982` by an exact comparison of all 512 fusion coefficients. Pointed-induction Fourier criteria exclude `R8_3_65` and `R8_3_1795` in every pointed associator class. The four remaining open cases admit sphericity and Galois pseudo-unitarity reductions and have no braided Morita representatives. The separate spherical-equivariantization proof excludes `R8_3_436` in general. A complete center-frame and Galois twist-orbit reduction, followed by an exact integer rotation certificate, excludes `R8_3_3472` over $\mathbb{C}$. The terminal contradiction replays with rational arithmetic and no optimization solver. Tensor comparisons and exact replay certificates accompany these results.

Research status. AI-generated research · independent verification pending. The results, proofs and accompanying data have not been independently verified by a human expert or formalized in Lean. Treat all claims and data with caution; exact computational checks do not establish unstated categorical hypotheses.

1. INTRODUCTION AND EXACT SCOPE

The fusion-ring census [11] supplies all 96 rank-eight based fusion rings of exact multiplicity one and all 877 of exact multiplicity two, and the supplied extension adds 3527 of exact multiplicity three. A fusion ring need not be the Grothendieck ring of a fusion category. We work over $\mathbb{C}$ and do not assume pivotality, sphericity, braiding, commutativity or unitarity when generally excluding a ring. A positive construction is a unitary category with the exact based-isomorphism type of its full multiplication tensor.

The lower-rank criteria and their finite verifiers are developed in [12, 13, 14, 16, 17, 18]. We cite their theorems and do not reprove them. These new manuscripts are cited by title, with no byline. Our rank-eight observation, Proposition 10.1, extracts ordinary formal codegrees from the Casimir spectrum even though the ring may be noncommutative. Known near-group, zero-spectrum and cyclotomic tests provide other general obstructions [25, 28, 21]. Standard quantum group, pointed, Tambara–Yamagami and representation categories supply the positive side [29, 41, 22, 42].

The accepted study of integral Grothendieck rings [19, 20] supplies a necessary divisibility test and explicit group-theoretical rank-eight models and exclusions. Being an integral or Drinfeld ring does not itself supply a fusion category, so only its explicit categorical constructions enter our positive list.

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Theorem 1.1 (Complete rank-eight classification through multiplicity two). *The rank-eight based fusion rings through multiplicity two have the following classification.*

<i>Exact multiplicity</i>	<i>Fusion rings</i>	<i>Excluded generally</i>	<i>Unitarily realized</i>
1	96	48	48
2	877	848	29
<i>Total</i>	973	896	77

Every excluded ring is ruled out over $\mathbb{C}$ with no pivotality assumption. Each positive model is matched to the entire tensor under a unit-preserving permutation, and is therefore a Grothendieck ring. Thus there are exactly 48 Grothendieck rings of exact multiplicity one and 29 of exact multiplicity two, and every one admits a unitary realization.

The general complex exclusion of R8_2_24 is proved in Section 20. The ring R8_1_3, of global FP dimension 20, is realized below by a twisted group-theoretical category. Its dimension vector agrees with that of the commutative ring $\text{Rep}(\text{Dic}_5)$, although its multiplication is noncommutative. Thus full tensor comparison remains essential even after a suitable dimension profile is found.

2. TRUSTED INPUT AND STABLE RECORDS

A rank- r fusion ring has a distinguished basis $B = (b_0 = 1, b_1, \dots, b_{r-1})$, structure constants $N_{ij}^k \in \mathbb{Z}_{\geq 0}$, and a duality involution $i \mapsto i^*$ satisfying

$$b_i b_j = \sum_k N_{ij}^k b_k, \quad N_{ij}^0 = \delta_{j,i^*}, \quad N_{ij}^k = N_{i^*k}^j = N_{kj^*}^i.$$

Its exact multiplicity is $\max_{i,j,k} N_{ij}^k$. A based isomorphism permutes the basis, fixes b_0 , and preserves every structure constant; it automatically preserves duality. We do not identify tensors by character table, spectrum or FP dimension vector alone.

The historical multiplicity-two source is the public *Fusion Ring Census* [11] at Git commit b86eb6dc6eeafe5fc5f6906d7b074e0322e9726. It contains `results/rank8/tables.txt.gz` and `results/rank8/counts.csv`, with audit file `verification/rank8/audit.json`. The SHA-256 of the uncompressed tensor file is

97b86d2de8ca0352c5c6e14686cd1d43dbd52180dc53b2ff1ccf7a61788c9600. The census verifies all duality types, associativity, nonnegative coefficients and uniqueness; its terminated-strata evidence justifies completeness through multiplicity two. We trust that source for counting and do not claim an independent exhaustive enumeration here.

Within each exact multiplicity m , sort full flattened $8 \times 8 \times 8$ tensors lexicographically and call the j th one `R8_m_j`. This is a deterministic local index, not an invariant of its original basis labelling. The files `data/decisions.json`, `data/realized_models.json` and `data/rank8_tables.txt.gz` give, respectively, one decision per input, the exact positive model tensors and the source multiplication tables. A SHA-256 of each JSON tensor prevents accidental source-record substitution. The comparison code minimizes the full tensor under all relabellings compatible with inexpensive intrinsic signatures; this is an isomorphism comparison, not a filter by numerical fingerprint.

3. THE MULTIPLICITY-THREE CONTINUATION

The supplied extension has 3527 rings of exact multiplicity three and 4500 cumulatively. Every tensor, identifier, decision and model through multiplicity two is preserved. The new raw tensor file has uncompressed SHA-256

7baaa0a2018e11a0464c3f19da9030195421f04a73806c22363cf9d5eeb92a84. The independently checked tensor axioms and preservation of earlier tensors do not constitute an independent exhaustive enumeration. The optimized producer source is now supplied and matches the completion receipt; enumeration completeness remains producer-reported.

The census release is pinned to Git commit 30046938fe2bd5cf4013c19eb0341d662f0d5dad. The source and audit are `results/rank8/tables.txt.gz` and `verification/rank8/audit.json` in that repository.

Theorem 3.1 (Current exact-multiplicity-three status). *The 3527 supplied rings of exact multiplicity three have the following recorded categorification status:*

<i>Status</i>	<i>Rings</i>
<i>General complex exclusion</i>	3507
<i>Unitary construction</i>	16
<i>Unresolved general categorification</i>	4

Positive identifications use full ordered tensor comparisons or, for R8_3_721, the exact coefficient bound and complete-census uniqueness proved in Theorem 3.16. A recorded construction proves existence, without claiming uniqueness up to tensor equivalence.

The source-bound combined ledger is `supplements/rank8/output/research_20261024/ledger.json`. Its exact-M3 first-exclusion allocation follows. Overlapping obstructions are retained in the evidence rather than counted twice.

First accepted criterion	Rings
RepA5 complement Morita relative cohomology	1
certified negative subring transfer	14
character labeled atomic modular STS signed induction	1
complete character labeled modular STS signed induction	5
complete distinguished modular S symmetric low frame obstruction	2
complete integer central cubic monodromy correlated rotation STS obstruction	1
complete labeled frame modular STS signed induction	1
complete low frame hull signed induction	2
cyclotomic	7
dimension84 adjoint rigidity	5
dnumber	2575
exact center indicator certificate	1
forced Morita dual integral center Gram cubic Verlinde	1
full character field labeled modular S symmetry signed induction	2
global convex induction lift	2
global duality induction lift	1
integral center Gram cubic Verlinde	1
integral drinfeld	34
involution module	19
lower rank subring	378
near group	1
nonsolvable dimension60 extension	1
normal split spectral support and annihilator transport	1
one spectrum	1
oriented signed	67
partial modular spectrum symmetric frame signed induction	1
pointed induction fourier fibre packing all indicators	1
pointed induction fourier phase all classes	1
quadratic casimir compression	64
rank7 certified subring	4
rank8 biquadratic lift	2
rank8 general biquadratic lift	1
rank8 noncomm dnumber	37

First accepted criterion	Rings
rank8 scalar oriented	3
rank8 spherical reduction	11
rank8 split module polynomial	44
rank two	10
rep a5 module rank	1
simple center prime power filtered joint induction	1
simultaneous C2 split module quotient	4
solvable dimension216 obstruction	4
solvable dimension80 rigidity	1
solvable dimension90 pointed base obstruction	2
spherical equivariantization formal codegree field	1
split module dimension280 obstruction	2
subring dimension	6
weak group unitarity lift	1
weighted induction	2
zero spectrum	179

The lower-multiplicity classification below retains its original scope and proof inputs. Its historical allocation is separate from this table.

3.1. Negative subring transfer and simultaneous quadratic fields. The new census is first compared with the current, general-scope negative rank-six and rank-seven catalogues. Each transfer certificate gives a dual-closed proper basis subset, verifies product closure, and supplies a unit-preserving permutation identifying every multiplication coefficient with the cited negative tensor. The tensor and its full published decision are hash bound. Any categorification would contain this full fusion subcategory, so the transfer excludes a general complex categorification. The recovered scan reproduces fourteen such transfers from catalogues of 531 rank-six and 1396 rank-seven negatives of multiplicity at most three.

Proposition 3.2 (Simultaneous quadratic compression). *Let R be a fusion ring and $B = \sum_i N_i N_i^t$ its regular Casimir matrix. For each real quadratic field $L = \mathbb{Q}(\sqrt{d})$ appearing among its irreducible quadratic characteristic-polynomial factors, write*

$$Q_d = f_d \sum_j m_j \frac{t_j^2 - 4n_j}{n_j^2}, \quad (x^2 - t_j x + n_j)^{m_j} \mid \det(xI - B),$$

where the sum includes exactly the factors generating L and f_d is the fundamental discriminant of L . Let r be the rank over $\mathbb{F}_2$ of all these quadratic squareclasses. A complex fusion categorification requires

$$2^{r-1} Q_d \leq 4$$

for every displayed field. If a categorification would admit a Galois-conjugate pseudo-unitary form, the stronger bound is $2^{r-1} Q_d \leq 1$.

Proof. The full- $I(1)$ compression theorem gives $[K : L] Q_d \leq 1$ for a spherical categorification with dimension field K . On a simple representation E , the Casimir eigenvalue is $(\dim E) f_E$ and its regular multiplicity is $(\dim E)^2$; hence the displayed spectral multiplicities already give the correct contribution, also for a noncommutative ring. Each formal codegree, and therefore each Casimir eigenvalue, lies in K . The compositum of the displayed quadratic fields has degree 2^r , so $[K : L] \geq 2^{r-1}$. For an arbitrary category, pass to its sphericalization. Its original fusion-algebra sector has Casimir $2B$, dividing the known reciprocal-square compression contribution by four. This

proves the first bound without assuming pivotality. A Galois-conjugate pseudo-unitary realization is spherical and yields the stronger bound directly. $\square$

The implementation checks the stronger hypothesis by the existing weak-integrality or two-distinct-Frobenius–Perron-dimensions reduction. It computes every polynomial factor, rational ratio, squareclass rank, and gate from the source tensor. In particular a scoped spherical obstruction alone is never promoted to general nonexistence. The proof source and replay are `criteria/COMPRESSION_AND_TRANSFER.md` and `criteria/quadratic_compression.py`.

3.2. Weakly integral quadratic trace fields. The relative-trace and tensor-rotation engines are extended to the case where the object dimensions generate a real quadratic field but the total Frobenius–Perron dimension D is rational. Weak integrality supplies pseudo-unitarity. The prime support of the Frobenius–Schur exponent is the prime support of D ; the quadratic conductor must be supported there. Write every dimension in the basis $1, \sqrt{d}$, retaining both coordinates even though $D = (D, 0)$. The central first- and second-moment and cyclic rotation equations remain identities in this field, with the same complete conductor-bounded relative-trace tables. The extension uses no division by the irrational coordinate of D .

All restriction-column choices and second-indicator assignments are retained. Numerical optimization only proposes a separator; exact integer replay verifies every inequality and the negative right-side pairing. The complete signed lifting remains required outside the independently checked general-to-positive gates. The additive adapter is `criteria/weak_integral_quadratic.py`; the existing frozen engines are left unchanged.

3.3. Correlated traces in biquadratic dimension fields. Let $K = \mathbb{Q}(\sqrt{q}, \sqrt{r})$, with coprime squarefree integers $q, r > 1$. For $s \in \{q, r, qr\}$ put $F_s = s$ if $s \equiv 1 \pmod{4}$, and $F_s = 4s$ otherwise, and let χ_s be the corresponding primitive quadratic character. Coordinates below use $(1, \sqrt{q}, \sqrt{r}, \sqrt{qr})$.

Proposition 3.3 (Complete finite-height relative traces). *The normalized relative trace of a primitive n th root to K has rational coordinate $\mu(n)/\varphi(n)$. Its $\sqrt{s}$ coordinate is zero unless $F_s \mid n$ and $\gcd(n/F_s, F_s) = 1$; otherwise it equals*

$$\frac{\epsilon_s \mu(n/F_s) \chi_s(n/F_s) \sqrt{F_s/s}}{\varphi(n)}.$$

There are four linked sign choices, with $\epsilon_{qr} = \epsilon_q \epsilon_r$. For the k th power of a primitive N th root, set $g = \gcd(N, k)$ and $n = N/g$, use the same fixed sign pair, and multiply the quadratic coordinate by $\chi_s(k/g)$.

Suppose the prime support of twist orders is contained in P . For a height h , put

$$B_h = \prod_{p \in P} p^{\max(1, v_p(F_q), v_p(F_r), v_p(F_{qr})) + \lfloor \log_p h \rfloor}.$$

Every joint relative trace of powers $1, \dots, h$ is represented by an order dividing B_h , with one of the four linked sign choices, or by the joint zero vector.

Proof. The rational projection is the Ramanujan sum $\mu(n)$. The quadratic projections are the primitive quadratic Gauss sums $\tau(\chi_s) = \sqrt{F_s}$ followed by the coprime-part Ramanujan sum. Dividing by $\varphi(n)\sqrt{s}$ gives the displayed formula. The three signs come from one field automorphism and therefore satisfy the stated product relation. Reduction of the root order gives the power rule; independent signs for different powers would discard essential correlations.

If $v_p(N)$ exceeds the exponent in B_h , then for every $k \leq h$ the p -valuation of $N/\gcd(N, k)$ exceeds one and every conductor valuation. The rational Möbius term vanishes. Each quadratic term vanishes either by noncoprimality with its conductor or by a zero Möbius value. All retained powers consequently have zero trace. Every other order divides B_h . $\square$

For a pseudo-unitary category the spherical Cauchy theorem [61] supplies P from the norm of the global dimension, after the exact integrality and conductor-support checks. These are bounds on retained traces, not on the unknown categorical exponent. The field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ at height six gives $B_h = 288$; $(q, r) = (3, 7), (3, 5), (3, 13)$ at height four give respectively 1008, 720, 1872 for the prime supports recorded in the certificates.

Weighted induction and exact separation. The full induction Gram matrix is computed from the tensor, including noncommutative rings. For each simple Grothendieck-algebra representation E , its distinguished central constituent has dimension D/f_E and multiplicity $\dim E$ in $I(1)$; the corresponding restriction column starts with $\dim E$. Its outer square contributes once to the Gram matrix. Thus a two-dimensional representation is handled with first entry two, rather than a commutative first-entry-one shortcut. All compatible low columns and all residual Gram-bounded nonnegative columns are enumerated. Their dimensions satisfy the necessary integrality and d -number tests.

Nonnegative weights for high columns and the complete linked trace types obey the Gram equations, indicator identities and cyclic-rotation traces on $\text{Hom}(\mathbf{1}, X^m)$ through the chosen height. Every pseudo-unitary categorification supplies a feasible point. An exact rational Farkas witness therefore proves a contradiction; the numerical solver only proposes the witness.

General scope and independent checks. Such a rotation contradiction by itself has pseudo-unitary scope. General exclusion additionally checks every original dimension character and every oriented signed quotient in sphericalization, using exact character, parity and gauge-equivalence certificates. The accepted applications include R8_3_347, R8_3_916 and R8_3_1348. The independent checker constructs the cyclotomic quotient $\mathbb{Q}[x]/\Phi_{B_h}$ and its quadratic Gauss sums, checks their squares and correlated product, and compares every bounded-order trace and retained power with the formula. This verifies the finite identities; the preceding argument proves completeness beyond the box.

3.4. A convex relaxation of central induction. The following relaxation avoids enumerating all simultaneous choices of restriction columns for the constituents of $I(\mathbf{1})$. It uses the central-induction and relative-trace identities already established above.

Proposition 3.4 (Convex central-induction obstruction). *Let R be a commutative fusion ring, let D be its Frobenius–Perron dimension, and suppose that its formal codegrees split in a real field K for which complete joint relative traces of the required roots of unity have been obtained. For each non-Frobenius–Perron character E , let $\mathcal{C}_E$ be the complete set of possible restriction columns of the central constituent of dimension $\beta_E = D/f_E$ of $I(\mathbf{1})$. Replace the choice of one column by variables*

$$y_{E,c} \geq 0, \quad \sum_{c \in \mathcal{C}_E} y_{E,c} = 1.$$

Retain the Gram, first and second central trace, and tensor-rotation equations, using $y_{E,c}cc^t$ for the low Gram contribution and $y_{E,c}c_i\beta_E$ for each low twist moment. Fix the central unit column separately. For high columns, use every nonnegative integral column allowed by the full nonunit Gram block and the necessary central-dimension conditions, paired with every joint relative-trace type. If, for every admissible second-indicator assignment, the resulting rational system $Ax = b$, $x \geq 0$, has an exact integer separator $A^t u \geq 0$, $b^t u < 0$, then R has no pseudo-unitary categorification. If the complete spherical-character and oriented-sphericalization checks also succeed, then R has no complex fusion categorification.

Proof. A hypothetical pseudo-unitary category selects one actual column from each $\mathcal{C}_E$, hence supplies a one-hot solution for the corresponding simplex variables. Since R is commutative, each character labels a distinct simple constituent of $I(\mathbf{1})$ with multiplicity one and twist one. Thus its Gram and trace contributions are exactly those stated. Every actual high column is admitted by the full Gram block, whose complement is a sum of column squares. The complete relative-trace tables include every possible twist and cyclic-rotation spectrum. Consequently the category supplies a nonnegative solution of all the displayed equations. An exact separator contradicts $u^t Ax = u^t b < 0$.

The final assertion uses the separately checked complete lifting argument; the convex relaxation itself does not assert pivotality or unitarity of an arbitrary fusion category. $\square$

At height four this criterion excludes R8_3_1453 generally. The relaxation has 15 777 variables and 127 equations; all sixteen second-indicator assignments have exact separators, and the full oriented lifting succeeds. Three fixed-low specializations reproduce the frozen Gram, moment, and rotation equations exactly. Known realized controls R8_3_334, R8_3_342, and R8_3_475 survive. The proof, implementation, and validation receipts are stored in `criteria/GLOBAL_CONVEX_INDUCTION.md`, `criteria/global_induction.py`, and `criteria/global_specialization_R8_3_1453.json`. The same relaxation over $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ excludes R8_3_920 generally. Its height-four system has 97 768 variables and 197 equations; both admissible second-indicator assignments have exact separators and complete signed lifting succeeds. Three fixed-low specializations, selected from all 6891 low frames, reproduce 182 inherited equations each. See `criteria/global_specialization_R8_3_920.json`. The biquadratic dimension filters are additionally checked on 324 exact fixtures against independent rational multiplication-matrix characteristic polynomials, including the coprime-radical fields used in the generalized trace adapter. One further necessary condition pairs high labels under duality. In the ribbon center the dual of a simple has the same dimension and twist, and its restriction vector is obtained by the source duality permutation. Thus labels (v, τ) and (v^*, τ) have equal multiplicity. Replacing their two columns by their sum gives a stronger nonnegative relaxation; all low and auxiliary rotation variables may remain unchanged. The complete high set and its involution are checked exactly. This refinement has a separate certificate kind and isolated replay, preserving the earlier uncontracted certificates. Complete coefficient checks and known positive controls are recorded in `criteria/global_duality_validation.json`; the proof is `criteria/GLOBAL_DUALITY.md`. Resource limits or an unsuccessful exact reconstruction of a numerical separator provide no obstruction. This duality refinement excludes R8_3_986 over $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Its 197-equation system has 52 718 variables, compared with 97 768 before pairing the high labels. Both admissible second-indicator assignments have exact separators, and all sixteen oriented signed branches are excluded by the complete lifting certificate. The source-bound certificate is stored with the `global_family` campaign and has a separate isolated exact replay.

3.5. A uniform split-module polynomial test.

Proposition 3.5 (Fixed-point difference-space obstruction). *Let R be a commutative fusion ring, g a nontrivial invertible basis element of order two fixing at least one basis element, and X a self-dual object. Let F be the basis elements fixed by g , and let p be any integer polynomial with $p(N_X) = 0$. A categorification of R requires a symmetric integer matrix S , indexed by F , satisfying*

$$S_{V,W} \in \{-N_{X,V}^W, -N_{X,V}^W + 2, \dots, N_{X,V}^W\}, \quad p(S) = 0.$$

Consequently, exhaustive failure of these equations, even after reduction modulo a prime, excludes a complex fusion categorification.

Proof. A simple object fixed by g supplies a rank-one module over the pointed subcategory generated by g , trivializing its associator class. Hence $A = \mathbf{1} \oplus g$ is a connected separable algebra. In $\mathcal{C}_A$, each fixed simple V splits into two module simples V_+, V_- , while each free g -orbit induces one module simple. The induction image is spanned by the free-orbit simples and the sums $V_+ + V_-$. The action of self-dual X is symmetric, so its orthogonal complement, spanned by $V_+ - V_-$, is invariant. Induction and symmetry force the block between two fixed pairs to have the form $\begin{pmatrix} u & a-u \\ a-u & u \end{pmatrix}$, where $a = N_{X,V}^W$ and $0 \leq u \leq a$ is an integer. Its action on differences has entry $2u - a$. The faithful regular representation makes $p(N_X) = 0$ a ring identity, which holds on every module and hence on this invariant difference space. This proves the necessary condition. $\square$

Swapping the two labels of a fixed pair conjugates S by a diagonal sign matrix. On a spanning forest of the odd-entry graph, all entries are nonzero and can therefore be made positive. This gives

a complete sign reduction. The implementation verifies the source polynomial identity and every finite case exactly; a separate replay uses unbounded integer arithmetic and a different spanning forest. It gives new exclusions $R_{8,3,j}$ for $j \in \{1019, 1057, 1067, 1197, 2392, 2836\}$, and independently recovers the applicable dimension-84 and dimension-280 obstructions. See `noncommutative/SPLIT_MODULE_POLYNOMIAL.md` and its source-bound certificates.

3.6. Module ranks and dimension-sixty Morita transport.

Proposition 3.6. *The ring $R_{8_3_307}$ admits no complex categorification.*

Proof. Its positive dimensions, in source order, are $(1, \sqrt{10}, \sqrt{10}, 2\sqrt{10}, 5, 3, 3, 4)$. The integral basis $\{0, 4, 5, 6, 7\}$ is the neutral part of a faithful C_2 -grading and has dimension 60. It has no nontrivial proper fusion subring. Any neutral category $\mathcal{D}$ is therefore simple and integral, so [3, Theorem 9.12] gives $\mathcal{D} \simeq \text{Rep}(A_5)$.

The odd component is an invertible $\mathcal{D}$ -bimodule of rank three. By the subgroup/projective-representation classification of $\text{Rep}(A_5)$ -module categories [75, Theorem 2 and Remark 12], its objects have projective degrees $a(1, 1, 2)$: the exact action matrices have a one-dimensional simultaneous dimension eigenspace with primitive vector $(1, 1, 2)$. The certificate checks both its eigenvector equations and rank two of the combined coefficient matrix. For the defining subgroup $H \leq A_5$, $|H| = 6a^2 \mid 60$, whence $a = 1$ and $H \cong S_3$. A one-dimensional projective representation trivializes the multiplier.

The dual category has one family of simples for each H -double coset, parametrized by irreducible representations of its stabilizer. The three double cosets have sizes 6, 18, 36 and stabilizers $S_3, C_2, 1$. Its rank is consequently $3 + 2 + 1 = 6$. Invertibility requires its dual to be $\mathcal{D}$, of rank five, a contradiction. The finite certificate checks every order-six embedding through Sylow-three normalizers and the complete double-coset calculation. $\square$

Proposition 3.7 (Pointed transport from dimension sixty). *Let $\mathcal{C}$ be faithfully graded by a finite group, with a nonsolvable neutral category of dimension 60. Then $\mathcal{C}$ is group-theoretical and integral. In particular $R_{8_3_2132}$ has no complex categorification.*

Proof. By [3, Theorem 9.16], the neutral category is weakly group-theoretical. Choose a nilpotent Morita representative. The orders of its extension groups multiply to 60. If all these groups were solvable, Morita invariance and [3, Proposition 4.5] would imply solvability of the neutral category. Every group whose order is a proper divisor of 60 is solvable, while the only nonsolvable group of order 60 is A_5 . The representative must therefore be $\text{Vec}_{A_5}^\omega$, with arbitrary admissible associator ω .

Transport the entire grading under this neutral Morita equivalence, using [3, Lemma 3.4]. Each component becomes an invertible bimodule over $\text{Vec}_{A_5}^\omega$, whose one-sided dual is pointed. The pointed-dual criterion [79, Theorem 3.4] requires the subgroup defining the module category to be abelian and normal in A_5 , hence trivial. Each component therefore has 60 simples and dimension 60, so all its simples are invertible. The transported category is pointed; the original category is group-theoretical and thus integral. No assumption $\omega = 1$ has been made.

For $R_{8_3_2132}$ the dimensions are $(1, 1, \sqrt{30}, 5, 5, \sqrt{30}, 2, 2)$. Its integral neutral category $\mathcal{D}$ has dimension 60, equals its adjoint subcategory, and has two invertibles. If it were solvable, its last cyclic-prime construction step could not be an extension, since $\mathcal{D}_{\text{ad}} = \mathcal{D}$. It would be a C_2 -equivariantization, by [3, Proposition 4.4]. Multiplication by the nontrivial character has neutral orbits $\{0, 1\}, \{3, 4\}, \{6\}, \{7\}$. A fixed underlying simple yields two equivariant simples interchanged by that character; a free underlying orbit yields one character-fixed simple of twice its dimension. De-equivariantization would have dimensions 1, 1, 1, 1, 1, 5, the near-group ring $\text{NG}(C_5, 4)$. Its whole tensor matches the already excluded $R_{6_4_19}$; the dimension-thirty Morita certificate permits only simple dimensions one and two. Thus $\mathcal{D}$ is nonsolvable. The transport result makes the whole category integral, contradicting its odd dimension $\sqrt{30}$. $\square$

The source-bound replay checks all dimension and grading equations, adjoint closure, character orbits, the complete lower-rank tensor match and the finite normal-subgroup premise. This is a new application and unification of the dimension-sixty dichotomy and integrality reductions; it does not identify an arbitrary categorification from its ring alone.

Theorem 3.8 (Dimension-80 ring rigidity). *Every complex fusion category of type $(1, 5; 5, 3)$ has the based ring of $\text{Rep}((C_2)^4 \rtimes C_5)$ for the fixed-point-free action. Its maximum fusion multiplicity is two. Hence $R8_3_275$, $R8_3_276$, $R8_3_344$, and $R8_3_345$ are not categorifiable.*

Proof. The category is solvable by [3, Theorem 1.6]. Use its cyclic-prime extension/equivariantization sequence [3, Definition 1.2(ii), Proposition 4.4]. A final extension would have neutral dimension 40 or 16, neither of the form $a + 25b$ with $a \in \{1, 5\}$ and $0 \leq b \leq 3$. Thus $\mathcal{C} = \mathcal{D}^{C_5}$, since its invertibles force the equivariantization prime to be five, and $\text{FPdim } \mathcal{D} = 16$.

A fixed base simple of dimension d contributes five simples of dimension d , and a free orbit contributes one of dimension $5d$; cyclic stabilizers have trivial projective multipliers. Base dimensions are rational algebraic integers, hence integral. A fixed dimension-five simple is impossible since $25 > 16$, and all free-orbit simples must be invertible. Thus $\mathcal{D} = \text{Vec}_H^\omega$, $|H| = 16$, with only the unit fixed by C_5 . Every invariant subset containing the identity has size one modulo five. The nontrivial characteristic center of the 2-group H therefore has order sixteen; then its characteristic subgroup $H[2]$ also has order sixteen. Hence $H = (C_2)^4$. The order-five action has irreducible minimal polynomial $\Phi_5(t) = t^4 + t^3 + t^2 + t + 1$ over $\mathbb{F}_2$, so it is unique up to conjugacy.

The five unit characters act trivially on the three free-orbit simples X_O , and $X_O^* = X_O$. Reciprocity gives $[X_O X_P : \chi] = \delta_{O,P}$ for every unit character χ . For a nonzero orbit Q , the coefficient at X_Q is the number of pairs $(x, y) \in O \times P$ with $x + y = z$, for any fixed $z \in Q$. This follows by forgetting equivariance and is independent of the associator and action coherence. The exact three-orbit convolution certificate has maximum two, whereas the four source tensors have maximum three. $\square$

Theorem 3.9 (A dimension-216 solvability obstruction). *No complex fusion category has type $(1, 2; 2, 1; 3, 2; 8, 3)$. In particular, $R8_3_810$, $R8_3_812$, $R8_3_2492$, $R8_3_2494$, $R8_3_2620$, and $R8_3_2622$ are not categorifiable.*

Proof. Use the same solvability sequence. A final extension would have neutral dimension 108 or 72, neither expressible as $a + 4b + 9c + 64d$ with $a \in \{1, 2\}$, $b \leq 1$, $c \leq 2$, $d \leq 3$. Thus $\mathcal{C} = \mathcal{D}^{C_2}$, with base dimension 108. The cyclic orbit rule forces precisely the base types

$$A = (1, 3; 3, 1; 4, 6), \quad B = (1, 3; 3, 1; 4, 2; 8, 1).$$

Integrality descends by the orbit dimension formula and rational algebraic integrality. Neither base type permits subcategory dimension 54 or 36: enumerate $a + 9b + 16c + 64d$, $a \in \{1, 3\}$, within the displayed bounds. Hence $\mathcal{D} = \mathcal{E}^{C_3}$, with base dimension 36. Type B is impossible because its two dimension-four simples would have to arise from fixed objects in multiples of three. Type A forces $\mathcal{E}$ of type $(1, 4; 4, 2)$, and the actual C_3 action cycles its three nonidentity invertibles. Their group is therefore V_4 .

For a dimension-four simple X , reciprocity gives $16 = |\text{Stab}(X)| + 4m$. The stabilizer order divides four, so it is four. Thus all invertibles and a noninvertible belong to $\mathcal{E}_{\text{ad}}$, whose dimension is 20 or 36. The ratio $\text{FPdim}(\mathcal{E})/\text{FPdim}(\mathcal{E}_{\text{ad}}) = |U(\mathcal{E})|$ is integral; therefore $U(\mathcal{E}) = 1$. The final solvability step is consequently a C_2 equivariantization, giving a central Tannakian lift of one nontrivial invertible. Transport it under the actual C_3 action to obtain bosonic central lifts of all three nontrivial invertibles.

The kernel of $\text{Inv } Z(\mathcal{E}) \rightarrow \text{Inv } \mathcal{E}$ is $\text{Aut}_\otimes(\text{Id}_\mathcal{E}) = \text{Hom}(U(\mathcal{E}), \mathbb{C}^\times) = 1$. Hence these lifts form a closed V_4 , and its quadratic self-braiding form is identically one. This pointed braided subcategory is Tannakian. De-equivariantization gives $\mathcal{E} = \mathcal{F}^{V_4}$, where $\text{FPdim } \mathcal{F} = 9$ and integrality again descends. The integral prime-square theorem [21, 22] makes $\mathcal{F} = \text{Vec}_H^\omega$, with $H = C_9$ or C_3^2 .

An orbit of length two has cyclic stabilizer and produces forbidden dimension-two simples. A nonidentity fixed point has stabilizer V_4 ; its projective degrees are either 1, 1, 1, 1 or 2. These would produce four extra invertibles, beyond the four from the unit, or a forbidden dimension-two simple. Thus all nonidentity elements of H have free V_4 orbits. Each nontrivial involution therefore acts fixed-point-freely. An odd abelian group has only one fixed-point-free involution, namely inversion: $(f - 1)(f + 1) = 0$ and $\ker(f - 1) = 0$ imply $f = -1$. Three nonidentity elements of V_4 cannot all act by inversion. This contradiction proves the theorem. $\square$

Theorem 3.10 (A split-module obstruction in dimension 280). *The based rings $R8_3_1051$ and $R8_3_2415$ are not categorifiable over $\mathbb{C}$.*

Proof. The exact source certificate puts both rings in dimensions $[1, 1, 5, 5, 6, 8, 8, 8]$, with the nontrivial invertible g swapping the unit pair and the dimension-five pair and fixing the remaining four objects U_i . For a self-dual dimension-five object X , its coefficient block among the U_i is

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and the source has the exact based-ring identity $p(X) = 0$, where $p(t) = t(t - 5)(t - 1)(t + 1)(t + 2)$.

The fixed dimension-six simple supplies a rank-one module category for $\text{Vec}_{\mathbb{C}_2}^\omega$, so ω is a coboundary and the regular algebra $A = 1 + g$ exists. In the category of right A -modules, the two free g -orbits give two simples, and each fixed U_i splits into $U_{i,+}, U_{i,-}$; the stabilizer multiplier vanishes since $H^2(C_2, \mathbb{C}^\times) = 0$.

The action of self-dual X is symmetric. Induction is a module functor, so its block between two split pairs has equal row and column sums R_{ij} . That block is $\begin{pmatrix} u & R_{ij}-u \\ R_{ij}-u & u \end{pmatrix}$, with $0 \leq u \leq R_{ij}$. Coefficients connecting a free simple to the two members of a split pair agree, since the free simple is induced and induction sends a fixed object to the sum of its two summands. Thus the four pair differences form an invariant subspace.

After independently interchanging the two summands of each pair, the matrix on that subspace must be

$$S = \begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 0 & b & c \\ 1 & b & 0 & d \\ 1 & c & d & e \end{pmatrix}, \quad a, c, d, e \in \{-1, 1\}, \quad b \in \{-2, 0, 2\}.$$

Every module action must satisfy the same based-ring identity, hence $p(S) = 0$. The certificate evaluates the full polynomial over the integers for all 48 possibilities and records a nonzero entry in each case. It also verifies $p(N_X) = 0$ in each source regular representation, which is faithful. This gives the required contradiction without a unitarity assumption. $\square$

Theorem 3.11 (Dimension-84 adjoint rigidity). *No complex fusion category has dimension type $(1, 4; 2, 2; 6, 2)$ and $\mathcal{C}_{\text{ad}} = \mathcal{C}$. Consequently the five tensors $R8_3_76$, $R8_3_735$, $R8_3_738$, $R8_3_1422$, and $R8_3_1805$ are not categorifiable.*

Proof. By [77, Theorem 1.2], a category of dimension 84 is solvable or group-theoretical. Every group of order 84 is solvable: its Sylow 7 subgroup is normal and its quotient has order 12. Solvability is Morita invariant, so either alternative makes $\mathcal{C}$ solvable. Its last cyclic-prime construction step [3, Definition 1.2, Proposition 4.4] cannot be an extension because $\mathcal{C}_{\text{ad}} = \mathcal{C}$. It is therefore an equivariantization $\mathcal{C} = \mathcal{D}^{C_2}$: the prime divides the number four of invertibles. The base has dimension 42 and is group-theoretical by [3, Theorem 9.2]. It is integral, since the orbit dimension formula makes its simple dimensions rational algebraic integers. Exactly two base invertibles are

fixed by C_2 . Let g be the sign object of the equivariantization. If n_2, n_6 count its fixed simples of dimensions two and six, respectively, then $n_2, n_6 \in \{0, 2\}$ and the base type is

$$(1, 2 + 2n_2; 2, (2 - n_2)/2; 3, 2n_6; 6, (2 - n_6)/2).$$

Thus the four possible multisets of base dimensions are $\{1, 1, 2, 6\}$, $\{1^6, 6\}$, $\{1, 1, 2, 3^4\}$, and $\{1^6, 3^4\}$. For completeness, all groups of order 42 are $C_7 \rtimes C_6$ with action image of order 1, 2, 3, 6, and $C_7 \rtimes S_3$ with action image of order 1, 2. Exhausting their subgroups and double cosets gives the following complete list of group-theoretical bases of the four types. The first and third types never occur. Write $F_{21} = C_7 \rtimes C_3$ and $F_{42} = C_7 \rtimes C_6$ for the faithful semidirect products. Type $\{1^6, 6\}$ occurs only for $(G, H) = (F_{42}, C_6)$ or (F_{42}, F_{42}) . Type $\{1^6, 3^4\}$ occurs only for $G = F_{21} \times C_2$, with $|H| = 3, 6, 21, 42$, or $G = F_{42}$, with $|H| = 3, 21$, up to conjugacy. This finite audit includes arbitrary associators: every double-coset stabilizer has squarefree order and trivial Schur multiplier, by restriction/corestriction to its cyclic Sylow subgroups. Its projective character degrees are therefore ordinary character degrees. The exact subgroup and double-coset enumeration is part of the certificate. In the second type the category is $\text{Rep}(F_{42})$. For $H = C_6$, the restriction $H^3(F_{42}, \mathbb{C}^\times) \rightarrow H^3(C_6, \mathbb{C}^\times)$ is an isomorphism: the remaining summand $H^3(C_7, \mathbb{C}^\times)^{C_6}$ vanishes because the faithful multiplier acts by its square. Thus the required trivial restriction kills the associator, and the split exact-factorization construction is $\text{Rep}(F_{42})$; compare [64, Corollary 7.4]. In the fourth type the canonical adjoint subcategory is $\text{Rep}(F_{21})$. For $H = C_3$ or F_{21} , support in the normal index-two subgroup F_{21} gives a faithful C_2 grading whose neutral category has type $(1, 3; 3, 2)$. It is $\text{Rep}(F_{21})$ by [64, Theorem 6.3]. Since F_{21} is centerless, this neutral category equals its own adjoint, forcing equality with the adjoint of the whole category. For $G = F_{21} \times C_2$ and $H = C_6$ or G , coprime cohomology decomposition instead gives $\text{Rep}(F_{21}) \boxtimes \text{Rep}(C_2)$, with the same adjoint. These reductions do not assume a trivial ambient associator. A representation category of a squarefree-order group has a unique fiber functor: the subgroup supporting a nondegenerate projective representation must have square order [75, Section 3.4, Theorem 2 and Example (i)]. Hence every tensor autoequivalence of $\text{Rep}(F_{21})$ or $\text{Rep}(F_{42})$ comes from a group automorphism. In the presentation $a^7 = b^m = 1$, $bab^{-1} = a^\lambda$, with $m = 3$ or 6 and λ of order m , an automorphism $a \mapsto a^u$, $b \mapsto a^v b^w$ forces $\lambda^w = \lambda$. It therefore fixes the abelianization and every linear character. The second base type consequently has six fixed invertibles; in the fourth type the canonical adjoint supplies at least three fixed invertibles. Both contradict the required number two. For each displayed source tensor, the certificate verifies all dimension equations, full adjoint closure, and every possible sign permutation directly in the given basis. $\square$

Remark 3.12 (Earlier dimension-660 screening). The ring `R8_3_731` is integral of dimension 660 and has no nontrivial proper closed based subring. Its exact primary-three witness excludes unitary categorification. By [3, Proposition 9.11], a hypothetical weakly group-theoretical categorification would be a representation category of a nonabelian finite simple group, hence unitary, a contradiction. Thus any categorification of this ring would be nonunitary and not weakly group-theoretical. Independently, the ordinary representation ring scan finds no matching tensor among the 40 groups of order 660. These were the original restricted tests. The present continuation excludes general complex categorification as well; see Section 3.21.

Theorem 3.13 (A dimension-90 solvability obstruction). *A fusion ring of type $(1, 2; 2, 4; 6, 2)$ is not categorifiable if its adjoint subring is the whole ring and its nonidentity invertible fixes every noninvertible basis element. In particular, `R8_3_2904` and `R8_3_3082` admit no complex fusion categorification.*

Proof. Every category of dimension 90 is solvable [77, Corollary 4.16]. Its final cyclic-prime step [3, Proposition 4.4] cannot be an extension, since the universal grading is trivial. Two invertibles force this step to be a C_2 equivariantization, say $\mathcal{C} = \mathcal{D}^{C_2}$. The two lifts of a fixed simple of $\mathcal{D}$ are

exchanged by the sign character. Thus each of the six sign-fixed noninvertibles of $\mathcal{C}$ comes from a free orbit. Halving their dimensions gives $\mathcal{D}$ of type $(1, 9; 3, 4)$ and dimension 45.

By [45, Theorem 1.1], an integral category of dimension $45 = 5 \cdot 3^2$ is group-theoretical: neither exceptional alternative applies. In fact such a category is pointed. All groups of order 45 have normal Sylow subgroups, so are C_{45} or $C_3^2 \times C_5$. In the presentation $\mathcal{C}(G, \omega, H, \psi)$, every stabilizer is H because G is abelian. A cyclic H has trivial Schur multiplier. A noncyclic H contains C_3^2 and has index 1 or 5. Restriction is then injective on the 3-primary component of $H^3(G, \mathbb{C}^\times)$, by restriction-corestriction. Since $[\omega]|_H = 0$, the remaining class is 5-primary, whereas $H^2(H, \mathbb{C}^\times)$ is 3-primary. The stabilizer-cocycle construction is multiplicative in ω , and the two ψ contributions cancel in cohomology because conjugation is trivial; see [46, Section 5.1]. Consequently every stabilizer multiplier is trivial. Its irreducible projective representations are therefore one-dimensional, and the double-coset index is one. The category is pointed, contradicting the four dimension-three simples forced above.

The exact receipt checks both source tensors, all dimension-character identities, the complete adjoint closure and sign actions. It is independent of the dimension-order linear-programming certificates. $\square$

3.7. When a unitary obstruction is a general obstruction. The negative primary-three form is ordinarily only an obstruction to unitary categorification. We use it for a general exclusion only after establishing that the hypothetical category is weakly group-theoretical. This reduction uses exact integer dimension below 84, solvability for dimensions with at most two prime factors, or a faithful grading with neutral category in that range; weak group-theoreticity is closed under extensions [3, Proposition 4.1 and the small-dimension theorems]. Every weakly group-theoretical category is completely unitary [39, Theorem 5.20]. The primary-three contradiction then applies to its unitary realization and excludes general existence.

The source-bound verifier checks the dimension hypotheses, every nonzero structure constant of the grading, the neutral dimension and the grading dimension ratio before accepting this implication. In particular it uses no assertion that arbitrary weakly integral categories are unitarizable. When the reduction fails, the same negative form retains its unitary-only scope and does not change a general unresolved status.

3.8. Source-bound identifications with known constructions. The new positive inventory includes tensor products of previously constructed unitary categories and representation categories of finite groups. Each entry is checked by a unit-preserving basis bijection against all $8^3 = 512$ source coefficients, including noncommutative products. The certificate specifies the direction of the basis map. Equality of dimension vectors alone is never used as a model identification.

For the Frobenius-group entry R8_3_342, the finite orbit-convolution calculation constructs $\text{Rep}(C_{13} \rtimes C_6)$ with its faithful action. The $\text{PSL}_2(11)$ entry R8_3_724 is reconstructed from its exact finite-group class algebra. The E_8 module-category entry R8_3_3278 uses the standard unitary E_8 algebra and its module fusion rules [29]. The remaining existing near-group, Fibonacci and E_8 product entries use the unitary $\text{NG}(C_3, 3)$ construction [78, Proposition A.4.1] or the adjoint of the unitary E_8 construction [29], followed by the indicated Deligne product. Their complete tensor comparisons are in the model catalogue. These comparisons establish the based-ring identification; categorical existence uses the cited construction and its stated scope.

3.9. Unitary scope of the Izumi–Xu extension. The fusion rules of R8_3_448 are those of [78, Appendix A.6.1]; in particular $Vg = g^2V$ is essential. The unitary construction is supplied by the symmetrically self-dual 2221 planar algebra of [82, Section 5.1, Theorem 5.5]. The extension classification in [83, Theorem 4.9, Remarks 4.10–4.11] identifies the resulting graded extension with the displayed fusion rules: the two associator choices have the same tensor. Thus the earlier conservative pseudo-unitary label is upgraded to unitary existence.

3.10. Two equivariantizations of a near-group category. Use the normalized unitary $C_3 + 6$ solution of [36, Example 13.10], with $(x, y) = (\sqrt{\sqrt{3}/24}, 0)$. Lemma 13.1 describes actual automorphisms $\gamma_{\omega, \theta, u}$, with (ω, θ, u) and $(-\omega, \theta, -u)$ identical. The paragraph before Lemma 13.2 gives the strict conjugation tensor functor. Take $\theta = \text{id}$ and either $(\omega, u) = (1, \text{diag}(1, -1))$ or $(i, R_{\pi/2})$. Both have order two; the latter squares to $\gamma_{-1, \text{id}, -I} = 1$. Their actions on the six-dimensional self-fusion space have three eigenvalues of each sign. Their actions on the maps S_g have signs $+1$ and -1 , respectively. Hence the complete rules are

$$(g, e)(h, f) = (g+h, e+f), \quad (g, e)X_f = X_f(g, e) = X_{e+f}, \quad X_e X_f = \sum_{g \in C_3} (g, e+f+\epsilon) + 3X_0 + 3X_1.$$

The cases $\epsilon = 0, 1$ identify $R8_3_334, R8_3_274$ by all 512 coefficients. These are coherent unitary actions, not abstract based-ring automorphisms.

3.11. Cyclic permutation of three Fibonacci factors. The canonical C_3 action on $\text{Fib}^{\boxtimes 3}$ has two fixed simple triples, each splitting into three equivariant simples, and two free orbits. For a fixed output, the multiplicity is the character average on its fusion space; a free input orbit has zero nonidentity trace. For a free output the coefficient is the ordinary multiplicity at a representative. These formulas reconstruct all 512 coefficients of $R8_3_475$ and give its unitary realization.

3.12. A Morita dual of the nonsplit 2D2 extension. In the known nonsplit extension $R8_2_37$, take the Q-system $A = 1 + r$ from [55, Lemma 4.1]. The neutral component of ${}_A \mathcal{E}_A$ is TT_3 , with basis $1, p, q, pq, qp, pqp$ and relations $p^2 = 1 + p$, $q^2 = 1 + q$, $pqp = qpq$; see [76, Definition 2.5]. The odd free-bimodule Gram matrix, in the order (a, a^3, ar, a^3r) , is

$$\begin{pmatrix} 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 3 & 3 & 13 & 13 \\ 3 & 3 & 13 & 13 \end{pmatrix}.$$

Its nonnegative integral factorizations have the column $(1, 1, 3, 3)^T$ and either $(0, 0, 2, 2)^T$ or four copies of $(0, 0, 1, 1)^T$. After dividing underlying dimensions by $\text{FPdim}(A)$, the latter would give four odd simples of dimension φ , but multiplication by p requires the unavailable dimension φ^2 . The two remaining odd dimensions are therefore $2\varphi, 2\varphi^2$. Left and right multiplication by either p or q is forced to be $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Reciprocity determines odd-odd products and yields precisely $R8_3_715$.

3.13. The conformal G_2 algebra Morita dual. Let $\mathcal{C} = \mathcal{C}(G_2, 3)$ and let A be the conformal algebra for $G_{2,3} \subset E_{6,1}$. The category ${}_A \mathcal{C}_A$ is unitary by [81, Theorems 2.21, 2.28, Corollary 3.32]. Write $d = (3 + \sqrt{21})/2$. The two chiral Izumi–Xu subcategories have pointed local intersection C_3 , by [78, Appendix A.4] and [80, Section 3.3, equation (16), Definition 3.12, Remark 3.13]. Their noninvertibles X, Y satisfy $X^2 = 1 + g + g^2 + 3X$ and $Y^2 = 1 + g + g^2 + 3Y$. Rigidity gives $(XY, XY) = 3$; dimension equality forces three further simples Z_i , each of dimension $d + 1$.

The g -action on this triple is nontrivial: otherwise the symmetric integral multiplicity block B has row sums three and satisfies $B^2 + J = 3I + 3B$. Its trace is six, so its off-diagonal sum would be three, a parity contradiction. Write $Z_i = g^i Z_0$ with Z_0 self-dual. Associativity gives $XY = YX = \sum_i Z_i$, $XZ_i = Z_i X = Y + \sum_j Z_j$, and $YZ_i = Z_i Y = X + \sum_j Z_j$. The commutative orientation is impossible: the modular center has only self-dual simples, and [21, Lemma 8.49] would force all simples of the commutative based ring to be self-dual, contrary to $g^* = g^2$. Hence $Z_i g = g^{-1} Z_i$. Reciprocity reduces the remaining products to ten nonnegative triples with sum three; exact associativity leaves only $(1, 1, 1)$, so

$$Z_i Z_j = g^{i-j} + X + Y + \sum_k Z_k.$$

The reconstructed full tensor is $R8_3_458$.

3.14. A twisted group-theoretical construction.

Theorem 3.14 (A twisted A_6/A_5 model). *There are μ_2 -valued cochains $\omega \in Z^3(A_6, \mu_2)$ and $\psi \in C^2(A_5, \mu_2)$ with $d\psi = \omega|_{A_5}$ such that the unitary group-theoretical category $\mathcal{C}(A_6, \omega, A_5, \psi)$ realizes $R8_3_3342$. Its simple dimensions are 1, 3, 3, 4, 5, 10, 10, 10.*

Proof. We give an exact finite certificate for the relative cocycle and then derive the fusion rules. Let $P = \langle r, s \rangle \cong D_8$ in A_6 , on letters $0, \dots, 5$, with

$$r = (0123)(45), \quad s = (02)(45).$$

Write $r^i s^j = (i, j)$, and set $E = \{(i, j) : i \equiv j \pmod{2}\}$, $K = \{(i, j) : i \equiv 0 \pmod{2}\}$, and $t = s$. Then E fixes both 4 and 5, while K is the other Klein four subgroup. On K , put $x(i, j) = i/2$, $y(i, j) = j$, with values in $\mathbb{F}_2$. The normalized bar equations

$$dw = 0, \quad w|_E = 0, \quad w|_K = x^2 y + xy^2 + dv$$

have an explicitly saved solution: there are 352 variables, 2455 equations, and rank 312.

For each of the 360 elements $g \in A_6$, the certificate supplies a normalized 2-cochain witnessing equality, modulo coboundaries, of w and its g -conjugate on $P \cap gPg^{-1}$. The intersection orders 1, 2, 4, 8 occur 256, 64, 32, 8 times, respectively. Thus the Cartan–Eilenberg stable-elements theorem [84, Chapter XII, Section 10] supplies a class $c \in H^3(A_6, \mathbb{F}_2)$ restricting to $[w]$. For $H = A_5$ fixing 5, E is a Sylow 2-subgroup of odd index 15; restriction and transfer give injectivity of $H^3(H, \mathbb{F}_2)$ into $H^3(E, \mathbb{F}_2)$. Hence $c|_H = 0$. Choose a normalized trivializing 2-cochain on H , extend it to A_6 , and gauge the global cocycle to vanish identically on H . Exponentiation by $a \mapsto (-1)^a$ gives $\omega|_H = 1$, so we may take $\psi = 1$.

The off-diagonal stabilizer is $L = H \cap tHt^{-1} = A_4$. The multiplier convention of [46, Section 5.1], restricted to the involutions $a, b \in E$, has additive form

$$\alpha(a, b) = w(a, b, t) + w(a, bt, t^{-1}bt) + w(abt, t^{-1}bt, t^{-1}at).$$

For $u = r^2$ and $v = rs$, the exact affine certificate proves $\alpha(u, v) + \alpha(v, u) = 1$: it gives a sum of original equations equal to this functional plus 1. The identity therefore holds for every solution of the displayed affine system, including the restriction of the globally gauged cocycle. This last restriction need only be cohomologous to the saved local representative. Consequently the multiplier has commutator -1 on a commuting pair, so its class in $H^2(A_4, \mathbb{C}^\times)$ is nontrivial. This relative computation is essential: the nonzero Dickson cubic itself maps to zero in $H^3(K, \mathbb{C}^\times)$.

There are two double cosets. The diagonal sector is $\text{Rep}(A_5)$. The other sector has the three spin representations of the binary tetrahedral group $2T$:

$$\rho_i = \rho_0 \otimes \chi^i, \quad i \in \mathbb{Z}/3, \quad \det \rho_i = \chi^{2i},$$

where ρ_0 is the defining $SU(2)$ representation and χ generates the character group of A_4 . The non-trivial μ_2 extension is unique: $H^2(A_4, \mathbb{F}_2)$ is the one-dimensional C_3 -invariant part of $H^2(V_4, \mathbb{F}_2)$. The three spin representations have squared dimensions summing to 12, so they exhaust the projective simples. Their corresponding objects X_i have dimension $[A_5 : A_4] \dim \rho_i = 10$.

Every X_i is self-dual. Reversing the endpoints conjugates A_4 by an odd permutation and therefore inverts its character group. Representation duality also inverts determinants, so the two inversions cancel. All remaining transport scalars lie in μ_2 and hence do not change two-dimensional determinants. The three distinct determinants distinguish the X_i . The five diagonal simples are also self-dual; hence all fusion coefficients are symmetric in their three indices.

At distinct endpoints, composing two off-diagonal kernels sums over four intermediate points, a transitive A_4 -set with stabilizer C_3 . Thus the off-sector product is governed by induction from this C_3 . There is a unique μ_2 trivialization of a multiplier on C_3 , since $H^1(C_3, \mu_2) = H^2(C_3, \mu_2) = 0$. The order-three lift in $2T$ has eigenvalues ζ, ζ^2 on ρ_0 , so $\rho_0|_{C_3} = \chi + \chi^2$. This is preserved by either automorphism of C_3 ; a residual μ_2 transport character is necessarily trivial. Equivalently,

transport preserves the trivial determinant, whereas a nontrivial C_3 character twist would change it. Therefore

$$N_{X_0 X_0}^{X_0} = \langle \chi + \chi^2, (\chi + \chi^2)^2 \rangle_{C_3} = 2.$$

For any pair i, j , the tensor of their two-character restrictions has multiplicities $(2, 1, 1)$ in some order. Pairing with the three spin restrictions gives off-sector multiplicities $(2, 3, 3)$ in some order. Symmetry and the anchor $N_{X_0 X_0}^{X_0} = 2$ have a unique completion (enumerated among the 3^6 choices for unordered pairs):

$$N_{X_i X_j}^{X_k} = 3 - \delta_{i+j+k,0} \pmod{3}.$$

The restrictions of the five diagonal simples to A_4 are $1, 3, 3, 1 + 3, \chi + \chi^2 + 3$. As $3 \otimes \rho_i = \rho_0 + \rho_1 + \rho_2$, their actions on the three off-sector objects are $I, J, J, I + J, 2J - I$, where J is the all-ones matrix. Reciprocity supplies the diagonal coefficients of off/off products. The remaining tensor is the ordinary A_5 character tensor. This determines every coefficient.

The verifier constructs that tensor independently, checks the dimension and associativity equations, and matches all 512 coefficients to the authoritative source under the basis map $[0, 1, 2, 6, 7, 3, 4, 5]$ from model to source, whose source-to-model inverse is $[0, 1, 2, 5, 6, 7, 3, 4]$. All source hashes are recomputed. Finally, unit-modulus cocycles make the pointed category and its subgroup bimodule construction unitary. $\square$

Remark 3.15. The comparison tensor `R8_3_3344` has $N_{XX}^X = 3$ for each dimension-ten simple. Thus it is not the ring of this particular construction. This comparison is not a general obstruction to other constructions of that tensor.

3.15. A Q-system realization of `R8_3_721`. The earlier center calculations left this ring undecided. We now construct an actual unitary fusion category and identify its Grothendieck ring within the supplied complete census. The existence argument uses an inherited Q-system; agreement of modular data is not used to manufacture an algebra.

Theorem 3.16 (A cyclic generalized-Haagerup model). *Let $\mathcal{C}$ be the C_2 de-equivariantization of the unitary C_8 generalized-Haagerup category in [101, Example 4.4 and Section 6]. Let $A = \mathbf{1} \oplus \rho$ be the image of its Q-system from [101, equations (2.21)–(2.22)]. Then $\mathcal{D} = {}_A \mathcal{C}_A$ is a unitary fusion category of rank eight, with trivial group of invertibles and FP dimensions*

$$1, a, a, a + 1, a + 1, 2a + 1, 2a, 2a + 2, \quad a = \frac{3 + \sqrt{17}}{2}.$$

All its fusion coefficients are at most three. The exact completion of the mandatory induction equations below identifies $K_0(\mathcal{D}) \cong \text{R8_3_721}$.

We separate the categorical construction, the finite coefficient bound, and the exact fusion completion. The census is used only at the end to name the uniquely generated based-ring class.

Lemma 3.17 (An inherited-vertex obstruction). *Suppose an existing unitary generalized-Haagerup category for C_{4m} satisfies the de-equivariantization hypotheses of [101, Section 6], including $\epsilon_{2m}(h) = (-1)^h$, and has the Q-system $\mathbf{1} \oplus \rho$ whose nonzero $\rho \rightarrow \rho^2$ vertex is proportional to t_0 . In its C_2 de-equivariantization, the Morita dual of this inherited algebra has no nontrivial invertible object.*

Proof. Write g for the generator of the quotient group C_{2m} and $d = 2m + \sqrt{4m^2 + 1}$. An invertible bimodule has underlying FP dimension $1 + d$. Since the source simple dimensions are 1 and the irrational number d , its underlying object has one invertible and one noninvertible summand. If its invertible summand is g^j , free-module adjunction and equality of dimensions identify its underlying left and right modules with Ag^j and $g^j A$, respectively. Their noninvertible summands are $g^{-j}\rho$ and $g^j\rho$, so $2j = 0$ in C_{2m} .

For $j = 0$, the bimodule is an automorphism-twisted copy of A . A unit-preserving algebra automorphism acts by a scalar z on ρ . The nonzero $\rho^2 \rightarrow \mathbf{1}$ and $\rho^2 \rightarrow \rho$ multiplication components impose $z^2 = 1$ and $z^2 = z$, hence $z = 1$.

For $j = m$, use the crossed-product realization of [101, Section 6]. Its implementing unitary λ satisfies

$$\lambda^2 = 1, \quad \rho'(\lambda) = \lambda, \quad \alpha'_{2m} = \text{Ad}(\lambda), \quad \alpha'_m(t_0) = t_{2m}.$$

Conjugation by α'_m sends ρ' to $\text{Ad}(\lambda)\rho'$. The intertwiner $u = \lambda : \rho' \rightarrow \text{Ad}(\lambda)\rho'$ has $u \otimes u = 1$. Transporting the conjugate algebra to the object $\mathbf{1} \oplus \rho'$ changes its vertex line from $\mathbb{C}t_0$ to $\mathbb{C}t_{2m}\lambda$. These lines are orthogonal by the Cuntz relations. A unit-preserving object isomorphism acts scalarly on the simple ρ' summand and cannot identify the two vertex lines. Thus $g^m A g^{-m} \not\cong A$ as algebras. An invertible bimodule with underlying free module $g^m A$ would give precisely such an algebra isomorphism, contradicting this calculation. $\square$

This lemma is conditional on existence of the stated generalized-Haagerup category and Q-system. It makes no existence assertion for arbitrary m .

Proof of Theorem 3.16. In the case $m = 2$, the source category and Q-system are supplied by the cited construction. The strong tensor de-equivariantization functor preserves the Q-system, and its image ρ is simple. Hence $A = \mathbf{1} \oplus \rho$ is connected and separable, and its bimodule category exists and is unitary. The source fusion rules are

$$g^4 = 1, \quad g\rho = \rho g^{-1}, \quad \rho^2 = \mathbf{1} + 2 \sum_{i=0}^3 g^i \rho, \quad \text{FPdim}(\rho) = 4 + \sqrt{17}.$$

Morita equivalence gives $\text{FPdim}(\mathcal{D}) = 136 + 32\sqrt{17}$.

For free bimodule induction $I(X) = A \otimes X \otimes A$, adjunction gives

$$K_{ij} = \dim \text{Hom}_{\mathcal{D}}(I(X_i), I(X_j)) = [AX_j A : X_i].$$

Every simple bimodule occurs in some $I(X)$, since its action map from the free bimodule on its underlying object is a split epimorphism. The known tensor-unit row is $(1, 0, 0, 0, 1, 0, 0, 0)$ in source order $(1, g, g^2, g^3, \rho, g\rho, g^2\rho, g^3\rho)$. The remaining Gram matrix has rank four. Its complete finite factorization checks 40 pivot vectors and 14256 multiplicity choices; it yields seven row types and exactly four factors, of ranks 6, 8, 9, 11.

The known center in [101, Theorem 6.8 and Example 6.10], equivalently [102, Definition 6.1 and Section 6.1.6], removes ranks nine and eleven. Indeed its twist-one support consists of six objects, and the canonical Lagrangian multiplicity vector has form $(1, p, x, y, z, w)$ on that support. The equations $S\ell = \ell$ and $T\ell = \ell$ force

$$p = 1, \quad x + y = 2, \quad z + w = 2.$$

All nine nonnegative integral choices satisfy the full S equations, and

$$\text{rank}(\mathcal{D}) = \langle \ell, \ell \rangle = 2 + x^2 + y^2 + z^2 + w^2 \in \{6, 8, 10\}.$$

Thus only the rank-six and rank-eight Gram factors remain. The rank-six factor has an extra invertible bimodule with underlying object $g^2 + g^2\rho = g^2 A$. Lemma 3.17 excludes it. The actual dual therefore has the unique rank-eight factor, and dividing its underlying dimensions by $\text{FPdim}(A)$ gives the dimensions in the theorem.

It remains to bound its unknown fusion coefficients, without assuming a bound in the construction. Let $B_{ix} = [U(x) : X_i]$ be the unique rank-eight forgetful matrix. Free induction satisfies

$$I(X) \otimes_A I(Y) \cong I(X \otimes A \otimes Y).$$

For every source pair (i, j) and dual output z , this gives

$$\sum_{x,y} B_{ix} B_{jy} N_{xy}^z = \sum_{a \in \{1, \rho\}} \sum_{u,t} N_{ia}^c N_{uj}^t B_{tz}.$$

Together with the unit, duality, Frobenius reciprocity and both rational components of the dimension character, these are 832 exact linear equations before redundant rows are removed. There are four possible dualities on the two equal-dimension pairs; they give respectively 120, 112, 112, 104 coefficient orbits. All four are retained.

Write one reduced system as $En = b$, where $n \geq 0$ consists of those fusion coefficients. For every coordinate j , the supplied rational covector y_j satisfies

$$y_j E \geq e_j^\top, \quad y_j b \leq 3.$$

Therefore $n_j \leq y_j En = y_j b \leq 3$. The certificates check every inequality in exact rational arithmetic for all four dualities; numerical optimization is used only to propose the covectors. This proves the required uniform multiplicity bound for the actual category.

Finally, we identify the actual category directly from its mandatory linear equations, without using completeness of the supplied census. After Frobenius and duality orbits are merged, the four possible dualities on the two equal-dimension pairs have 120, 112, 112, 104 coefficient variables. Impose the unit equations, both rational dimension coordinates and all 512 free-induction identities

$$\sum_{x,y} B_{ix} B_{jy} N_{xy}^z = \sum_{u,t} \sum_{a \in \{1, \rho\}} N_{ia}^c N_{uj}^t B_{tz}.$$

The coefficient bounds above apply to every orbit in every duality branch. Rational elimination leaves 22, 19, 19, 16 free coordinates, each itself an actual fusion coefficient in $\{0, 1, 2, 3\}$. Every other coefficient has form $(c + \sum_j a_j x_j)/D$ with integral c, a_j and $D > 0$. For each remaining value of x_j , interval arithmetic over the other finite domains removes it only if the resulting affine interval cannot meet $[0, 3D]$; the necessary gcd congruence checks integrality. Repeating these deductions and branching over every retained value is exhaustive. There is no solver timeout, floating point or associativity pruning. The complete search is

a -pair duality	b -pair duality	nodes	solutions
fixed	fixed	3	0
fixed	exchanged	3	0
exchanged	fixed	5	2
exchanged	exchanged	3	0

Thus the necessary linear equations alone leave two tensors. Both satisfy all 4096 associativity identities, all induction identities, dimensions, unit and Frobenius reciprocity. They differ by exchanging b_0, b_1 , so form one based-ring class. Only now do we compare with the supplied census: all 512 coefficients match R8_3_721 under the basis map $(0, 1, 2, 4, 6, 3, 5, 7)$, up to the harmless equal-dimension relabeling. This proves the claimed identification independently of census completeness. $\square$

Exact evidence and scope. The rank enumeration is `evidence/rank8_next_20261021/semion_fmion/construction/r721_bimodule_rank.py`. The coefficient system and rational certificates are in the same evidence tree's `models/` directory:

`r721_dual_fusion.py` and `r721_dual_multiplicity_bound.py`.

The solver-free exact completion and complete branch traces are in

`evidence/correctness_20261025/r721_exact_completion.py`.

An independent bounded-completion implementation agrees. The earlier dimension crosswalk is corroboration only, not a completeness premise. Normal and optimized-Python receipts, independent reconstructions, the ordered tensor and the cited Q-system argument accompany these files.

The identification proves existence, and does not classify all associators on this ring or assert that the Morita class is new.

3.16. Exact model inventory.

Census ID	Construction	Scope
R8_3_274	Phase-rotation C_2 -equivariantization of $\text{NG}(C_3, 6)$	unitary
R8_3_331	$\text{NG}(C_3, 3) \boxtimes \text{Vec}(C_2)$	unitary
R8_3_334	Reflection C_2 -equivariantization of $\text{NG}(C_3, 6)$	unitary
R8_3_342	$\text{Rep}(C_{13} \rtimes C_6)$	unitary
R8_3_448	Unitary Izumi–Xu C_2 extension (2221)	unitary
R8_3_458	Morita dual of the conformal $G_{2,3} \subset E_{6,1}$ algebra	unitary
R8_3_471	$\text{NG}(C_3, 3) \boxtimes \text{Fib}$	unitary
R8_3_475	$(\text{Fib}^{\boxtimes 3})^{C_3}$ by cyclic permutation	unitary
R8_3_715	Morita dual of the nonsplit unitary $2D2$ extension	unitary
R8_3_724	$\text{Rep}(\text{PSL}_2(11))$	unitary
R8_3_1456	$\text{Ad}(E_8) \boxtimes \text{Vec}(C_2)$	unitary
R8_3_3278	Unitary E_8 fusion category	unitary
R8_3_3314	$\text{Ad}(E_8) \boxtimes \text{Fib}$	unitary
R8_3_3342	Twisted $\mathcal{C}(A_6, \omega, A_5, \psi)$ from a stable relative μ_2 class	unitary
R8_3_721	Inherited Q-system Morita dual in the C_2 de-equivariantization of $\text{GH}(C_8)$	unitary
R8_3_982	Triality-zero component of the known E_{21} category for $SU(3)_{21} \subset (E_7)_1$ (Theorem 7.1)	unitary

Every model in this inventory admits a unitary realization. For R8_3_448, the original pseudo-unitary existence result is supplemented by the unitary 2221 planar-algebra construction above.

3.17. The remaining general questions and replay scope. There remain 4 unresolved exact-M3 tensors. The source tensors and per-case attempted criteria are recorded in `data/remaining_cases.py` and the per-case audit `evidence/research_20261024/rank8_research_audit.json`. The inventory distinguishes an exact contradiction, a necessary-condition survivor, a time limit, and an inapplicable test. Passing these conditions supplies no coherent associator; an unsuccessful construction search supplies no exclusion. Any claim of categorical novelty requires a separate comparison up to tensor equivalence.

3.17.1. Individually investigated candidates. We call the remaining rings extensively investigated exotic candidates in the operational sense: each survived the documented case-specific campaign of known and newly developed exclusion tests and model searches, including complete finite domains and specified unbounded construction families. This term asserts neither categorifiability nor novelty of a future realization. A unitary exclusion does not settle the general complex question. The last column gives ranks of nontrivial proper fusion subrings, as checked on the exact source tensors.

Census ID	Commutative	Invertibles	Unitary status	Proper subring ranks
R8_3_928	no	2	unresolved	2
R8_3_941	no	2	unresolved	2
R8_3_1154	no	2	unresolved	2
R8_3_1768	yes	2	unresolved	2

Historical search stages. The following earlier campaigns retain their original proof domains and checkpoint results. Their historical unresolved lists are superseded by Theorem 3.1; in particular the known model of R8_3_982 is established in Theorem 7.1.

Asaeda–Haagerup and recent odd-order constructions. The ring R8_3_721 has the exact Asaeda–Haagerup global dimension $136 + 32\sqrt{17}$, so a dimension mismatch does not dispose of this lead. The complete known Morita class nevertheless gives no model: AH_1, AH_2, AH_3 have ranks 6, 9, 9, respectively [85, Section 4.3, Tables 1–3], while AH_4, AH_5, AH_6 have four invertibles and maximum multiplicity two [9, Lemmas 3.4, 3.7, 3.15 and Theorem 4.21]. The source ring has one invertible and maximum multiplicity three. The audit independently reconstructs the two rank-eight AH based tensors and matches all 512 coefficients to the existing positive controls R8_2_39 and R8_2_484. Of the 277 baseline survivors, fourteen have the AH global dimension, checked using exact positive dimension characters; none is a member of this known family.

The September 2026 constructions [86, Theorem 2.1] and [87, Theorems A, 4.4 and 6.3] were also checked. For odd $N \geq 3$, write $d^2 = Nd + 1$ with $d > 0$. The HI family has global dimension $D_N = N(1 + d^2)$, and the specified near-group family has global dimension ND_N . Exact Casimir characteristic-polynomial division, supplemented by integer spectral bounds, rules out both dimensions for every baseline survivor and every odd N ; the bounds $D_N > N^3$ and $ND_N > N^4$ certify the infinite tail. This also rules out their Morita duals, since Morita equivalence preserves FP dimension. The cited general existence statements are spherical and pseudo-unitary; the stated Morita classification has additional unitary, cyclic, untwisted and Q-system hypotheses. These completed model-family checks make no general nonexistence claim for the remaining rings. The aggregate command is

```
python supplements/rank8/software/verify_m3_continuation.py
-output frontier-validation/baseline_normal.json.
```

It checks the original M3 ledger against the source, baseline criteria, continuation certificates and explicit model tensors. Its full receipt binds the exact ledger digest, manifests and proof-source hashes. Under Python optimization, inherited assertion-based checkers run in an isolated interpreter with assertions enabled. Metadata-only mode checks these identities and reconciliation without replaying the mathematics.

3.18. Further investigation of the original frontier. The previous release left 34 exact-M3 rings unresolved. The historical frontier recorded 25 further general exclusion claims; the old R436 TPE premise was withdrawn, and its general exclusion now follows from an independent spherical-equivariantization proof. The current totals also include the later accepted constructions above, leaving 4 questions. The earlier ledger and replay receipts are retained byte for byte; accepted changes are recorded in a separate overlay ledger.

3.19. A simultaneous fixed-point module obstruction.

Theorem 3.18. *The fusion rings R8_3_1203, R8_3_1350, R8_3_2842, and R8_3_2885 have no categorification over $\mathbb{C}$.*

Proof. We give a general reduction and then its exact finite verification. Let g be an invertible object of order two fixing a simple object in a categorification of a commutative fusion ring. A fixed simple gives a rank-one module category over $\text{Vec}(C_2, \omega)$, so its module associativity trivializes $[\omega]$. Thus $A = \mathbf{1} \oplus g$ is a connected separable algebra. In the category $\mathcal{C}_A$ of right modules, a two-element g -orbit induces a single simple module, whereas a fixed simple V induces two simples V_+, V_- , each with underlying object V . These statements follow from induction adjunction and semisimplicity.

The span of induced objects is invariant under every action matrix M_X , and under its transpose M_X^* by rigidity. Its orthogonal complement, with basis $V_+ - V_-$ indexed by the fixed simples,

therefore carries matrices S_X satisfying

$$S_1 = I, \quad S_{X^*} = S_X^t, \quad S_X S_Y = \sum_Z N_{XY}^Z S_Z.$$

The corresponding two-by-two module block has both row and column sums $a = N_{XV}^W$; hence it is $\begin{pmatrix} u & a-u \\ a-u & u \end{pmatrix}$. Consequently $(S_X)_{VW} \in \{-a, -a+2, \dots, a\}$. The matrix S_g is diagonal with entries ± 1 . If $gX = X$ and N_{XV}^V is odd, then $S_g S_X = S_X$ forces $(S_g)_{VV} = 1$. For the four rings in the theorem, every fixed vertex admits such an X , and therefore $S_g = I$.

The fixed indices are 4, 5, 6, 7, with g at index 3 in the first two rings and index 1 in the last two. The representation S factors through $R_{\mathbb{Q}}/(g-1)$, a six-dimensional rational algebra generated by the self-dual basis object of index 4. The annihilating polynomial for its image is

$$\begin{cases} (t^2 - 14t + 4)(t^4 + 2t^3 - 6t^2 - 12t - 4), & 1203, 2842, \\ (t^2 - 12t - 14)(t^4 - 10t^2 + 12t - 2), & 1350, 2885. \end{cases}$$

Every S_X is a specified rational polynomial in S_4 . The exact polynomial identities are checked independently in the original rank-eight regular representation by multiplication by $I + N_g$.

We enumerate all symmetric integral 4×4 matrices S_4 having the prescribed bounds and parities, test the displayed polynomial, and reconstruct the other S_X . A simultaneous bounded integral representation never occurs:

ring	all matrices	polynomial survivors	simultaneous survivors
1203	110592	32	0
1350	62208	16	0
2842	110592	32	0
2885	62208	16	0

All computations use exact rational or unbounded integer arithmetic; the finite bounds come from the module-block argument. The discovery enumerator uses a simultaneous diagonal-sign gauge and is replayed by an independent enumerator without this reduction. Both retain their checks under optimized Python. As a positive control, the known model $R8_3_342 = \text{Rep}(C_{13} \rtimes C_6)$ passes every simultaneous multiplication, duality, bound and parity check on its two-dimensional difference representation. The evidence and two read-only verifiers are in `evidence/rank8_frontend_20261021/quadratic/`. $\square$

The argument uses only semisimple module-category theory (see Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transformation Groups **8** (2003), 177–206, and Etingof–Gelaki–Nikshych–Ostrik, *Tensor Categories*, Chapter 7). In particular, neither pivotality nor unitality is assumed. The conclusions for 1350 and 2885 strengthen the previous unitary-only obstructions.

Theorem 3.19. *The fusion ring $R8_3_921$ has no categorification over $\mathbb{C}$.*

Proof. Use the same split module, now with g at index 5 and fixed indices 3, 4, 6, 7. Put $B = S_3$. Since index 4 is dual to 3 and the ring is commutative, $BB^t = B^t B$. Thus B is a real normal matrix. All sixteen fixed-block coefficients of index 3 are odd, and its four diagonal coefficients are 1. Hence $|\text{tr } B| \leq 4$ and $\|B\|_F^2 \geq 16$. Its squarefree source annihilator is

$$p(t) = t(t^2 - 8t + 6)(t^2 + 2t + 4).$$

The characteristic polynomial of this rational four-dimensional matrix is $t^a(t^2 - 8t + 6)^b(t^2 + 2t + 4)^c$, where $a + 2b + 2c = 4$. Its trace is $8b - 2c$, and normality gives squared Frobenius norm $52b + 8c$. The trace bound rules out $b > 0$; the norm bound then forces $c = 2$. Thus $q(B) = 0$, where $q(t) = t^2 + 2t + 4$. The exact identity in the regular fusion representation

$$q(N_3)(-7N_1 - 7N_2 - 3N_3 + 30N_6 - 7N_7) = 148N_6$$

therefore forces $S_6 = 0$. This contradicts the odd fixed-block entry $N_{6,3}^3 = 1$. The argument needs no condition $S_g = I$. The verifier `normal_split_spectral.py` checks all source identities and all six characteristic-factor possibilities. Its independent full-box search examines all 131072 bounded matrices: 2096 are normal and 16 satisfy $p(B) = 0$, and every one of these satisfies $q(B) = 0$. The matrix norm here is an ordinary exact linear-algebra invariant; it introduces no categorical unitarity assumption. $\square$

3.20. A spherical-equivariantization obstruction for inversion.

Theorem 3.20. *The fusion ring $R8_3_436$ has no complex fusion categorification. This conclusion uses no near-group associator classification or unitarization theorem.*

Proof. The source tensor has a faithful C_2 grading with neutral basis $[0, 1, 2, 5]$ and odd basis $[3, 4, 6, 7]$. Write $g = X_1$, $X = X_5$ and $t = X_3$. Its exact products give

$$g^3 = 1, \quad gX = Xg = X, \quad X^2 = 1 + g + g^2 + 3X, \quad t^2 = 1, \quad tgt^{-1} = g^{-1}, \quad tXt^{-1} = X.$$

Suppose $\mathcal{C}$ categorifies the tensor, and put $\mathcal{D} = \mathcal{C}_e$. The pointed category generated by t may have a nontrivial associator class $\omega \in H^3(C_2, \mathbb{C}^*)$. Multiply the associator of the whole graded category by the inflation of ω^{-1} . This preserves the fusion tensor and the neutral category, and makes $1, t$ a monoidal section of the grading. Conjugation by that section gives a coherent categorical C_2 action on $\mathcal{D}$, interchanging g, g^2 and fixing X . Coherence is supplied by the section; it is not inferred from a bare order-two permutation or autoequivalence.

Thornton's sphericity theorem for near-group categories [93, Theorem 1.3] applies without a braiding hypothesis. The spherical dimensions satisfy

$$\dim g = 1, \quad d = \dim X, \quad d^2 = 3 + 3d.$$

The pivotal structure on $\mathcal{D}$ is unique: a tensor automorphism of the identity has scalars λ_g, λ_X ; the relations $gX = X$ and $X \subset X^2$ force both to be one. Every tensor autoequivalence therefore preserves its spherical structure. The equivariantization $\mathcal{E} = \mathcal{D}^{C_2}$ is consequently spherical, with its underlying pivotal maps, and the forgetful functor preserves dimensions.

The five simples of $\mathcal{E}$ are $1, \varepsilon, Y, A, B$, with forgetful images $1, 1, g \oplus g^2, X, X$, respectively. Fixed simples have two lifts because projective C_2 representations over $\mathbb{C}$ can be linearized. The central sign object ε fixes Y and exchanges A, B . Forgetful images and Frobenius reciprocity give

$$Y^2 = 1 + \varepsilon + Y, \quad YA = AY = YB = BY = A + B.$$

Since $B = \varepsilon A$, these relations prove commutativity of the whole ring. If $A^* = B$, the remaining rules would have $A^2 = B^2 = \varepsilon + Y + aA + bB$ with $a + b = 3$; duality would require $a = b$, an impossibility. Thus A, B are self-dual and

$$A^2 = B^2 = 1 + Y + aA + bB, \quad AB = \varepsilon + Y + bA + aB, \quad a, b \geq 0, \quad a + b = 3.$$

All four ordered pairs are retained. For $s = a - b \in \{-3, -1, 1, 3\}$, the ring has scalar character

$$\chi(1) = 1, \quad \chi(\varepsilon) = -1, \quad \chi(Y) = 0, \quad \chi(A) = x, \quad \chi(B) = -x, \quad x^2 = 1 + sx.$$

Its formal codegree is

$$f = 2 + 2x^2 = s^2 + 4 \pm s\sqrt{s^2 + 4}.$$

It belongs to $\mathbb{Q}(\sqrt{5}) \setminus \mathbb{Q}$ for $|s| = 1$, and to $\mathbb{Q}(\sqrt{13}) \setminus \mathbb{Q}$ for $|s| = 3$. But the spherical dimension field of $\mathcal{E}$ is $\mathbb{Q}(\sqrt{21})$, for either root d . Ostrik's formal-codegree field theorem [74, Theorem 2.13 and Corollary 2.15] forbids this: the corresponding central simple has dimension $\dim(\mathcal{E})/f$, whose forgetful image is an integral combination of the simple dimensions of $\mathcal{E}$. Thus f must lie in its dimension field. This contradiction proves the theorem for arbitrary complex categorifications. $\square$

Remark 3.21 (Proof scope and exact certificate). The previous universal TPE scalar-table parameterization premise is withdrawn. Its scalar calculation does not prove that missing categorical statement. The proof above is independent of it and explicitly handles the pointed associator anomaly. Its source tensor has SHA-256

0f6d68de67a7287db7dd32a786237f7af9f35ec7c26b3ada2ba50012ff5455c7. The checker in evidence/r436_repair_20261024/ verifies the grading, inversion, every rank-five branch, formal codegrees and the nonintegral field-norm contradictions. The categorical argument has a separate review.

3.21. A uniform $\text{Rep}(A_5)$ complement obstruction. The integral ring R8_3_3344 is excluded over $\mathbb{C}$. The argument separates a structural reduction from a complete relative cocycle calculation; neither step assumes unitarity.

Structural reduction. Suppose a fusion category $\mathcal{C}$ of dimension 360 contains $\mathcal{D} \simeq \text{Rep}(A_5)$, with three complementary simples of dimension 10, on which the left and right action matrices of the simples of dimensions 1, 3, 3, 4, 5 are

$$I, \quad J, \quad J, \quad I + J, \quad 2J - I,$$

where J is the 3×3 all-ones matrix. Then $\mathcal{C}$ is group-theoretical. Indeed, the complement $\mathcal{M}$ is an indecomposable $\text{Rep}(A_5 \times A_5)$ -module category. By Ostrik's classification it is $\text{Rep}_\alpha(K)$ for a subgroup $K \leq A_5 \times A_5$. The displayed action forces all three projective irreducible degrees to equal an integer q , hence $|K| = 3q^2$. Neither projection of K can be A_5 : Goursat's lemma reduces this case to a graph or a direct product, and every projective sector of A_5 has rank at least four. The spectrum $\{3, 0, 0\}$ of the degree-three action forces each projection to contain a 3-cycle. Thus its proper projections have orders 3, 6, 12, leaving $|K| = 3, 12, 48$.

Set $A = \text{Fun}(A_5)$ and $\mathcal{E} = {}_A\mathcal{C}_A$. Its diagonal subcategory is Vec_{A_5} . The complementary A -bimodules are projectively K -equivariant vector bundles on the free K -set $A_5 \times A_5$. They number $3600/|K|$ and form a transitive $A_5 \times A_5$ -set. Their common Frobenius–Perron dimension is therefore $q/2$, since their squared dimensions sum to 300. The possibilities are

$$(|K|, q, \text{rank}, \text{FPdim}) = (3, 1, 1200, \tfrac{1}{2}), \quad (12, 2, 300, 1), \quad (48, 4, 75, 2).$$

The first is impossible and the second makes $\mathcal{E}$ pointed. In the third, Goursat's lemma gives $K = A_4 \times_{C_3} A_4$. Each complementary simple has left stabilizer V_4 , so XX^* is the sum of its four stabilizing invertibles. Consequently $\mathcal{E}_{\text{ad}} = \text{Vec}_{A_5}$, since the conjugates of V_4 generate A_5 . Each universal-grading component is an invertible Vec_{A_5} -bimodule. But a component with left stabilizer V_4 has dual $\mathcal{C}(A_5, 1, V_4, \psi)$, which is not pointed: distinct Sylow 2-subgroups of A_5 intersect trivially, giving a simple of dimension $[V_4 : 1] = 4$ by the double-coset formula. This contradicts invertibility and proves the reduction.

The unique proper subring of R8_3_3344 has basis $[0, 1, 2, 6, 7]$ and the $\text{Rep}(A_5)$ tensor. Every categorification of this simple dimension-60 subring is $\text{Rep}(A_5)$ by Etingof–Nikshych–Ostrik, *Weakly group-theoretical and solvable fusion categories*, Theorem 9.12. The full tensor verifies the hypotheses above.

All group-theoretical possibilities. The ring has no nontrivial invertible and trivial universal grading, so its center has no nontrivial invertible. In a presentation $\mathcal{C}(L, \omega, H, \psi)$, this forces L to be perfect. An exhaustive check of the 162 groups of order 360 leaves only $L = A_6$. The identity double-coset sector $\text{Rep}(H)$ forces H perfect as well. The perfect-group catalogue on the divisors of 360, together with the index-three action excluding order 120, leaves $H = 1, A_5, A_6$. The endpoints have the wrong ranks. The index-six coset action identifies every A_5 embedding with the usual point stabilizer up to an automorphism of A_6 .

The restriction $H^2(A_6, \mathbb{C}^\times) \rightarrow H^2(A_5, \mathbb{C}^\times)$ is surjective, and

$$\ker(H^3(A_6, \mathbb{C}^\times) \rightarrow H^3(A_5, \mathbb{C}^\times)) \simeq C_2.$$

Here is a reproducible calculation rather than an assumed cohomology table. Normalized mod-2 bar cohomology on a Sylow D_8 , with all 360 stable-element conjugation conditions, gives 2 stable classes in degree 2 and 4 in degree 3. The restriction kernels on the point-fixing V_4 have orders 1 and 2, respectively. The degree-two result and the coefficient exact sequence prove the surjectivity above and bound the relative 2-part by C_2 . Transfer annihilates the relative kernel by the index 6. For its 3-part, restriction to the Sylow C_3^2 embeds it in $H^4(C_3^2, \mathbb{Z}) = \langle x^2, xy, y^2 \rangle_{\mathbb{F}_3}$. An explicit normalizer rotation leaves only the line $\langle x^2 + y^2 \rangle$, which restricts injectively to the coordinate C_3 in A_5 . Thus the relative 3-part vanishes. The previously certified relative μ_2 cocycle has a nontrivial A_4 stabilizer multiplier and supplies the nonzero relative class.

There are consequently only two relative gauge classes of (ω, ψ) . The trivial one gives rank 9. The nontrivial one is the already verified model R8_3_3342. Its three dimension-ten simples satisfy $N_{X_i X_i}^{X_i} = 2$, whereas all three corresponding coefficients of R8_3_3344 equal 3. Hence no group-theoretical model exists for the latter ring. The structural reduction proves its noncategorifiability over $\mathbb{C}$.

A general obstruction for the dimension-660 ring. The ring R8_3_731 is also excluded over $\mathbb{C}$. Its source dimensions are 1, 5, 5, 10, 10, 11, 12, 12 and its based ring is simple. Any categorification is integral, hence pseudo-unitary and spherical. It cannot be braided: simplicity and the prime-dimensional simple of dimension 5 exclude the nondegenerate case by Etingof–Nikshych–Ostrik Corollary 7.2, while the symmetric case would be a group representation category. The exhaustive ordinary-group scan treats all 40 groups of order 660 and finds no matching tensor. Liu–Palcoux–Ren, *Interpolated family of non group-like simple integral fusion rings of Lie type*, Theorem 5.1, therefore forces its hypothetical Drinfeld center to be simple. Its dimensions are integral, so Corollary 7.2 excludes every nontrivial central simple of prime-power dimension. This strategy is already proposed in their Remark 5.15; the new result is the complete finite induction and rotation certificate below.

The formal codegrees are 660, 4, 5, 5, 11, 11, 12, 12, so the distinguished central simple dimensions are 1, 165, 132, 132, 60, 60, 55, 55. The exact Gram ellipsoid gives 38058 vectors, whose seven required unit-multiplicity dimension slices have sizes 65, 61, 61, 31, 31, 37, 37. Complete exact row-sum and positive-semidefinite completion leaves precisely 450 low induction frames. The required duality invariance of the restriction-column multiset excludes 434. For each remaining frame, all admissible high restriction columns are enumerated exactly, with central dimension dividing 660 and not a prime power. Exact Gram-completion Farkas certificates exclude another 13 frames.

The last three frames, with indices 293, 301, 303, have respectively 280, 210, 324 admissible high columns. Their second indicators are all forced, namely $(0, 0, 0, 0, 1, 1, 1)$ in nonunit source order. Joint Gram, twist-moment, and tensor-rotation equations through order six use 53 complete normalized rational trace patterns. Divisors of 19800 together with the zero pattern exhaust these patterns; this is a trace enumeration bound, not a claimed exponent bound. The three integer systems have 182 equations each and respectively 14917, 11207, 17249 variables. Each has a saved integer Farkas vector y satisfying $A^t y \geq 0$ and $b^t y < 0$. The read-only verifier reconstructs all 450 frames and all three integer systems and checks every certificate exactly, without calling an LP solver. Thus $434 + 13 + 3 = 450$ branches are excluded, proving general noncategorifiability. The previous unitary-only obstruction is now supplemented by this independent general obstruction. The printed F_{660} in the triangular-prism paper remains R8_3_722, a different tensor.

Evidence. The full proof, exact source hashes, all subgroup and cohomology receipts, and normal/optimized Python replays are in `evidence/rank8_frontier_20261021/integral/`. The independent review recomputes the mod-2 calculation without importing the discovery verifier. The categorical inputs are Ostrik’s module-category classification (§3.4, Theorem 2), Etingof–Nikshych–Ostrik’s invertibility of graded components (*Fusion categories and homotopy theory*, Theorem 6.1), Gelaki–Naidu’s double-coset simple-object formula (Theorem 5.1), and Cartan–Eilenberg stable elements and restriction/transfer.

3.22. Exact structural and infinite-family construction filters. The following results concern specified construction domains; they are not general categorification exclusions for the rings that remain open. The adjoint closure is the full ring for `R8_3_65`, `R8_3_439`, `R8_3_567`, `R8_3_901` and `R8_3_1463`, excluding a nontrivial faithful group grading of these particular categories. The two latter rings have a unique Fibonacci generator. In a putative Fibonacci Deligne factorization, the complementary simple basis would be forced to be $[0, 1, 2, 7]$ and $[0, 1, 3, 6]$, respectively; neither basis is closed under multiplication. They also cannot be braided: the Fibonacci subcategory would be nondegenerate, since a symmetric fusion category is integral, and would split off by [4, Corollary 2.46 and Theorem 3.13]. We do not infer a coherent central lift from based-ring commutativity. The generalized near-group `R8_3_65` has a free C_4 action on its four noninvertibles and noninvertible FP dimension satisfying $d^2 = 1 + 11d$. The classification [96, Theorem 1] therefore excludes a braided categorification as well.

There is a stronger unbraided construction restriction for `R8_3_65`. Any canonical $\text{Rep}(G)$ of an equivariantization must lie in its pointed C_4 subcategory, since its other simple dimensions are irrational. Hence every nontrivial possibility contains a central $\text{Rep}(C_2)$. Its sign object acts freely on all eight source simples. The orbit description of equivariantization forces a rank-four de-equivariantization of global dimension $125 + 55\sqrt{5} > 24 + 12\sqrt{3}$, contrary to the unrestricted rank-four bound proved in the rank-four companion, Corollary `finite:cor:sharp-bounds`. Thus every nontrivial finite-group equivariantization is excluded for this ring. In contrast, the complete rank-two based-module equations have exactly two solutions up to exchanging vertices:

$$N_\rho = \begin{pmatrix} 0 & 1 \\ 1 & 11 \end{pmatrix}, \quad N_\rho = \begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix}.$$

An actual module category with either action would trivialize the pointed $H^3(C_4, \mathbb{C}^\times)$ class. Neither that vanishing nor an associator for the second action is inferred from these matrices.

For `R8_3_439` and `R8_3_567`, the complete C_3 split-module calculation permits exactly two conjugate nontrivial Fourier systems. The subgroup algebra $A = \bigoplus_{g \in C_3} g$ exists because a fixed noninvertible trivializes its pointed third-cohomology class. Its Morita dual has six invertibles and two noninvertibles. The right deck actions and the full left fusion action force the dual noninvertible matrix via an integer linear system of rank 64 in 64 unknowns. The eight dual actions are independent, and all 512 resulting coefficients recover the original respective source ring. This closes this particular Morita route at the level of based rings; it asserts neither equivalence of associators nor existence of a categorification.

We also enumerate every possible underlying object of a connected separable algebra in seventeen source rings, supplementing the six quadratic cases treated separately. The coefficient bounds $a_0 = 1$, $a_i = a_i^*$ and $0 \leq a_i \leq \lfloor \text{FPdim}(X_i) \rfloor$, invertible stabilization, and the entrywise nonnegative positive-semidefinite induction residual Gram condition leave the following numbers of objects:

source index	65	439	455	567	721	901	903	928	941
survivors	30	28	21	22	45	18	27	24	38
source index	982	985	1154	1463	1615	1617	1660	1661	
survivors	31	31	81	26	49	39	45	49	

Every retained object is independently checked with rational arithmetic, and ten complete finite domains are independently re-enumerated in Python; the remaining domains use the same exactly bounded native enumerator. For the noncommutative tensors the induction matrix uses right multiplication by the candidate algebra. No surviving object is asserted to carry an associative multiplication. For `R8_3_65`, every object $1 + X$ with X noninvertible fails this necessary filter, leaving only the order-two pointed choice among objects of the form $1 + X$.

For the new cyclic family $\text{NG}(C_n, n)$ of [95, Theorem 2.10 and Remark 2.11], and indeed for every categorification of those rings, the FP global dimension D_n has

$$\text{Tr}(D_n) = n(n+4), \quad \text{Nm}(D_n) = n^2(n+4).$$

If a target quadratic FP dimension D equals sD_n with $s \in \mathbb{Q}_{>0}$, then necessarily

$$n = \frac{\text{Tr}(D)^2}{\text{Nm}(D)} - 4, \quad s = \frac{\text{Tr}(D)}{n(n+4)}.$$

Thus an exact calculation replaces a bounded search over n . Direct and Morita equivalence require $s = 1$; a group extension or equivariantization requires integral s . If the target is a de-equivariantization by a group of order $k = 1/s$, then $k \mid n$, its rank is $n/k + k$, and its invertible group is cyclic of order n/k . To see this, the integral subcategory $\text{Rep}(G)$ in the near-group category must be pointed, so G is cyclic. All its stabilizers have trivial second cohomology. The equivariantization orbit description [97, Corollary 2.13 and Remark 3.1] then forces every invertible orbit to be fixed and the unique noninvertible orbit to be free. Finally, $\text{NG}(C_n, n)$ has trivial universal grading, excluding proper neutral-component extraction.

The arithmetic tests and the independent spherical-equivariantization exclusion of R8_3_436 exclude all five specified single-operation routes for the original frontier. No universal near-group unitarization theorem is used. For example, R8_3_65 forces $(n, s) = (121, 4/121)$; R8_3_439 and R8_3_567 force $(n, k) = (12, 2)$, whose de-equivariantization would have cyclic, rather than S_3 , invertibles. Arbitrary chains of operations, Morita changes inserted into those chains, and general noncentral algebra condensations are outside this claim.

For R8_3_455, a separate exact Fourier calculation excludes the self-Morita-duality required of the neutral category within the strict Evans–Gannon $\text{HI}(C_3)$ matrix presentation, including its known Galois companions. We retain the gap between an arbitrary abstract complex neutral category with vanishing pointed cocycle obstructions and that strict presentation. This conditional calculation does not change the general status of R8_3_455.

3.23. Closing the remaining near-group and quantum-group model branches. There is a useful Morita-invariant construction obstruction. If a category has no nontrivial invertible basis object commuting with all simples and has full adjoint subcategory, its center has no nontrivial invertibles. Indeed a central invertible must forget to the unit; its half-braiding is a tensor automorphism of the identity and is trivial on every constituent of $X \otimes X^*$, hence on the full adjoint closure. Both R8_3_439 and R8_3_567 satisfy these hypotheses exactly. Morita equivalence preserves the braided center [7, Introduction].

The complete $SU(2)$ product-dimension sieve retains just the branch $2A_{12}$ for these two rings: one adjoint $SU(2)_{10}$ factor and total integer multiplier two. Full $SU(2)_{10}$ has a faithful C_2 grading; each other allowed realization of this dimension branch uses precisely one dimension-two extension, equivariantization or Deligne factor. Each supplies a nontrivial central invertible, preserved under any Morita changes. Thus the whole retained branch is excluded, including all ADE module-category Morita duals of $SU(2)_{10}$. Together with the thirty dimension obstructions this gives thirty-two $SU(2)$ family exclusions among the original thirty-four rings. The remaining two dimension matches are integral rings already excluded independently. No assertion about arbitrary algebra condensation or other Lie types is made.

The actual inversion action of C_2 on the known $\text{NG}(C_3, 6)$ also deserves an explicit check. At the normalized solution, [36, Lemma 13.1 and Example 13.10] gives the involution $(-\text{id}, R_{\pi/4}L)$, with $L = \text{diag}(1, -1)$ and $(R_{\pi/4}L)^2 = I$. Its crossed product is coherent, but its tensor has multiplicity six and a nontrivial C_2 grading, so it cannot be either source tensor. Under the additional hypothesis that the full pointed S_3 associator class is trivial, condensation of $\bigoplus_{g \in S_3} g$ instead forces a rank-five dual with dimensions $1, 1, 2, 3 + 2\sqrt{3}, 3 + 2\sqrt{3}$. Complete associativity and rigidity leave five based tensors. Full coefficient matching and fresh replay of four frozen rank-five exclusion certificates leave

only $R5_3_40$. The known equivariantization realizing that based ring has a central $\text{Rep}(C_2)$ and so cannot be this dual. This structural filter alone does not decide other associators; the centerless Morita obstruction in Section 3.41 excludes the required dual.

The new indicator calculation now decides which pointed-cocycle branch is necessary. After the proved Galois-pseudo-unitary reduction, the complete joint-rotation test on the unique full-character- S low frame leaves only

$$(\nu_2(X_1), \dots, \nu_2(X_7)) = \begin{cases} (0, 0, 1, 1, 1, 1, 1), & R8_3_439, \\ (0, 0, 0, 0, -1, -1, -1), & R8_3_567. \end{cases}$$

All fourteen other indicator branches have exact separating witnesses. For an order-two invertible in the canonical positive spherical structure, its indicator is the sign of its pointed $H^3(C_2, \mathbb{C}^\times)$ class; see [91, Definition 2.2 and Proposition 4.1] with trivial subgroup. The C_3 restriction is already trivialized by a fixed noninvertible. Restriction followed by corestriction is multiplication by the subgroup index for arbitrary subgroups [90, Section 3.3, equation (3.12)]; hence the two trivial Sylow restrictions imply $2\omega = 3\omega = 0$ and $\omega = 0$. Galois transport preserves vanishing and transports the subgroup algebra. Thus every complex categorification of $R8_3_439$ has the rank-five Morita dual just described, necessarily with a tensor category structure inequivalent to the known equivariantization. For $R8_3_567$, the C_2 restriction is nontrivial and the full S_3 subgroup algebra is impossible. Its C_3 condensation remains available. Neither conclusion proves existence or nonexistence.

Finally let D_0 be the irrational global FP dimension of an arbitrary near-group category with group order n and multiplicity m . Then $\text{Tr}(D_0) = 4n + m^2$ and $\text{Nm}(D_0) = n(4n + m^2)$. For $R8_3_721$, $D = 136 + 32\sqrt{17}$ has trace 272 and norm 1088. An equality $D = tD_0$ with positive integer t therefore forces $n = 4/t$ and $m^2 = 256/t$. The complete possibilities are $(n, m, t) = (4, 16, 1)$ and $(1, 8, 4)$; $t = 2$ gives a nonsquare. Both groups of order four are excluded by full tensor matches to $R5_16_1909$ and $R5_16_1653$ and fresh frozen certificate replay. The rank-two case is excluded by [23, Main Theorem]. This exhausts all near-group orders and multiplicities with integer dimension multipliers and Morita changes for this target, but not dimension-decreasing operations or arbitrary nonintegral products.

3.24. Complete quadratic module and construction domains. For the six residual indices 1055, 1768, 1795, 3258, 3413, 3472, connected rank-two based modules admit a complete finite classification. The positive Frobenius–Perron character and its quadratic conjugate exhaust their two-dimensional rational representation. If a self-dual generator has dimension polynomial $t^2 - Tt + U$, its action is

$$\begin{pmatrix} a & b \\ b & T - a \end{pmatrix}, \quad 0 \leq a \leq T, \quad b^2 = a(T - a) - U > 0.$$

Vertex interchange allows $a \geq T - a$. All other action matrices follow from rational affine expressions in the generator dimension. Exact nonnegative integrality, duality, and all fusion products leave respectively 1, 1, 2, 1, 1, 2 based-module classes. Their full actions and candidate internal-end objects are recorded. These data do not solve module associativity and are not constructions of categories.

For index 1055 the forced $1+g$ -module difference representation has rank two. Normality and the source annihilator reduce the first generator to zero; all 1296 remaining bounded simultaneous possibilities give exactly four representations, each with $S_g = -I$. In a central Tannakian condensation the tensor quotient sends g to the unit, which would instead force $S_g = I$. Thus this condensation route is impossible.

An exact search through all 128 unit-containing subsets shows that the only proper nontrivial fusion subring of indices 1055, 1768, 1795 is the pointed C_2 subring. Indices 3258, 3413, 3472 have no proper nontrivial fusion subring. The irrational total dimensions therefore rule out every nontrivial faithful finite-group grading: its neutral component would have integral dimension one or two. A

nontrivial equivariantization contains $\text{Rep}(G)$; the same subring audit excludes it for the latter three rings and leaves only $G = C_2$ for the former three.

For 1768 and 1795 the sign invertible acts freely on all eight simples. The orbit/projective-stabilizer description of equivariantization simples then forces a hypothetical C_2 -de-equivariantization to have rank four: a free base orbit would give a sign-fixed simple, while a fixed base simple gives two equivariant lifts. Its dimension would be respectively $222 + 36\sqrt{37}$ or $270 + 120\sqrt{5}$, both greater than 45. The unrestricted rank-four companion theorem bounds that dimension by $24 + 12\sqrt{3} < 45$. This explicitly cited theorem dependency rules out both routes; its source digest and the exact quotient tensors are pinned in the evidence. Together with the S_g argument above, all six residuals are excluded from nontrivial finite-group equivariantization constructions. No central lift is inferred from commutativity of the fusion ring alone.

There is also an unbounded arithmetic sieve for generalized-near-group construction towers. If a base has G invertibles, stabilizer order s , noninvertible orbit size $\ell = G/s$, common irrational dimension d , and total noninvertible multiplicity K , then

$$d^2 = s + Kd, \quad D_0 = \ell(2s + Kd), \quad \frac{N(D_0)}{\text{Tr}(D_0)} = G.$$

Pointed products, finite-group extensions, and equivariantizations multiply dimension by a positive integer k , while Morita equivalence preserves it. Consequently $G_D = N(D)/\text{Tr}(D)$ must be integral, $k \mid G_D$, $s \mid G_D/k$, and $K^2 = \text{Tr}(D)/(k\ell) - 4s$. These divisor and square tests exhaust the stated family without a parameter cutoff. Division by de-equivariantization is outside its scope.

For 1795,3258,3413, the values $G_D = 10/3, 45/7, 45/7$ exclude the whole stated family. For 1055 and 1768, $G_D = 6$ and all matches have $s = 1, K = 12$; the tuples (k, G, ℓ) are

$$(1, 6, 6), \quad (2, 3, 3), \quad (3, 2, 2), \quad (6, 1, 1).$$

The last is excluded by the rank-two classification and the third by the rank-four bound, its base dimension being $148 + 24\sqrt{37} > 45$. For 3472, $G_D = 7$, $s = 1, K = 14$, and the matches are $(1, 7, 7), (7, 1, 1)$; the latter is again rank-two impossible. The remaining matches are dimension constraints only, without an asserted base category or Morita equivalence.

3.25. Exact character-field coverage. For the seven indices below, the source element $H = \sum_{j=0}^7 (j+1)^2 X_j$ has squarefree degree-eight characteristic polynomial and generates the rational algebra. Exact interpolation writes every basis element as a rational polynomial in H . Factoring its polynomial over the quadratic FP field therefore proves the following full-character coverage, independently of numerical root approximations.

Index	FP radicand	Characters in FP field	Other factor degrees
65	5	4	2, 2
1055	37	6	2
1768	37	8	none
1795	5	8	none
3258	5	8	none
3413	5	5	3
3472	2	2	3, 3

Thus a modular-eigenvalue calculation confined to the real FP field has complete character coverage for 1768,1795,3258. The other four need larger splitting fields. The following exact adapters now supply all four of them, with every source character polynomial checked to split completely:

index	$[L : \mathbb{Q}]$	defining polynomial of u
65	4	$u^4 - 8u^2 + 36$
1055	4	$u^4 - 68u^2 + 1600$
3413	6	$u^6 - 21u^4 + 2u^3 + 84u^2 + 24u - 19$
3472	6	$u^6 + 2u^5 - 9u^4 - 14u^3 + 10u^2 + 8u + 1$

For 65 and 1055, the fields are respectively $\mathbb{Q}(\sqrt{5}, i)$ and $\mathbb{Q}(\sqrt{37}, \sqrt{-3})$, with u the sum of the displayed radicals. Complex conjugation fixes the real radical and negates the imaginary one; its multiplicativity and involution identities are checked exactly. For 3413 and 3472, take $u = \sqrt{q} + \theta$, with respectively $(q, p_\theta) = (5, t^3 - 3t + 1)$ and $(2, t^3 + t^2 - 2t - 1)$.

For $\lambda \in L$, algebraic integrality is equivalent to the rational matrix of multiplication by λ having monic integral characteristic polynomial: necessity follows from the minimal polynomial, and sufficiency because that characteristic polynomial annihilates λ . This tests the full ring of integers without assuming that the primitive power basis is integral. The adapters check all eight character rows, their products, dualities, orthogonality, the FP embedding, and direct multiplication-operator controls. All twelve distinct restriction columns of the known center of Fibonacci $\boxtimes \text{Vec}(C_2 \times C_2)$ are retained in both overfield adapters. These are complete arithmetic applicability results; categorical obstruction claims retain the hypotheses of the associated center theorem.

3.26. A complete underlying-object filter for connected algebras. The following necessary conditions require neither pivotality nor unitarity. Let A be a connected separable algebra, $m = A$ its simple regular right module, and $a_i = [A : X_i]$. Free-module adjunction and rigidity identify a_i with $\dim \text{Hom}(X_i \otimes m, m)$ and then with $\dim \text{Hom}(m, X_i^* \otimes m)$. Semisimplicity permits reversing the two arguments without changing this dimension. Hence $a_i = a_{i^*}$. The positive Frobenius–Perron eigenvector of the module action gives $a_i \leq \text{FPdim}(X_i)$. If an invertible g occurs in A , the nonzero map $g \otimes m \rightarrow m$ is an isomorphism, so $gA = A$ at the level of underlying objects.

Write B for the integral multiplicity matrix of free induction. Adjunction gives $B^\top B = N_A$, where N_A denotes right multiplication by A (the source rings below are commutative). The row corresponding to m is a , so

$$N_A - aa^\top \text{ is entrywise nonnegative and positive semidefinite.}$$

Thus all underlying objects of connected separable algebras occur in the finite domain $a_0 = 1$, $a_i = a_{i^*}$, $0 \leq a_i \leq \lfloor \text{FPdim}(X_i) \rfloor$. Exhausting this entire domain, imposing invertible stabilization, and applying the two Gram conditions gives the following complete filter.

Source index	Bounded self-dual vectors	Entrywise survivors	PSD survivors
1055	794976	35913	85
1768	9539712	249139	115
1795	12984608	416274	87
3258	580608	32098	99
3413	580608	35404	60
3472	168591360	4841427	70

The evidence stores every surviving coefficient vector. Integer fraction-free elimination has a proved arithmetic bound; independent rational elimination checks every survivor. A complete independent 392-vector control domain for $\text{Rep}(C_{13} \rtimes C_6)$ reproduces all 30 survivors, including the regular function algebra with zero residual Gram matrix. For source index 3472, only $1 + X_3$ among all $1 + X_i$, $i > 0$, survives. The unit algebra and all internal-end candidates from the rank-two based-module search survive. A surviving object need not admit an algebra structure: integral Gram factorization and the module/algebra associativity equations remain further requirements.

3.27. Real STS constraints from a quadratic CM character field. Suppose the complete character field is a CM extension $L = E(\sqrt{-r})$ of degree two over the Frobenius–Perron field $E = \mathbb{Q}(\sqrt{q})$. Write $\pi(x) = (x + \bar{x})/2$. For distinguished central simples the modular identity is

$$\sum_Z \theta_Z \chi_A(v_Z) \chi_B(v_Z) = f_B \chi_A(v_B).$$

Apply the normalized trace to E from a common field containing L and the twists. Distinguished twists are one. Complete high dual pairs have equal twists and conjugate character products, so

their summed coefficient lies in E and factors through the existing FP-field relative twist trace. Consequently the low choice (v, G) has coefficient

$$\pi(\chi_A(v)\chi_B(v) - \delta_{G,B}f_B\chi_A(v)),$$

with right-hand side $\pi(\delta_{B,FP}f_B - 1)$. Thus these additional real STS rows require no enlarged phase field. The separate full-character-field S -symmetry equations are retained.

The isolated implementation checks conjugation invariance of every high-orbit coefficient and reconstructs every coefficient in the exact FP basis. Its independent $\text{Fib} \boxtimes \text{Vec}_{C_4}$ control checks an exact feasible primal, all modular columns and all factored-oracle scores against direct arithmetic. This strengthening applies to `R8_3_65`, whose field is $\mathbb{Q}(\sqrt{5}, i)$. It does not apply to `R8_3_3472`: there the complete real character field is strictly larger than the FP field, and an STS calculation would require a separate extension of the relative-trace domain.

For `R8_3_65`, the complete height-four relaxation has 283 rows and 6521 columns. It admits an exactly verified nonnegative rational solution. Thus this additional modular constraint is exhausted at the stated height and does not exclude the ring; the feasible relaxation does not construct a fusion category. Both the positive-control receipt and the target primal replay pass with Python assertions disabled.

3.28. A complete noncommutative Galois–pseudo-unitary reduction.

Proposition 3.22. *Every complex fusion categorification of `R8_3_721` or `R8_3_1154` admits a spherical structure and has a pseudo-unitary Galois conjugate.*

Proof. For each ring, the complexified fusion algebra is $\mathbb{C}^4 \oplus M_2(\mathbb{C})$. The common kernel of all commutators in the regular representation consists exactly of the scalar blocks. The four characters obtained there exhaust all possible dimension characters; a matrix block has no one-dimensional representation. Its formal codegree is nevertheless included: if its regular Casimir eigenvalue is λ , with multiplicity four, its formal codegree is $\lambda/2$. Exact multiplication identities and the complete Casimir factorization verify this accounting. For each original ring, every scalar-character orbit except the FP orbit is excluded by a zero dimension or formal-codegree norm divisibility.

For an arbitrary categorification take its canonical sphericalization [74, Section 5.1] and choose dual-compatible lifts. The signed quotient by the sign object $\varepsilon = -1$ satisfies the full parity intervals $|T_{ij}^k| \leq N_{ij}^k$ and $T_{ij}^k \equiv N_{ij}^k \pmod{2}$, cyclic trace symmetry, dual reversal, and every ordered associativity equation. We impose neither commutativity nor a modulo-four indicator normalization. Complete enumeration gives 64 signed tensors for `R8_3_721`, all gauge-equivalent to the original tensor, and 288 for `R8_3_1154`. The latter comprise one 64-member original gauge class and five other classes of sizes 64, 64, 32, 32, 32. All scalar-character factors of the five other classes are excluded by zero dimensions or exact norm divisibility. Both sign sectors' formal codegrees, including matrix blocks, are used: sphericalization has dimension 2δ and formal codegrees $2f$, so central induction requires δ/f to be integral [74, Theorem 2.13 and Corollary 2.14].

Thus every possible signed quotient is gauge-original. Switching the lifts by its dual-compatible gauge makes the signed tensor equal to N . The negative lift multiplicities are then zero, so the chosen lifts form a spherical fusion subcategory whose forgetful functor to the original category is a tensor equivalence. Its dimension character lies in the FP Galois orbit. A number-field model [21, Theorem 2.30 and the following remark] and Galois conjugation therefore give a spherical category with positive FP dimensions, hence a pseudo-unitary category with the same fusion rules. $\square$

The source-bound replay is `central_integrality/general_galois_pu_gate.py` in the frontier evidence directory; its primary calculation incorporates the independent unrestricted signed enumeration. The earlier normalized enumeration is retained only as a cross-check. Consequently the central-support, partial modular, integral Gram, joint-rotation, and low Verlinde constraints for these two rings are necessary for general complex existence after Galois transport. Passing

these constraints supplies neither a model nor unitarity. This proposition alone decides neither ring; `R8_3_721` is realized separately in Theorem 3.16.

3.29. Spherical gates for all seven noncommutative frontiers.

Proposition 3.23. *Every complex fusion categorification of any of `R8_3_439`, `R8_3_455`, `R8_3_567`, `R8_3_721`, `R8_3_928`, `R8_3_941`, and `R8_3_1154` admits a spherical structure and has a pseudo-unitary Galois conjugate.*

Proof. The quadratic cases are proved in Section 3.28. The same unrestricted signed construction gives the following complete domains in the two quartic cases:

ring	ordered signed tensors	gauge-original	nongauge excluded
<code>R8_3_928</code>	32	16	16
<code>R8_3_941</code>	128	64	64

No commutativity or modulo-four indicator restriction is imposed. Each nongauge part is one dual-compatible sign-gauge class. Its spherical dimension has polynomial $t^4 - 320t^3 + 18688t^2 - 139264t + 278528$, whereas the ordinary scalar formal codegrees have polynomial $t^4 - 576t^3 + 11520t^2 - 73728t + 147456$. Induction in the sphericalization requires their quotient to be an algebraic integer. In a common number field its norm is a positive integer power of $278528/147456 = 17/9$, a contradiction.

The scalar algebra of each ordinary ring is the irreducible quartic FP orbit, and the full complex algebra is $\mathbb{C}^4 \oplus M_2(\mathbb{C})$. An independent check constructs all five central idempotents and verifies regular ranks $(1, 1, 1, 1, 4)$. For the nongauge signed tensors, rational regular projectors verify a separable quartic scalar algebra and a four-dimensional matrix algebra with one-dimensional center. Its Casimir eigenvalue is 8, hence its formal codegree is 4. Thus no dimension character or matrix formal codegree is omitted. All surviving signed tensors are gauge-original; the spherical-section and Galois-transport argument of the preceding proposition applies.

For 439, 455, 567, the unrestricted signed domains have respectively 256, 32, 64 tensors, of which 64, 32, 32 are gauge-original. Rational regular projectors again verify every scalar and matrix component, including the matrix formal codegrees. The last nongauge class of both 439 and 567 has a rational scalar character of dimension $\delta = 24$. In a sphericalization with dimension field K , every central simple dimension belongs to K , since its forgetful image is a sum of simples. For an ordinary-sector formal codegree f , induced-unit decomposition therefore gives $\dim(A_E) = (2\delta)/(2f) = \delta/f \in K$; see [74, Theorem 2.13 and Corollary 2.15]. Hence $f \in K$. Here $K = \mathbb{Q}$, whereas an ordinary formal codegree has irreducible polynomial $t^2 - 96t + 576$. This excludes the last nongauge class without assuming pseudo-unitarity. All other nongauge classes are excluded by zero, duality or norm conditions. The ordinary scalar-character sieve leaves precisely the FP orbit, completing the same spherical-section and Galois-transport proof. $\square$

The canonical quartic replay is `noncommutative_followup/verify_scope_gates.py`. The additional three-case replay is `noncommutative_followup/additional_scope_gates.py`. Together the seven gates make the following pseudo-unitary calculations necessary for general complex existence after Galois transport; they make no assertion of unitarity or existence.

Write $B = MM^\top$ for the induction Gram matrix and the central restriction matrix. Every restriction column is orthogonal to $\ker B$. Imposing this exact central-image condition before enumeration leaves respectively 13, 1, 1, 59 weighted distinguished frames for 721, 928, 941, 1154; ordinary Gram and moment bounds leave 1, 1, 1, 4. The two-dimensional representation contributes one distinguished column with unit entry 2, whose outer square is counted once.

For an irreducible trace character χ_E , put $m_E = \dim E$. The distinguished central simple A_E has twist 1, and its complete unnormalized S row is given by [98, Equation (39)]:

$$S_{A_E, Z} = \frac{D}{f_E m_E} \chi_E(FZ), \quad D = \text{FPdim}(\mathcal{C}).$$

The matrix-block character is included. Symmetry and $STS = DT^{-1}ST^{-1}$ impose

$$\sum_Z \chi_E(FZ) \chi_F(FZ) \theta_Z = f_F m_F \chi_E(FA_F).$$

These supplement the linear indicator equations and the joint divisor-power rotation constraints. They remove one of two symmetric assignments for 721 and one of the last two frames for 1154. The surviving 928 and 941 branches have exact rational witnesses. The central eigenvalues $S_{A_E, Z} / \dim Z$ are also tested for algebraic integrality.

For 721, complete integral Gram enumeration has four patterns. Each survives joint rotation through height 12 and the combined height-6 rotation and STS system with its multiplicities fixed. For selfdual distinguished simples, the ordinary Verlinde formula gives

$$N_{A_E, A_F}^{A_G} = D^{-2} \sum_Z \frac{S_{A_E, Z} S_{A_F, Z} S_{A_G, Z}}{\dim Z}.$$

All twenty coefficients are nonnegative integers for every pattern; the corresponding forgetful tensor equations remain exactly feasible. For 1154, the complete integral Gram domain has 14211 patterns. Cubic Verlinde integrality leaves 1238; all pass the ten exact rational forgetful tensor cones. The subsequent fixed-pattern rotation calculation is given separately below.

The exact pointed-bimodule stabilizer test also admits both C_2 associator classes wherever applicable. The forced pseudo-unitary invertible indicators are $-1, +1, -1$ for 928, 941, 1154, respectively. These checks do not construct the required module coherences or associators. Four higher-height numerical attempts remain explicitly undecided and are not counted as completed obstructions. Exact feasible relaxations, including those above, are not categorical models.

3.30. An all-level quantum-group product and Morita-family sieve.

Lemma 3.24. *Let $a_1, \dots, a_r$ be totally positive algebraic numbers, each chosen as a largest conjugate. Then $\mathbb{Q}(\prod_i a_i)$ is the compositum of the fields $\mathbb{Q}(a_i)$.*

Proof. In a common Galois closure, an automorphism fixing the product has $0 < \sigma(a_i) \leq a_i$ for every i . Equality of products forces equality in each factor. Thus the product stabilizer is the intersection of the individual stabilizers, and the assertion follows from Galois theory. $\square$

For $n = k + 2 \geq 3$, the adjoint $SU(2)_k$ category has global dimension

$$A_n = \frac{n}{2 - 2 \cos(2\pi/n)}.$$

The full category has dimension $2A_n$; see [92, p. 8] and the equal-component theorem [21, Proposition 8.20]. All conjugates of A_n are positive, A_n is largest, and $[\mathbb{Q}(A_n) : \mathbb{Q}] = \varphi(n)/2$. Consequently a proposed identity $D = m \prod_n A_n^{e_n}$, with $m \in \mathbb{Z}_{>0}$, forces every factor field into $\mathbb{Q}(D)$. The verifier uses the possibly larger FP field, a safe enlargement of the search domain, and verifies its normality exactly.

If this field has degree d , then $\varphi(n) \mid 2d$. Every prime divisor p of n satisfies $p - 1 \mid 2d$, and its exponent is bounded by $\varphi(p^a) \leq 2d$. These conditions give a complete finite domain with no level cutoff. Exact real cyclotomic resultants and factorization check field containment. Rational factors $A_3 = 1$, $A_4 = 2$, $A_6 = 6$ are absorbed into m ; all remaining factors exceed one. The inequalities $A_n^{e_n} \leq D$ bound all exponents exactly, after which every product and integer quotient is checked.

The replay `noncommutative_followup/su2_all_frontier.py` gives 30 family exclusions among the 34 original frontier rings. The retained dimension matches are precisely

$$\begin{array}{l|l} R8_3_439, R8_3_567 & D = 2A_{12} \\ R8_3_731 & D = 660 \\ R8_3_3344 & D = 360. \end{array}$$

The two integral cases use the empty product. None of these matches is a constructed category. The center-invertible obstruction in Section 3.23 subsequently excludes the two nonintegral matches throughout the stated categorical operation family, giving 32 family exclusions in total.

The excluded family includes arbitrary finite Deligne products of the full and adjoint $SU(2)$ categories at every positive level, followed by any finite sequence of finite-group extensions, equivariantizations, products with integer-FP-dimension categories, and Morita equivalences. The first three operations multiply global dimension by a positive integer; Morita equivalence preserves it [21, Corollary 8.14]. The statement excludes this construction family only. Other Lie types, arbitrary algebra condensation, dimension-decreasing de-equivariantization, and additional nonintegral product factors are outside its scope.

The generalized near-group dimension argument also extends uniformly to the 34 original frontier rings. Write $G = \ell s$ for the invertible group order, ℓ for the noninvertible orbit size, and s for the stabilizer order. If $d^2 = s + Kd$, then an irrational positive integer dimension multiple $D = k\ell(s + d^2)$ is quadratic and satisfies

$$\frac{\text{Norm}(D)}{\text{Trace}(D)} = kG, \quad K^2 = \frac{\text{Trace}(D)}{k\ell} - 4s.$$

Thus k divides the forced integer $G_D = \text{Norm}(D)/\text{Trace}(D)$, and s divides G_D/k . Exhausting these divisors gives every possible parameter tuple, with no rank cutoff. The exact replay agrees with all six earlier quadratic computations. It yields 20 dimension-family exclusions, 12 complete finite parameter lists, and two integral targets outside this quadratic test. The parameter lists are not models. The family here has a single generalized near-group source and a positive integer multiplier; products of distinct nonintegral sources and dimension-decreasing operations are not included. The canonical table is `noncommutative_followup/generalized_near_group_all_frontier.py`.

3.31. Earlier central tests for three further noncommutative rings. The same central-image and full-character S symmetry computation has 5, 1, 5 initial frames for 439, 455, 567 and leaves one frame in each case, with two symmetric assignments. For 567 the scalar characters do not all lie in the real FP field; the computation adjoins i and verifies complete character orthogonality in the resulting exact quartic field. No character is discarded for being nonreal.

On the unique frame for each of 439 and 567, all eight second-indicator branches are tested against joint tensor rotations through height six. Fourteen exact dual certificates exclude seven branches per ring, and two exact primal witnesses certify the remaining relaxations. In the original basis, the surviving tuples for objects $1, \dots, 7$ are

$$\nu_2^{439} = (0, 0, 1, 1, 1, 1, 1), \quad \nu_2^{567} = (0, 0, 0, 0, -1, -1, -1).$$

In particular, the three odd S_3 invertibles have indicator $+1$ in 439 and -1 in 567, in the pseudo-unitary Galois representative. The canonical replays are:

`noncommutative_followup/additional_central_constraints.py`
`noncommutative_followup/additional_joint_rotation.py`

The surviving frames and witnesses remain necessary conditions, not categorical constructions.

Proposition 3.25 (A two-prime norm obstruction for integral subcategories). *Let $\mathcal{B}$ be a fusion category with no nontrivial invertible simple objects. Suppose that its integral-dimensional simple objects span a fusion subcategory $\mathcal{E}$, and that the absolute field norm of $\text{FPdim}(\mathcal{B})$ has at most two distinct prime divisors. Then $\mathcal{E} = \text{Vec}$.*

Proof. Write $N = \text{FPdim}(\mathcal{E}) \in \mathbb{Z}_{>0}$ and $K = \mathbb{Q}(\text{FPdim}(\mathcal{B}))$. Subcategory divisibility [21, Proposition 8.15] implies that $\text{FPdim}(\mathcal{B})/N$ is an algebraic integer. Taking its norm shows that every prime divisor of N divides the norm of $\text{FPdim}(\mathcal{B})$. Thus $N = p^a q^b$ for at most two primes. The category $\mathcal{E}$ is solvable by [3, Theorem 1.6]. If nontrivial, it has a nontrivial invertible simple by [3, Proposition 4.5(iv)], contradicting the hypothesis on $\mathcal{B}$. $\square$

For any hypothetical category $\mathcal{C}$ with ring `R8_3_439` or `R8_3_567`, the already verified center-invertible obstruction gives $\text{Inv } Z(\mathcal{C}) = \{\mathbf{1}\}$. Its simple dimensions are six copies of 1 and two copies of $d = 3 + 2\sqrt{3}$, so $D = \text{FPdim}(\mathcal{C}) = 48 + 24\sqrt{3}$. Every central simple restricts to an object of dimension $a + bd$ with $a, b \in \mathbb{Z}_{\geq 0}$. It has integral dimension precisely when $b = 0$. These objects form a fusion subcategory: their forgetful images lie in the pointed source subcategory, which is closed under tensor products and duals. Since $\text{FPdim } Z(\mathcal{C}) = D^2$ [21, Proposition 8.12], its integral subcategory dimension N satisfies

$$\text{Norm}_{\mathbb{Q}(\sqrt{3})/\mathbb{Q}}(D^2/N) = \frac{576^2}{N^2} \in \mathbb{Z}, \quad N \mid 576 = 2^6 3^2.$$

Proposition 3.25 therefore shows that the tensor unit is the only integral-dimensional central simple. This conclusion requires no pivotal or pseudo-unitary hypothesis.

The same center condition holds for the $R_{5,3,40}$ Morita dual forced by the full- S_3 algebra in a hypothetical `R8_3_439` category. Its source dimensions are 1, 1, 2, d , d , and the integral source objects form the $\text{Rep}(S_3)$ based subcategory. The same proof removes every nonunit central restriction column supported on 1, s , V . This assertion is about the specified Morita dual; it does not impose trivial center invertibles on arbitrary categorifications of $R_{5,3,40}$.

3.32. Complete fixed-Gram rotation tests for `R8_3_1154`. For `R8_3_1154`, the complete central-support and low-frame calculation leaves frame 31, the second-indicator tuple $(0, 0, -1, 1, 1, 1, 1)$, and sixteen high restriction types. Exact affine bounds enumerate all 14,211 nonnegative integral Gram completions; every completion is checked against the original 28 Gram equations. Of these, exactly 1,238 have nonnegative integral distinguished low Verlinde coefficients, and all of these also satisfy the quadratic spectral multiplicity conditions and the ten fixed low-product rational cones.

We next fix each of these 1,238 integral multiplicity vectors in the joint rotation system through order six, together with all distinguished character STS equations. Four exact separating rays exclude eight vectors. The remaining 1,230 vectors have exact nonnegative rational solutions. A source-bound receipt covers every vector in 434 certificate families: each separating ray is checked on the complete column domain, and every primal family is checked in the original matrix equation at each vector assigned to it. No pattern is left to a numerical feasibility decision. The canonical read-only replay is `central_integrality/fixed_gram_rotation_1154.py`.

The Galois-pseudo-unitary reduction makes this a necessary-condition calculation for general complex existence after Galois transport. The finite bound is rotation order six, not a bound on the categorical exponent. The surviving rational solutions do not supply central fusion rules or coherent associators, and `R8_3_1154` remains open.

3.33. Integer central restriction and cubic constraints.

Proposition 3.26 (Integer central necessary system). *Let $\mathcal{C}$ be pseudo-unitary with commutative Grothendieck ring, $D = \text{FPdim}(\mathcal{C})$, characters χ_E , and formal codegrees f_E . Put $\beta_E = D/f_E$, and let A_E denote the distinguished simple of $Z(\mathcal{C})$ belonging to E . A complete finite restriction column domain gives a finite system of integer equations with bounded nonnegative integer variables that includes restriction Gram identities, distinguished S -symmetry, duality, and*

$$N_{A_E, A_F}^{A_G} = \sum_Z \frac{\beta_E \beta_F \beta_{G^*} \chi_E(FZ) \chi_F(FZ) \chi_{G^*}(FZ)}{D^2 \text{FPdim}(Z)}.$$

Every categorification supplies an integer solution of this system.

Proof. By [98, Lemma 4.3 and Corollary 4.5], $S_{A_E, Z} = \beta_E \chi_E(FZ)$ in the unnormalized modular convention. The displayed identity is the Verlinde formula for the center of dimension D^2 . Introduce a nonnegative integer selection variable for each allowed restriction of each distinguished simple, with sum one for each character. The tensor unit is fixed separately; its character is located by equality with the positive dimension character. High variables count actual central simples

per complete restriction-duality orbit. Gram, modular symmetry and low-duality equations are linear in these variables. Expanding the displayed formula in the full character field makes every nonconstant coordinate zero and its constant coordinate a nonnegative integer auxiliary variable. Diagonal Gram bounds and the sum of positive constant contributions give finite upper bounds. Clearing denominators produces the stated system. No integrality is imposed on relative twist-trace weights. $\square$

The eigenvalues $\beta_E \chi_E(FZ)/\text{FPdim}(Z)$ of the integer fusion matrix of A_E also have equal multiplicities throughout each full Galois orbit. These yield further exact linear equations. Integer interval and gcd propagation are necessary tests. A covector annihilating all columns modulo an integer $m > 1$, but not the right-hand side, certifies nonmembership in the column lattice; rational annihilators are checked in the same manner. Signed integer lattice membership alone is weaker than a nonnegative bounded solution. Numerical optimization only proposes vectors, which are checked against every exact original equation and bound. Any negative pseudo-unitary result still requires a complete signed and dimension-character argument before it is promoted to a general-complex exclusion. The actual center of $\text{Fib} \boxtimes \text{Vec}_{C_4}$ provides a 64-simple positive integer control with complex character values. The dimension identity further gives the exact bound

$$N_{A_E, A_F}^{A_G} \leq \left\lfloor \frac{\beta_E \beta_F}{\beta_G} \right\rfloor.$$

If a candidate eigenvalue has a Galois conjugate absent from the complete remaining domain, its nonnegative multiplicity is zero; this rule may be iterated, retaining the fixed constituents. Row content division, exact proportional-row identification, and zero-row deletion preserve the integer solution set. Their signed row scales also lift any compact-system annihilator back to a covector checked on every original column.

3.34. Joint Galois spectra and exact fusion bounds. The integer central reconstruction can be strengthened without enlarging the twist field. Let A_E be the distinguished simple objects in $I(\mathbf{1})$, let $\beta_E = \text{FPdim}(A_E)$, and suppose that a normal number field L contains the complete character table and the Frobenius–Perron dimensions. Write $m_E = \dim(E)$ for the degree of the irreducible representation affording the full matrix character χ_E . For a central simple with restriction vector v , put

$$\Lambda(v) = \left(\frac{\beta_E \chi_E(v)}{m_E \text{FPdim}(v)} \right)_{E \neq \text{FP}}.$$

In a pseudo-unitary categorification, the multiset of these complete tuples, counted with central-simple multiplicities, is invariant under $\text{Gal}(L/\mathbb{Q})$. In the commutative implementations here, every $m_E = 1$; the degree factor is required for full characters of noncommutative source rings.

Indeed, the distinguished fusion matrices are commuting integral matrices in the braided center and are simultaneously diagonalized by its modular matrix. Choose an integer linear form separating the finite union of the candidate tuples and all their Galois conjugates. Such a form exists because only finitely many proper rational hyperplanes are forbidden. The corresponding integer linear combination of the fusion matrices has an integral characteristic polynomial. Equality of multiplicities of its conjugate eigenvalues, together with separation, proves the assertion for complete tuples. The distinguished labels remain fixed; the same field automorphism acts on every tuple coordinate. No braiding of the source category is assumed.

Consequently, complete-tuple multiplicities give additive integer equations in the low choices and high duality-orbit counts. Fixed constituents contribute to the right-hand side. A tuple whose conjugate is absent from the remaining candidate support forces every nonnegative variable containing that tuple to vanish; this deletion may be iterated. Missing conjugates of a fixed constituent instead give an obstruction. These joint equations strengthen the separate Galois multiplicity equations for individual distinguished matrices.

The nonnegative integer variables recording distinguished fusion coefficients also satisfy the exact bounds

$$0 \leq N_{A_E, A_F}^{A_G} \leq \left\lfloor \frac{\beta_E \beta_F}{\beta_G} \right\rfloor.$$

The floor is evaluated in the positive Frobenius–Perron embedding by exact sign comparisons. The joint-spectrum and bound adapters retain the actual 64-simple center of $\text{Fib} \boxtimes \text{Vec}(C_4)$, including its nonreal characters, in normal and assertion-disabled replays. These are necessary reconstruction conditions: a surviving integer or rational solution does not supply an associator, while a numerical infeasibility report alone is never an exclusion certificate.

3.35. A pointed integral-center criterion.

Proposition 3.27. *Let $\mathcal{C}$ be pseudo-unitary. Suppose its simple dimensions are $a_i + b_i\sqrt{q}$, with $b_i \geq 0$, and $b_i = 0$ precisely for invertible simples. If the tensor unit is the only integral-dimensional simple constituent of $I(\mathbf{1})$, then the integral-dimensional subcategory of $Z(\mathcal{C})$ is pointed, and the forgetful functor is injective on its invertible objects.*

Proof. An integral-dimensional central simple restricts to the pointed source subcategory, since the irrational coefficients of source dimensions are nonnegative. By adjunction, such a simple can contain the source unit after forgetting only if it is itself the unit. For an integral central simple Z , every constituent of ZZ^* is integral: its restriction belongs to the pointed source subcategory. Hence

$$1 = [\mathbf{1} : ZZ^*] = [\mathbf{1} : F(ZZ^*)] = \sum_{g \in G(\mathcal{C})} [g : F(Z)]^2.$$

Thus $F(Z)$ is a single invertible and Z is invertible. Two such invertibles with the same restriction have quotient restricting to the unit, so their quotient is the unit. $\square$

For commutative source rings, the hypothesis on $I(\mathbf{1})$ is checked on the complete distinguished dimension list D/f_E . The criterion removes every integral high restriction vector other than a single source invertible. Its exact quadratic hypotheses and every deleted column are recorded for R8_3_1768 and R8_3_1795.

3.36. Source-unit identities for distinguished products. Let the source ring be commutative, and write $v_E = [F(A_E)]$ for a distinguished central simple. Its multiplicity in $I(\mathbf{1})$ is $m_E = 1$, so $(v_E)_0 = 1$; every other central simple has source-unit restriction multiplicity zero. Write E^* for the dual character label, and let v^* denote source duality on a restriction vector. Taking the source-unit coefficient of $F(A_E A_F)$ gives the exact identity

$$\delta_{F, E^*} + \sum_{G \neq \text{FP}} N_{A_E, A_F}^{A_G} = \langle v_E, v_F^* \rangle.$$

In particular, for each of the seven nonunit distinguished labels,

$$1 + \sum_{G \neq \text{FP}} N_{A_E, A_{E^*}}^{A_G} = \|v_E\|^2.$$

The right side is linear in the atomic one-hot selection variables: $\|v_E\|^2 = \sum_v x_{E,v} \|v\|^2$. For the other unordered pairs,

$$2 \left(\delta_{F, E^*} + \sum_{G \neq \text{FP}} N_{A_E, A_F}^{A_G} \right) \leq \|v_E\|^2 + \|v_F\|^2,$$

since the difference is $\|v_E - v_F^*\|^2$. These become exact integer equations by adjoining nonnegative integer slack variables. Their upper bounds are the sums of the largest allowed squared restriction norms. For a fixed complete frame the norms are constants. Thus the same adapter applies to atomic choices and complete frames, including nonselfdual character labels. These rows can strengthen a

fractional atomic relaxation; they are necessary consequences of an actual category and are not additional existence assertions. The actual 64-simple center of $\text{Fib} \boxtimes \text{Vec}_{C_4}$ satisfies every new identity and bound in exact arithmetic, with both normal and optimized replay.

Compatibility of integer restrictions with phases. An integer solution of the preceding system and a rational solution of the rotation and distinguished- STS cone need not describe the same restriction multiplicities. We therefore also specialize the phase cone to selected integer solutions. Fix each selected low column with coefficient one, retain all relative-phase columns for every high restriction-duality orbit O of positive multiplicity n_O , and add

$$\sum_{\tau} x_{O,\tau} = n_O.$$

All other high columns vanish, while proper-divisor rotation columns remain. Every categorical center with this restriction pattern satisfies the specialized system. The compressed trace weights $x_{O,\tau}$ remain nonnegative rational variables; they are not assumed integral. Exact primals are also expanded back into the original full phase cone, and exact separators are checked on every specialized column. Rejection of a selected integer pattern does not exclude other integer patterns or the source ring.

Complete fixed-frame arithmetic screen for **R8_3_982**. The distinguished- S constraints leave exactly four labeled low frames. For each frame we enumerate every high restriction vector allowed by the residual Gram form, then impose integer Gram, distinguished cubic, Galois-balance and exact Frobenius–Perron bounds. The four signed column lattices contain their right-hand sides. Two frames admit explicitly checked nonnegative integer solutions, each with 25 positive high restriction-duality orbits and proposed center rank 56. Both solutions also satisfy every joint-Galois balance equation and all 28 exact source-unit product identities among the seven nontrivial distinguished objects.

These two integer solutions do not survive the phase compatibility test. For each, the height-4 specialization has 366 equations and 885 variables; exact separators cover all eight indicator choices. This excludes the two displayed restriction patterns, not other integer solutions. For the other two frames, the strengthened bounded continuous systems have exact rational primals, checked against every original equation and bound. Their reduced systems have 504 equations and 361 variables. Numerical integer infeasibility proposals are retained only as search diagnostics. Thus this complete four-frame arithmetic screen and the two complete selected-pattern phase checks left **R8_3_982** unresolved at that checkpoint. They provide neither a category model nor a general noncategorifiability theorem; the later realization is Theorem 7.1.

3.37. Global phase cuts on integer reconstruction data. The integer reconstruction and the phase equations can be coupled by exact separating inequalities. Fix a complete phase operator, with low columns C_l , high columns $C_{o,t}$ indexed by restriction-duality orbits and complete relative-trace types, and auxiliary rotation columns. An actual integer pattern has selected low counts x_l and high orbit counts n_o ; its nonnegative phase weights satisfy $\sum_t w_{o,t} = n_o$.

If an integer covector y pairs nonnegatively with every auxiliary column, define

$$a_l = yC_l, \quad m_o = \min_t yC_{o,t}.$$

For each allowed indicator choice ε , every compatible integer pattern satisfies

$$\sum_l a_l x_l + \sum_o m_o n_o \leq yb_\varepsilon.$$

This follows by applying y to the phase equation and bounding each complete high fiber by its minimum times its total mass. The minima are checked over every original orbit, including orbits absent from the pattern that suggested y . A common inequality valid for all indicator choices uses $\max_\varepsilon yb_\varepsilon$ on the right. The low columns retain their character labels.

For nonnegative integer variables, a valid inequality $a \cdot x \leq b$ also implies

$$\sum_j \lfloor a_j/d \rfloor x_j \leq \lfloor b/d \rfloor \quad (d \in \mathbb{Z}_{>0}).$$

This exact weakening, valid also for negative coefficients, permits smaller coefficients in a numerical proposal. The original full-domain covector, every indicator right-hand side, the divisor, and all rounded coefficients remain independently checkable. An integer slack converts the inequality into an equality, with a finite upper bound obtained from the existing variable box.

The finite application to R8_3_65 considers all four indicator choices and allows at most ten such global cuts. Every proposed integer pattern is checked against the complete integer equations, every phase separator is checked against the complete specialized phase domain, and each added global cut must strictly reject its originating pattern. A numerical failure to find the next integer point remains inconclusive; it is not an exclusion certificate.

The recorded R8_3_65 campaign added one exact uniform cut. It rejects the initial integer reconstruction for all four indicator choices, with exact violation 512 after integer rounding. The strengthened system has 1476 equations and 524 nonnegative integer variables, including source-unit product constraints and the conditional pure-invertible restriction bounds. Its finite-box and modulo-2, 3 checks pass. The next bounded integer proposal was inconclusive, so this result excludes the selected reconstruction and establishes a global necessary inequality; it does not exclude the fusion ring.

At this earlier exploratory stage, the R8_3_3472 joint-spectrum, cubic, Frobenius–Perron-bound, and source-unit-product system has 3347 equations and 13242 variables. The complete support and finite-box filters and the modulo-2, 3 tests pass. The bounded linear search produced no exact separator, and the bounded integer proposal produced no checked witness. These outcomes are explicitly inconclusive. No complete Hermite-lattice or bounded integer enumeration is asserted, and this character field’s larger real trace-field STS problem lay outside that tested phase domain. These historical outcomes are superseded by the complete general exclusion in Theorem 8.1.

3.38. Integral central multiplicities and forced pointed lifts. The two noncommutative rings R8_3_928 and R8_3_941 admit a common four-coordinate description of their residual central Gram domain:

$$Q = \begin{pmatrix} 8 & 8 & 8 & 12 \\ 8 & 50 & 58 & 73 \\ 8 & 58 & 74 & 89 \\ 12 & 73 & 89 & 117 \end{pmatrix}.$$

Their unique distinguished induction frame has 65 possible high restriction columns. All pass the full-character partial- S algebraic-integrality test. The twenty distinguished cubic Verlinde expressions coincide in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. An exact rational reduction followed by two independent bounded integer enumerations yields precisely 343 solutions of the full Gram and nonnegative integral cubic equations. The native search visits 1665 nodes and the independent Python finite-set search 561 nodes; every resulting vector is checked against the original equations.

All 343 vectors contain exactly one central simple restricting to the nontrivial source invertible g . Since each source is its own adjoint subcategory, the forgetful kernel is trivial. The corresponding central simple G is therefore invertible of order two, and its lift over g is unique. Tensoring the five distinguished objects by G forces five additional distinct central simples, with canonical high indices 40, 41, 42, 43, 57. All other high restriction types that occur are fixed as restriction vectors by multiplication by g . This proves a statement about the center, without asserting a tensor-factor decomposition of the nonbraided source.

For R8_3_928 the already certified positive-spherical indicator is $\nu_2(g) = -1$, so $\theta_G = \pm i$. Monodromy with an order-two invertible has values ± 1 . Balancing consequently prevents G from fixing a central simple. Multiplicities of the fixed restriction types must be even, leaving exactly 15 of

the 343 vectors. Their paired second and sixth twist moments vanish; the five translated distinguished objects have second and sixth moments -1 . The subsequent finite systems retain the actual correlation between the two twists in each semion orbit.

For **R8_3_941** the indicator is $+1$, and $\theta_G = \pm 1$. The positive-twist branch would give central $\text{Rep}(C_2)$. The sign character acts freely on all eight source simples, so de-equivariantization has four simples. Its exact quotient tensor is the historical rank-four ring $F_{6,19}$, and its dimension is

$$72 + 48\sqrt{2} + 40\sqrt{3} + 28\sqrt{6} > 72.$$

This contradicts the unrestricted rank-four companion bound $24 + 12\sqrt{3} < 45$ (Corollary **finite:cor:sharp-bounds**). Hence $\theta_G = -1$: the lift is a fermion. The ordinary rank-four bound has been used only in the Tannakian branch.

Write $q_Z = \tilde{S}_{G,Z}/d_Z \in \{\pm 1\}$. Positive-charge simples occur in distinct pairs with opposite twists; negative-charge simples may be fixed. For each fixed restriction type write $2p_v + q_v = m_v$, where p_v counts positive-charge pairs and q_v counts negative-charge individuals. The five low charges and their translated charges are $+, +, +, +, -$. In the real character field, orthogonality and the fermion–distinguished STS equations give

$$\sum_Z q_Z d_Z \chi_E(FZ) = 0, \quad \sum_Z \theta_Z q_Z d_Z \chi_E(FZ) = -D \chi_E(g).$$

Fixed contributions are evaluated from the actual translated restrictions; for the two-dimensional character one must retain the factor $\chi_E(g)/\dim(E) = -1$. The complete 41 664-assignment integer charge box, independently reduced to 446 free-variable trials, has 93 solutions on 68 Gram patterns. The height-six correlated cones keep all 38 individual trace types and all 28 opposite-pair types. They also retain the original rotation and distinguished STS equations. An actual $Z(\text{Ising})$ control tests both negative-charge fixed objects and positive-charge pairs, including all fermion-row normalizations. The complete correlated campaign excludes 77 allocations by exact dual certificates and retains 16 allocations, on 16 distinct Gram patterns, with exact nonnegative rational witnesses. Thus this strengthened finite test does not exclude **R8_3_941**. The subsequent supported modular tests at primes 2, 3, 5, 7, 11, 13 find no obstruction. Bounded integer searches provide twelve nonnegative integer witnesses, each checked against every original equation; four integer feasibility questions remain unresolved. Numerical negative search outcomes are not used as exclusion certificates. The additional permutation and mixed-fusion-trace gate leaves fourteen correlated moment witnesses, eleven with exact integer counts and three unresolved integer-feasibility cases; its complete finite domain is described in the accompanying refinement.

The fermionic quotient has four m-type simple orbits. Its coefficient tensor has only the trivial and full fusion subrings, so it cannot be a direct tensor product of two rank-two superfusion categories. This is a scoped construction obstruction; no general classification of rank-four superfusion categories is invoked. Surviving exact integral or moment data, wherever recorded in the accompanying receipts, do not supply super associators or a categorical realization.

A permutation-trace refinement. The forced central fermion G in the pseudo-unitary Galois conjugate of **R8_3_941** supplies another necessary test. Since N_G is an involutive permutation matrix and has monodromy eigenvalues ± 1 , its number of fixed simples equals $P - N$, where P and N count the two charges. Only negative-charge simples can be fixed by balancing. The ten known distinguished objects and translates form five nonfixed pairs. If p_j counts the positive-charge pairs and q_j the remaining negative-charge high simples of restriction type j , then

$$P = 8 + 2 \sum_j p_j, \quad N = 2 + \sum_j q_j, \quad \sum_j (q_j \bmod 2) \leq P - N \leq \sum_j q_j, \quad P - N \equiv \sum_j q_j \pmod{2}.$$

The complete 93-allocation domain retains 71 allocations. Intersecting with the sixteen exact paired-phase survivors removes allocations 4 and 86: they respectively require -2 fixed objects, or thirteen fixed objects among only seven eligible ones.

Products of fusion matrices also have nonnegative integral traces. The receipt tests all 420 quantities $\text{Tr}(N_G^e \prod_{i=1}^4 N_{A_i}^{a_i})$, with $e \in \{0, 1\}$ and $\sum_i a_i \leq 6$, on every allocation, using the full simultaneous modular eigenvalues. An actual $Z(\text{Ising})$ integer-matrix control checks the normalization. No further survivor is removed. The final intersection leaves fourteen rational paired-cone witnesses, eleven of them with exact nonnegative integer witnesses, and three unresolved integer-feasibility questions. These are finite-system survivals, not constructions of fusion categories.

3.39. Forced central involutions and correlated semion traces. The complete integer Gram and distinguished Verlinde enumerations for R8_3_928 and R8_3_941 each leave 343 restriction patterns. Their previously verified Galois–pseudo-unitary gates permit the following necessary conditions to be used for general complex categorifiability.

Proposition 3.28. *Every categorification of either quartic ring, after this Galois transport, has a unique central invertible G whose restriction is the nontrivial central source invertible g . It satisfies $G^2 = 1$. For R8_3_928 one has $\theta_G^2 = -1$, whereas for R8_3_941 one has $\theta_G^2 = 1$.*

Proof. The complete distinguished restriction list contains only the tensor unit with restriction supported on $\mathbf{1}$. Adjunction therefore makes the forgetful kernel Vec . If a central simple restricts to mg , its product with its dual restricts to $m^2\mathbf{1}$; the kernel statement and rigidity force $m = 1$. Two such lifts differ by a kernel invertible, so there is at most one. All 343 complete integer patterns force exactly one such lift. The same kernel argument gives $G^2 = 1$.

The complete earlier rotation certificates force $\nu_2(g) = -1$ and $+1$, respectively. The forgetful functor preserves the canonical positive pivotal structures. On the pointed order-two subcategory, the indicator is the normalized cocycle value $\omega(g, g, g) \in \{1, -1\}$; this also follows from [91, Definition 2.2 and Proposition 4.1] with trivial subgroup. The order-two hexagon gives $c(G, G)^2 = \omega(g, g, g)$, and balancing gives $1 = \theta_G^2 c(G, G)^2$. Hence $\theta_G^2 = \nu_2(g)$. $\square$

For the semionic case, the monodromy charge of any central simple Z is ± 1 , since it is an eigenvalue of the involution fusion matrix N_G . Balancing gives

$$\theta_{GZ} = \theta_G M_{G,Z} \theta_Z.$$

Thus G has no fixed simple. The 40 high restriction vectors fixed by multiplication by g must have even multiplicity; exactly 15 of the 343 integer patterns remain. The five distinguished low objects translate to five distinct high objects of multiplicity one, with twists $\pm i$. Their normalized relative moments at powers $1, \dots, 6$ are $(0, -1, 0, 1, 0, -1)$, and the second and sixth total moments of every fixed restriction type vanish.

Imposing these conditions together with the full height-six tensor rotation equations, all distinguished STS equations, and the fixed integer Gram pattern gives exact Farkas contradictions for five patterns and exact rational feasible vectors for ten. This calculation uses the matrix-character normalization $D/(f_E \dim E)$ throughout.

The remaining correlation can be retained directly. Write $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and let tr_K denote normalized relative trace. Each unordered G -orbit contributes a joint tuple

$$(\text{tr}_K(\theta^k + (i\theta)^k))_{k=1}^6.$$

The opposite sign is covered by interchanging the two roots. Proposition 3.3 gives the finite order box 288, together with the all-zero tail. Multiplication by i cannot remove a prime-power valuation exceeding this bound. Direct exact averaging of all powers of ζ_{288} over the 24 automorphisms fixing K produces exactly 30 joint pair types. A pair variable has multiplicity mass two. Replacing each fixed restriction block by these pair columns preserves every Gram, rotation, moment, and STS row; the other high columns and auxiliary rotation variables are retained once. All ten patterns still have exact rational feasible vectors in the resulting 873-equation, 2227-variable systems.

The simultaneous eigenvalue tuple

$$\left(\frac{D \chi_E(FZ)}{f_E \dim(E) \dim(Z)} \right)_E$$

has Galois-invariant multiplicity: the distinguished center fusion matrices are commuting integer matrices, and Galois transport preserves the dimensions of their joint eigenspaces. Testing all four automorphisms of K leaves 255 of the original 343 patterns for each quartic ring; all 15 semion-parity patterns pass this additional condition. The same full-tuple test preserves all four complete integer patterns for R8_3_721 and all 1238 cubic/spectral integer patterns for R8_3_1154, closing this finite test across the four remaining noncommutative cases. An actual $\mathcal{Z}(\text{Fib})$ control checks the joint-spectrum normalization.

The pair variables themselves count actual orbits and hence are integers; normalized relative trace changes their values, not their multiplicities. The auxiliary rotation variables similarly count individual eigenvalues. A bounded integer search finds exact nonnegative integer witnesses for four of the ten remaining patterns, verified against all 873 equations. The other six integer-feasibility questions remain uncertified: numerical infeasibility is not used as an obstruction. Exact modular lattice tests at 2, 3, 5, 7, 11, 13, including removal of every variable forced zero by nonnegative rows, do not resolve them.

These vectors are feasible necessary data, not constructions. In particular, the forced central semion does not imply a tensor factorization of the original category. The source-bound read-only entry point for both finite campaigns is

`r928_semion_rotation/verify_all.py`.

The independent orbit and correlated-trace checkers are registered separately. Actual double-semion center data control the STS and twist normalizations.

Example 3.29 (A scope control for fermionic dimension arguments). A free central fermion, even in a unitary category and with an all- M -type superquotient, need not admit a scalar ring character taking that fermion to -1 . This supplies a control against a proposed obstruction for R8_3_941, rather than a model for that ring.

Let $V = C_2^2$ and $b(g, h) = g_1 h_1 + g_2 h_2 + g_2 h_1$ modulo two. The extension with product $(g, a)(h, c) = (g + h, a + c + b(g, h))$ is Q_8 , with central involution $f = (0, 1)$. In lexicographic order on $\{1, 2, 3\}^3$, using the encoding $g = g_1 + 2g_2$, take the normalized function $A : V^3 \rightarrow \mathbb{Z}/4$ whose nonunit table is

$$(1, 3, 0, 0, 3, 1, 3, 0, 1, 0, 3, 1, 2, 1, 1, 0, 0, 0, 3, 2, 3, 0, 0, 0, 3, 0, 1).$$

It satisfies

$$A(g, h, k) + A(g, h + k, l) + A(h, k, l) - A(g + h, k, l) - A(g, h, k + l) = 2b(g, h)b(k, l) \pmod{4}.$$

Consequently $\omega((g, a), (h, c), (k, d)) = i^{A(g, h, k) + 2db(g, h)}$ is a normalized unitary three-cocycle. This is the pointed supercocycle construction of [89, Theorem 5.4 and Corollary 5.8]. The receipt also checks all 8^4 ordinary pentagons directly.

The half-braiding $c_{f, (g, a)} = (-1)^a$ satisfies all central hexagons and $c_{f, f} = -1$. Every associator involving two copies of f is one, so the tensor-square half-braiding is $c_{f, x}^2 = 1$ and $f \otimes f$ is the unit in the center. The restriction of ω to $\langle f \rangle$ is trivial. In the positive spherical structure the twist is $\theta_f = c_{f, f} = -1$; compare [4, Section 2.11.5, equation (26)]. The underlying f -action is free, hence the fermionic quotient has four M -type simple orbits. Nevertheless f is a commutator in Q_8 , so every scalar character of $\mathbb{C}[Q_8]$ sends f to $+1$. Thus the absence of a character with value -1 is not by itself a categorification obstruction. No novelty claim for this pointed construction is intended.

Proposition 3.30 (Complete low-frame convex hull). *Let R be a commutative fusion ring and fix its positive dimension character. Let B be the induction Gram matrix. For each distinguished nonunit constituent of $I(1)$, let its finite set of possible restriction columns be fixed by its formal codegree.*

Enumerate all joint low frames L with the prescribed unit row, retaining only those for which $B - LL^\top$ is entrywise nonnegative and positive semidefinite. Columns with identical allowed sets may be ordered to remove permutation duplicates. In any finite central-induction linear relaxation, replace the independent low-column choices by the convex hull of these complete frames. The resulting system remains necessary for pseudo-unitary categorification.

Proof. Every actual low restriction matrix has one representative in the enumeration: the remaining center simples express $B - LL^\top$ as a sum of Gram matrices of nonnegative integral columns. Let a_c denote the full constraint column of a low constituent, including its Gram and twist-moment contributions. Assign a variable λ_L to each complete frame and use the column $a_L = \sum_{c \in L} a_c$, omitting the separately fixed tensor unit. The original low-choice equations become $\sum_L \lambda_L = 1$. An actual category gives a one-hot choice of L , so it satisfies the new system. All other columns and the rotation equations are unchanged. $\square$

The implementation stores these columns implicitly. A row functional is evaluated on every complete frame by summing its constituent scores; high columns use an exact factorization into Gram and relative-trace terms. Column generation proposes row functionals on restricted cones, but an exclusion is accepted only after checking every column of the complete finite domain over the integers. Conversely, a recorded nonnegative rational primal solution is checked against every equation. Such a solution establishes survival of this necessary-condition system, not existence of a category.

Applied to R8_3_925, the complete-frame system at rotation height 4 has an exact separating functional for each of the two admissible second-indicator assignments. The full oriented signed lifting certificate closes every remaining sphericalization branch, giving an exclusion over $\mathbb{C}$ without a pivotality assumption. The source-bound certificate is `global_induction/frame_results/R8_3_925_h4.json`; its canonical exact-only replay is `global_induction/verify_accepted_frames.py`.

The same complete-frame criterion at height 6 excludes R8_3_1739. Its domain has 6891 low frames, 2863 high restriction columns, 38 joint relative-trace types, 323 equations, and 115762 variables. Both admissible second-indicator assignments have exact full-domain separators. The complete oriented signed lift also verifies, so this is again an exclusion over $\mathbb{C}$. The source-bound certificate is `global_induction/frame_results/R8_3_1739_h6.json`, replayed by the same canonical endpoint.

Proposition 3.31 (Partial modular spectrum and complete character labeling). *Let R be the commutative Grothendieck ring of a pseudo-unitary fusion category, with global dimension D , irreducible characters χ_E , and formal codegrees f_E . If v is the restriction of a simple object of its center, then*

$$\frac{D\chi_E(v)}{f_E \dim(v)}$$

is an algebraic integer for every E . Moreover, the distinguished low restriction columns admit a bijective labeling v_E satisfying $\dim(v_E) = D/f_E$ and

$$(D/f_E)\chi_E(v_F) = (D/f_F)\chi_F(v_E).$$

Proof. In the unnormalized modular matrix of the center, Deshpande’s induction–restriction character formula, equation (39) of *On centers of bimodule categories and induction-restriction functors* (arXiv:1611.04486), gives $S_{A_E, Z} = (D/f_E)\chi_E(v_Z)$. Symmetry and the modular character formula imply that $S_{A_E, Z}/\dim(Z)$ is a fusion-ring character value, hence integral. Symmetry restricted to the distinguished objects gives the second assertion. $\square$

The implementation reconstructs and verifies the complete character table in $\mathbb{Q}(\zeta_{24})$, whose integer ring is $\mathbb{Z}[\zeta_{24}]$, and exhausts all dimension-compatible bijective low-column labelings. Thus neither filter loses a possible categorification. For R8_3_926, the first filter retains 2723 of 2863 high restriction candidates, and symmetric labeling retains 11 of 6891 complete low frames. Exact

complete-domain separation of both indicator cases, followed by all 16 oriented signed branches, excludes this ring over $\mathbb{C}$. The source-bound certificate is `global_induction/spectral_results/R8_3_926_h4.json`.

Proposition 3.32 (Distinguished modular symmetry). *Let R be a commutative fusion ring, with positive dimension D , ring characters χ_E , and formal codegrees f_E . Enumerate every possible complete restriction frame of the distinguished central simples in $I(\mathbf{1})$, including all dimension-compatible character labels. A pseudo-unitary categorification requires at least one such labeling with*

$$\frac{D}{f_E} \chi_E(v_F) = \frac{D}{f_F} \chi_F(v_E) \quad \text{for all } E, F.$$

Proof. The distinguished central simple A_E has dimension D/f_E and $S(A_E, Z) = (D/f_E) \chi_E(FZ)$ [98, Lemma 4.3 and Corollary 4.5]. The assertion is therefore the symmetry of the unnormalized modular matrix. Every actual frame has a representative in the complete low-column enumeration, since its remaining restriction columns give an entrywise nonnegative positive semidefinite residual Gram matrix. The dimension-compatible bijection search retains every possible character labeling. $\square$

For `R8_3_1463`, the seven raw candidate sets have sizes 20, 9, 4, 4, 12, 12, 6. All 2024 complete frames fail this symmetry test. The exact character table splits in $\mathbb{Q}(\sqrt{3}, \sqrt{5})$. The ordinary character alternatives and all four oriented signed branches also close, giving an exclusion over $\mathbb{C}$ without a pivotality assumption. The canonical exact replay is `global_induction/verify_accepted_generic.py`; no high-column, phase, STS, or numerical solver computation is required for this proof.

3.40. A complete octic low- S obstruction for `R8_3_901`. All eight characters of `R8_3_901` lie in $L = \mathbb{Q}(\sqrt{3}, \sqrt{5}, i)$. The primitive element $u = \sqrt{3} + \sqrt{5} + i$ satisfies

$$u^8 - 28u^6 + 238u^4 + 372u^2 + 441 = 0.$$

The exact character adapter checks all multiplication, duality and orthogonality identities in this field. For the distinguished central simples, impose $\beta_E \chi_E(v_F) = \beta_F \chi_F(v_E)$ in all eight rational coordinates, with no conjugation in this symmetric identity.

The seven raw restriction-column domains have sizes 6, 9, 4, 4, 12, 12, 20. The complete induction Gram enumeration leaves 2024 low frames. Each has exactly four dimension-compatible character bijections, and none of the 8096 bijections satisfies the symmetry identities. This uses neither high restriction columns nor a partial- S integrality filter nor linear programming. A separate checker enumerates full bijections before testing symmetry, independently of the primary recursive pruning. Its known $\text{Fib} \boxtimes \text{Vec}_{C_4}$ low-frame control has 576 permitted bijections, all of which satisfy the identities.

The complete general lift has two ordinary quartic character factors: the Frobenius–Perron Galois orbit and an orbit with unequal dimensions on a dual pair. Of the 32 oriented signed branches, 16 are gauge equivalent to the original ring. Each remaining branch has one character orbit violating duality and one excluded by the norm condition at the prime 3, with global-dimension polynomial

$$t^4 - 240t^3 + 4560t^2 - 28800t + 57600.$$

Consequently `R8_3_901` admits no complex fusion-category categorification, without a pivotality or unitarity assumption. The source-bound replays are `quadratic/octic_low_s_general.py` and `quadratic/review_octic_low_s.py`.

Proposition 3.33 (Character-labeled central induction). *Let R be a commutative fusion ring, K its positive dimension field, and L a verified splitting field of its ring characters. Put $\beta_E = D/f_E$. In a pseudo-unitary categorification, every central restriction column v satisfies*

$$\frac{\beta_E \chi_E(v)}{\dim(v)} \in \mathcal{O}_L \quad \text{for every } E.$$

Label each low constituent by its character and impose $\beta_E \chi_E(v_F) = \beta_F \chi_F(v_E)$ in all rational coordinates of L . These constraints may be imposed either on complete labeled frames or on independent low-choice simplices, with the latter giving a weaker necessary relaxation.

Proof. The character formula for central induction gives $S(A_E, Z) = \beta_E \chi_E(FZ)$ [98, Lemma 4.3 and Corollary 4.5]. Symmetry of S gives the low equalities, and $S(A_E, Z)/\dim(Z)$ is a fusion eigenvalue of the center, hence an algebraic integer. Every actual labeled frame gives one-hot choices in the independent simplices. Integrality in L is tested by the integer coefficients of the multiplication characteristic polynomial; this does not assume that the chosen primitive power basis is an integral basis. $\square$

If the required character products lie in the relative-trace field, one may additionally impose the center relation

$$\sum_Z \theta_Z \chi_E(v_Z) \chi_F(v_Z) = f_F \chi_E(v_F).$$

This is the unnormalized STS relation, using the center Gauss sum D and the twist-one distinguished objects. Complete high duality pairs are combined before taking a relative trace. Their twists agree and their character coefficients are conjugates, so the pair coefficient is real. Membership in the actual trace field is checked exactly before factoring it out of the trace.

3.41. A centerless rank-five Morita partner excludes R8_3_439. The preceding pointed-associator calculation makes full S_3 condensation unconditional for R8_3_439. The complete dual based-ring enumeration and frozen rank-five exclusions leave only $R_{5,3,40}$. Its basis $(1, s, V, P, Q)$ is self-dual, its dimensions are $(1, 1, 2, 3 + 2\sqrt{3}, 3 + 2\sqrt{3})$, and its dimension is $D = 48 + 24\sqrt{3}$. The Galois reduction permits us to take the original category pseudo-unitary; its Morita dual is then pseudo-unitary by Morita invariance of categorical and Frobenius–Perron dimensions [21, Theorem 2.15 and Corollary 8.14]; its canonical spherical structure is provided by [21, Proposition 8.23]. Its center has no nontrivial invertibles. The integral-center criterion therefore forbids every nonunit integral-dimensional central simple.

The formal codegrees of this dual ring are $D, D' = 48 - 24\sqrt{3}, 3, 4, 4$. The complete induced-unit search gives 24 low frames, of which five pass the ordinary moment conditions and just one passes full distinguished S symmetry:

$$(1, 0, 0, 0, 0), \quad (1, 0, 0, 1, 1), \quad (1, 1, 1, 2, 2), \quad (1, 0, 1, 1, 2), \quad (1, 0, 1, 2, 1).$$

The two assignments of the equal-codegree characters both fix the simple A corresponding to D' . The remaining Gram matrix is

$$Q = \begin{pmatrix} 4 & 2 & 4 & 4 \\ 2 & 6 & 7 & 7 \\ 4 & 7 & 34 & 31 \\ 4 & 7 & 31 & 34 \end{pmatrix}.$$

The exact high domain uses positivity of $Q - vv^\top$, the central simple dimension d -number condition [24, Theorem 1.8], and algebraic integrality of D/d_Z [21, Proposition 3.3]. There are 38 possible high restriction vectors. Four have integral dimension and one violates algebraic integrality of a distinguished S -ratio. Two independent complete integer enumeration algorithms find exactly 58 multiplicity patterns on the remaining vectors.

Let $\chi = (1, 1, 2, 3 - 2\sqrt{3}, 3 - 2\sqrt{3})$ be the conjugate-FP character. The distinguished simple A is self-dual: it is the unique simple of dimension $D/D' = 7 + 4\sqrt{3}$ among the constituents of the self-dual object $I(1)$ [74, Theorem 2.13]. The distinguished-row formula [98, equation (39)] and the Verlinde formula give

$$N_{A,A}^A = \frac{D}{(D')^3} \sum_{Z \in \text{Irr } Z(C)} \frac{\chi(FZ)^3}{\text{FPdim}(Z)}.$$

Counting the five low simples once and each high restriction with its integer multiplicity gives the following complete list:

$$\begin{array}{c|ccccc} N_{A,A}^A & (7 - \sqrt{3})/6 & 1/3 & (19 - 5\sqrt{3})/12 & (6 - \sqrt{3})/3 & (1 + 2\sqrt{3})/3 \\ \hline \text{number of patterns} & 16 & 16 & 16 & 6 & 4 \end{array}$$

No entry is an integer. Thus no possible associator on the forced rank-five dual can have the required center. Consequently R8_3_439 admits no complex fusion-category realization.

The exact replay reconstructs the complete low-frame domain, both integer enumerations, and every cubic value; a Fibonacci-center control checks the cubic normalization. Its canonical entry point is

`evidence/rank8_frontier_20261021/r439_centerless_rank5/verlinde_exclusion.py`

The earlier rational height-six rotation and *STS* relaxation has a feasible branch; the integer cubic test supplies the additional obstruction needed here.

3.42. A direct cubic central obstruction for R8_3_567. For R8_3_567, the general Galois-pseudo-unitary gate and the integral-center criterion permit the same integer central test directly on the rank-eight ring. Its global dimension is $D = 48 + 24\sqrt{3}$; its formal codegrees with representation dimensions are $(D, 1)$, $(48 - 24\sqrt{3}, 1)$, $(12, 1)$, $(12, 1)$, $(3, 2)$. The full character table is computed in $\mathbb{Q}(\sqrt{3}, i)$. Complete distinguished S symmetry leaves one low frame:

$$\begin{aligned} & (1, 0, 0, 0, 0, 0, 0, 0), \quad (1, 0, 0, 1, 1, 0, 0, 0), \\ & (1, 0, 0, 0, 1, 0, 0, 0), \quad (1, 0, 0, 1, 0, 0, 0, 0), \quad (2, 1, 1, 2, 2, 0, 0, 0). \end{aligned}$$

The last simple occurs twice in $I(1)$, but once in sums over central simples. The distinguished row has the normalization

$$S_{A_E, Z} = \frac{D}{f_E \dim E} \chi_E(FZ)$$

from [98, equation (39)] and character orthogonality.

The exact Gram bound, central dimension d -number condition [24, Theorem 1.8], and algebraic integrality of D/d_Z [21, Proposition 3.3] give 45 high restriction vectors. Two have integral dimension and are forbidden. Nine further vectors violate algebraic integrality of $S_{A_E, Z}/d_Z$; the complex character rows are retained in this test. The remaining 34 vectors have the form (u, u, p, q, w, w, w) , and their complete Gram equation reduces to

$$\sum_Z m_Z \begin{pmatrix} u_Z \\ w_Z \\ p_Z \\ q_Z \end{pmatrix} (u_Z \quad w_Z \quad p_Z \quad q_Z) = \begin{pmatrix} 4 & 0 & 4 & 4 \\ 0 & 4 & 6 & 6 \\ 4 & 6 & 42 & 31 \\ 4 & 6 & 31 & 42 \end{pmatrix}.$$

Two independent complete integer searches agree on 1,071 solutions; duality of restrictions leaves exactly 155.

For the self-dual distinguished simple A corresponding to $D' = 48 - 24\sqrt{3}$, the cubic coefficient is again

$$N_{A,A}^A = \frac{D}{(D')^3} \sum_Z \frac{\chi(FZ)^3}{d_Z}.$$

Its complete list consists of 84 values $1/3$ and 71 values $4/3$. Every integer Gram completion therefore contradicts the Verlinde formula. Thus R8_3_567 has no complex fusion-category realization. The read-only exact replay, including both enumerations and a known Fibonacci-center normalization control, is

`evidence/rank8_frontier_20261021/r567_centerless_integer/verlinde_exclusion.py`

3.43. Integer central data and correlated rotations exclude R8_3_455. The general Galois-pseudo-unitary gate also applies to R8_3_455. Its dimension field is $\mathbb{Q}(\sqrt{3}, \sqrt{13})$, with $D = 39 + 9\sqrt{13}$. The complete low/ S search has one frame and 13 high restriction vectors. Two independent complete integer Gram searches give 812 patterns. Twenty low cubic Verlinde coefficients leave 24; all pass the required field-Galois eigenvalue multiplicity balance. When two scalar characters share the FP codegree, the FP character is explicitly identified as the tensor-unit row.

Two distinguished central simples forget to **1**. Besides the unit, the other is an invertible G with $G^2 = \mathbf{1}$ and $\theta_G = 1$. If $S_{G,Z}/d_Z = -1$, balancing gives $\theta_{GZ} = -\theta_Z$, so $GZ \not\cong Z$. Since $F(GZ) = F(Z)$, each such restriction column has even multiplicity. This leaves 12 patterns. All 120 low-pair products admit nonnegative integer aggregate forgetful decompositions; the obstruction requires further coupling to the twists.

The quadratic conductors are 12, 13, 156, and the Cauchy theorem [61] supplies twist-order prime support $\{2, 3, 13\}$. Proposition 3.3 at height six gives

$$B_6 = 2^4 \cdot 3^2 \cdot 13 = 1872,$$

and exactly 55 joint relative-trace types in basis $(1, \sqrt{3}, \sqrt{13}, \sqrt{39})$. This bounds the retained trace domain, not the categorical exponent. The four sign choices in the quadratic Gauss projections are linked across all retained powers.

For every remaining integer pattern, all four second-indicator choices are tested against the full divisor-power tensor rotations through height six, the full distinguished STS equations, and the fixed integer high multiplicities. The two equal-codegree assignments give identical equations because their restriction columns coincide. Each of the 48 systems has 381 equations and 792 nonnegative variables. One common exact integer Farkas vector verifies $A^\top y \geq 0$ and $b^\top y < 0$ in every case. Thus R8_3_455 has no complex fusion-category realization.

The field adapter is checked against the complete earlier $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ trace domain, independent cyclotomic relative traces, and the actual center $Z(\text{Vec}_{C_2})$ through height six. The canonical read-only replay is

`evidence/rank8_frontier_20261021/r455_center_integer/verify_all.py`

The same complete labeled-frame STS criterion excludes R8_3_903. Its complete low-column enumeration and exact character-label tests leave four labeled choices. Algebraic integrality retains 2919 high restriction vectors, forming 1532 duality orbits. At height four the necessary system has 341 equations and 52127 columns. Two exact integer functionals separate all eight permitted indicator assignments. The spherical alternatives and all 64 oriented signed branches also close, so the obstruction holds for general complex fusion categories. The complete sparse phase-I algorithm is used only for discovery; `global_induction/verify_accepted_complete.py` reconstructs and verifies the exact whole-domain certificates without numerical optimization.

The complete labeled-frame STS criterion also excludes R8_3_1615. Four labeled low choices survive the complete restriction and character-label tests. The exact integrality filter retains 3003 high restriction vectors, each fixed by duality. The height-four system has 341 equations and 102141 columns. Five exact integer functionals separate all eight permitted indicator assignments. The spherical alternatives and all eight oriented signed branches close, giving an obstruction for general complex fusion categories. The exact-only canonical replay is `global_induction/verify_accepted_complete.py`.

For R8_3_1617, the complete labeled-frame STS test leaves six labeled low choices and 3835 high restriction vectors, each fixed by duality. The height-four system has 341 equations and 130431 columns. Two exact integer functionals separate the two permitted indicator assignments. The spherical alternatives and both oriented signed branches close, proving an exclusion for general complex fusion categories. The exact-only canonical replay is `global_induction/verify_accepted_complete.py`.

For R8_3_1660, four labeled low choices and 2937 high restriction vectors survive the complete restriction and character tests. All high vectors are fixed by duality. The height-four *STS* system has 341 equations and 99897 columns. Three exact integer functionals separate all eight permitted indicator assignments. The spherical alternatives and all eight oriented signed branches close, proving an exclusion for general complex fusion categories. The exact-only canonical replay is `global_induction/verify_accepted_complete.py`.

For R8_3_1661, the complete labeled-frame *STS* test leaves four labeled low choices and 3051 high restriction vectors, each fixed by duality. The height-four system has 341 equations and 103773 columns. Two exact integer functionals separate the two permitted indicator assignments. The spherical alternatives and all six oriented signed branches close, proving an exclusion for general complex fusion categories. The exact-only canonical replay is `global_induction/verify_accepted_complete.py`.

This complete labeled-frame system at height 4 excludes R8_3_985. The modular filters leave four labeled low frames from 118696 complete low frames and 3045 high restriction vectors. Both indicator assignments have exact full-domain separating functionals; all 24 oriented signed branches also close. The canonical replay is `global_induction/verify_accepted_sts.py`.

The full-field atomic symmetry system excludes R8_3_3413 at height 4. Its characters split in the real degree-six field $\mathbb{Q}(\sqrt{5}, \zeta_9 + \zeta_9^{-1})$. The cone has 253 equations, 4485 variables, 367 high restriction vectors and ten joint trace types. Its sole indicator assignment has an exact separating functional, and its complete signed lift verifies. This proof uses full-field symmetry and integrality, without factoring character products outside the phase-trace field. The canonical replay is `global_induction/verify_accepted_atomic.py`.

The full-field atomic spectral criterion also excludes R8_3_1055. Its characters split in the field

$$\mathbb{Q}(\sqrt{37}, \sqrt{-3}).$$

Exact algebraic integrality leaves 566 high restriction vectors in 352 duality orbits. The height-four cone has 211 equations and 16695 columns; its full complex *S*-symmetry equations do not require a phase-trace extension. Both indicator assignments have exact separating functionals. The spherical alternatives and all four oriented signed branches also close, giving a general complex exclusion. Independent normal and optimized replays check every column for both rays. The canonical exact-only replay is `global_induction/verify_accepted_atomic.py`.

The character-labeled atomic *STS* relaxation also excludes R8_3_3258. All eight characters split in its real dimension field $\mathbb{Q}(\sqrt{5})$. The exact spectral filter leaves 1471 high restriction vectors, and the height-four cone has 241 equations, 17619 variables, and ten joint phase-trace types. Its sole indicator assignment has an exact full-domain separating functional. The spherical alternatives and the complete oriented signed lift also verify, so this is an exclusion over $\mathbb{C}$. The canonical exact-only replay is `global_induction/verify_accepted_atomic_sts.py`.

4. A FURTHER SIMULTANEOUS CENTER-PRODUCT TEST

This earlier exact continuation changed no general exclusion or realization count. It strengthened the necessary center data for R8_3_721 and R8_3_1154. A surviving tuple below is not a center fusion ring or a categorical model. The later Q-system construction of R8_3_721 appears in Theorem 3.16. The previously verified signed-sphericalization and Galois-pseudo-unitarity gates for these two rings make the pseudo-unitary tests necessary for general complex categorification.

Lemma 4.1 (Simultaneous pair-product Gram condition). *Let $\mathcal{C}$ be pseudo-unitary, put $D = \text{FPdim}(\mathcal{C})$, and let $A_0 = \mathbf{1}, A_1, \dots, A_s$ be self-dual simple objects of its center whose complete unnormalized S rows are known. If d_Z is the dimension of a central simple Z , then*

$$K_{ij,kl} = \frac{1}{D^2} \sum_Z \frac{S_{A_i,Z} S_{A_j,Z} S_{A_k,Z} S_{A_l,Z}}{d_Z^2} = \sum_Y N_{A_i,A_j}^Y N_{A_k,A_l}^Y.$$

Subtract the summands for $Y = A_0, \dots, A_s$ to obtain G . Let M have the restriction vectors of all individual remaining central simples as columns, and let T have the residual forgetful pair products as columns. Let n_{high} be the number of remaining central simples. Their pair multiplicities form a matrix

$$X \in \mathbb{Z}_{\geq 0}^{n_{\text{high}} \times \binom{s+1}{2}}, \quad MX = T, \quad X^t X = G.$$

Repeated restriction vectors are repeated columns of M ; they are not identified as central simples.

Proof. Apply the ordinary Verlinde formula twice and use $S^2 = D^2 C$, where C is charge conjugation. Self-duality of the four displayed objects gives the stated bilinear pairing. No assumption of a C^* structure is used. Subtracting the known simple coefficients gives the Gram identity, and applying the tensor forgetful functor to each pair gives $MX = T$. $\square$

The normalization uses the center dimension D^2 . In the noncommutative source case the complete distinguished row is $S_{A_E, Z} = D\chi_E(FZ)/(f_E \dim E)$; the matrix-block constituent is counted once as a central simple. As an independent control, the electric objects of $Z(\text{Vec}(S_3))$ reproduce all 27 ordered cubic coefficients and all 81 ordered fourth moments of $\text{Rep}(S_3)$ [100, Section 2.2]. In particular the source block with $\dim E = 2$ and $f_E = 3$ has normalization $6/(3 \cdot 2) = 1$. The categorical input is ordinary Verlinde, equivalently the regular module specialization of the twisted Verlinde formula [99, Theorem 1.1].

4.1. Forced pair products for R8_3_721. All five distinguished objects are self-dual: their complete source trace characters satisfy $\chi_E(x^*) = \chi_E(x)$, so the corresponding central S rows are invariant under duality, and nondegeneracy identifies each object with its dual. Start with the four complete integral Gram patterns already obtained in the frozen calculation, and retain both symmetric character assignments. All 35 fourth moments for the four nonunit low objects are nonnegative integers in each of these eight cases.

Order pairs as 11, 12, 13, 14, 22, 23, 24, 33, 34, 44. In every case the residual G has rank four and M has rank three. If X_0 is the exact minimum-norm solution of $MX = T$, then

$$G - X_0^t X_0 = \frac{2}{3} h^t h, \quad h = (0, 0, 0, 0, 1, 1, -1, 1, -1, 1).$$

Thus every solution is $X = X_0 + th$, where $Mt = 0$ and $\sum t_i^2 = 2/3$. Each t_i lies in $(-1, 1)$; integrality of one coordinate fixes it modulo $\mathbb{Z}$. Enforcing integrality and nonnegativity in all ten coordinates gives a unique minimum-square choice in each row. The sum of those minima is already $2/3$. Consequently the entire integral matrix X is unique. Direct exact substitution checks both original identities.

Here are its row types, with the high-column indices of the frozen Gram enumeration:

type	11	12	13	14	22	23	24	33	34	44
0	2	2	2	1	2	3	1	2	1	1
1	4	4	4	2	4	6	2	4	2	2
3	2	2	2	1	3	2	0	3	0	0
4	4	4	4	2	4	4	2	4	2	0
5	4	4	4	2	4	4	2	4	2	1
8	8	8	8	4	8	8	4	8	4	0

Denote the type-0-or-1 group by **A**, the type-3 singleton by **B**, the type-4-or-8 group by **C**, and the eight type-5 simples by **D**. The sizes m_A, m_C are independently one or four: a singleton **A** uses type 1 and four copies use type 0; a singleton **C** uses type 8 and four copies use type 4. The resulting center ranks are 16, 19, 19, 22.

4.2. Complete low-action extension of these patterns. Write N_i for fusion by A_i . The known vectors $e_0, \dots, e_4$ and $N_i N_j e_0$, $1 \leq i \leq j \leq 4$, span a nine-dimensional space. Put $u = 33 + 8\sqrt{17}$ and $u' = 33 - 8\sqrt{17}$. The complete simultaneous eigenvalue tuples of (N_1, N_2, N_3, N_4) are

tuple	multiplicity
$(u, u+1, u+1, (u+1)/2)$	1
$(u', u'+1, u'+1, (u'+1)/2)$	1
$(1, -2, 2, 1), (1, 2, -2, 1), (1, 2, 2, 5)$	1 each
$(1, -2, -2, 1)$	m_A
$(1, 2, 2, -3)$	1
$(1, 0, 0, -1)$	m_C
$(-1, 0, 0, 0)$	8

Every tuple has positive vacuum weight $\sum d_Z^2/D^2$. The vacuum cyclic space therefore has dimension nine, so it is exactly the already known span. Select its basis columns V , remembering the corresponding words p_α of degree at most two. With $G_0 = V^t V$, its forced action, extended by zero on the orthogonal complement, is

$$F_i = V G_0^{-1} B_i G_0^{-1} V^t, \quad (B_i)_{\alpha\beta} = D^{-2} \sum_Z d_Z^2 p_\alpha(\lambda_Z) \lambda_{i,Z} p_\beta(\lambda_Z).$$

These degree-at-most-five moments give exact rational matrices. Let $Q = I - V G_0^{-1} V^t$. Its range is the sum of the within-group difference spaces of **A**, **C**, **D**. The remaining simultaneous spectral projectors P_A, P_C, P_D have ranks $m_A - 1, m_C - 1, 7$, are mutually orthogonal, and sum to Q . Every possible extension is consequently

$$\begin{aligned} N_1 &= F_1 - Q + 2(P_A + P_C), & N_2 &= F_2 - 2P_A, \\ N_3 &= F_3 - 2P_A, & N_4 &= F_4 + P_A - P_C. \end{aligned} \tag{1}$$

This describes the entire remaining spectral domain, not a choice of canonical projectors.

Proposition 4.2. *The patterns $(m_A, m_C) = (1, 4)$ and $(4, 1)$ are impossible. The pattern $(1, 1)$ has a unique nonnegative integral tuple of the four necessary low fusion actions. The pattern $(4, 4)$ has exactly 46 labelled such tuples, forming three classes under independent permutations within the equal-restriction groups. All surviving tuples satisfy the exact forgetful intertwining relations. These assertions do not construct a full center fusion ring or a category.*

Proof. For $(1, 4)$, $P_A = 0$, and the **C** block of the forced matrix $N_2 = F_2$ is $(9/2)J_4$, contradicting integrality. For $(4, 1)$, $P_C = 0$, and the fixed **A** block of $N_1 + N_2 = F_1 + F_2 - Q$ is $(5/2)J_4 - I_4$, again contradicting integrality. Both statements hold for both symmetric character assignments. For $(1, 1)$ both projectors vanish; exact substitution in (1) gives the asserted unique tuple.

It remains to give a complete finite analysis when both sizes are four. The fixed within-group blocks of F_1, F_2, F_3, F_4 are respectively

$$\mathbf{A}: \quad (5/4)J_4, J_4, J_4, (3/4)J_4; \quad \mathbf{C}: \quad (17/4)J_4, (9/2)J_4, (9/2)J_4, (9/4)J_4.$$

Use the elementary identity $E_{ii} = \sum_j E_{ij}^2$ for an orthogonal projector E . Since $0 \leq P_A, P_C, P \leq Q$ as orthogonal subprojectors, their diagonal entries on **A** and **C** are at most $Q_{ii} = 3/4$. All hidden projectors annihilate the constant vector of each high group.

Set $P = P_A + P_C$. Integrality of N_1 forces its diagonal entries in either group to be $1/4$ or $3/4$, and its within-group off-diagonal entries to be odd multiples of $1/4$. In either case the square of the diagonal plus three times $(1/4)^2$ already equals the diagonal. Hence every within-group off-diagonal absolute value is $1/4$, and all entries outside that group vanish.

Nonnegative integrality of N_2 gives $(P_A)_{ii} \in \{0, 1/2\}$ on **A**, with half-integral within-group entries. A zero diagonal gives a zero row. A diagonal $1/2$ and zero group row sum force an off-diagonal entry at most $-1/2$; its norm identity is then exhausted. Thus P_A on **A** is a sum Q_A of disjoint

matching projectors $(e_j - e_k)(e_j - e_k)^\dagger/2$, with no cross entries. On $\mathbb{C}$, integrality gives diagonal $1/4$ or $3/4$ and odd-quarter off-diagonal entries. The same norm argument gives off-diagonal absolute values $1/4$ and no cross entries.

The diagonal integrality of $N_4 = F_4 + 2P_A - P$ now forces every diagonal of P in both groups to be $3/4$. Zero row sums force all off-diagonals to be $-1/4$. Therefore $P = E_A + E_C$, the two canonical difference projectors; their ranks exhaust its rank six, so it vanishes on D . On $\mathbb{C}$ the complement $E_C - P_A$ has diagonal zero or $1/2$ and off-diagonal zero or $-1/2$, so it too is a matching projector, say Q_C . The rank-three condition on P_A says the two matchings have equal size. Exhaustively,

$$P_A = Q_A + E_C - Q_C, \quad P_C = E_A - Q_A + Q_C, \quad P_D = E_D.$$

There are 1, 6, 3 matchings on four labelled vertices of sizes 0, 1, 2, giving $1 + 6^2 + 3^2 = 46$ choices and three permutation classes. All these matrices pass direct exact substitution in (1).

For every surviving tuple the verifier checks symmetry, nonnegative integrality, the unit column, all forced pair columns, pairwise commutation, projector ranks and orthogonality, spectral trace and square trace, and $M_{\text{full}}N_i = L_{F(A_i)}M_{\text{full}}$. This proves the stated necessary-action result. $\square$

The high-block traces and squared Frobenius norms provide compact checks:

center rank	$(\text{tr } H_i)_{i=1}^4$	$(\text{tr } H_i^2)_{i=1}^4$
16	(48, 56, 56, 28)	(2798, 2880, 2880, 746)
22	(54, 50, 50, 28)	(2804, 2892, 2892, 752)

The full matrices, source bindings, matching domains and exact replays are retained with `quarti c_center_challenge.py` and `r721_center_extension.py`. Checks remain active under Python optimization. The two surviving Gram patterns still require products of arbitrary high objects, a full based ring and categorical coherence.

4.3. A bounded refinement for R_{1154} . We apply Lemma 4.1 to the complete domain of 1,238 cubic-surviving integral Gram patterns at low index 31. These patterns are indexed in the frozen 14,211-pattern integral Gram file; there are 16 ordered high restriction types. The exact verifier rebuilds the cubic coefficient rows and the complete survivor list, checks all 35 scalar fourth moments, and verifies the ordered high restriction domain. All patterns have the same matrix $Q = MM^\top$ of rank 3. The simultaneous pair-product Gram matrix G also has rank 3.

Let X_0 be the minimum-norm solution of $MX = T$. Use the pair order

$$(11, 12, 13, 14, 22, 23, 24, 33, 34, 44).$$

Put

$$h = (1, 0, 1, -1, 0, 0, 0, 1, -1, 1).$$

For every pattern the exact residual is

$$G - X_0^\top X_0 = \alpha h^\top h, \quad \alpha \in \{92/205, 262/205, 672/205, 1082/205\}.$$

In particular, all 1,238 patterns pass positivity and the residual rank bound. Every required non-negative integral factorization must therefore have

$$X = X_0 + th, \quad Mt = 0, \quad \sum_z t_z^2 = \alpha.$$

Indeed, the minimum-norm solution is orthogonal to $\ker M$, so subtracting its Gram leaves the Gram of $X - X_0$; the displayed rank-one form forces each row of this difference to be a scalar multiple of h .

For an individual high row, set $q = (X_0)_{z,11}$. Since $h_{11} = 1$, integrality forces $t_z = n - q$ with $n \in \mathbb{Z}$. The other entries give integer congruences and nonnegativity bounds $L \leq n \leq U$, all checked exactly. The minimum of $(n - q)^2$ on this integer interval occurs at the clipped floor or ceiling of q . Summing these minima with the exact high multiplicities gives a necessary lower bound for α . This excludes 892 patterns.

A second screen retains the finite individual domains satisfying $(n - q)^2 \leq \alpha$. It uses lower and upper bounds for the sum of squares and coordinatewise interval bounds for $Mt = 0$. A candidate value is removed when fixing that one row makes a lower norm bound or a linear interval inconsistent; the same removal is valid for all interchangeable rows of that restriction type. Iterating these necessary tests requires at most three passes, with at most four candidate values per coordinate. No coupled integer assignment is enumerated. A further 54 patterns are excluded: 34 by an upper norm bound, 16 by a linear interval bound, and 4 by a lower norm bound.

α	patterns	minimum-square exclusions	further exclusions	remaining
92/205	8	8	0	0
262/205	864	852	6	6
672/205	300	32	48	220
1082/205	66	0	0	66
total	1238	892	54	292

The eight exclusions in the frozen height-six campaign have indices

$$207, 208, 376, 377, 898, 899, 1407, 1408,$$

and are exactly the first row of the table. Thus the combined exclusion set has size 946, including 938 additional patterns beyond that campaign; all 292 surviving domains also survived height six. This is a refinement of necessary central data, not a fusion-ring exclusion or a construction of a center. Joint integral solvability of $Mt = 0$ and the exact norm equation remains untested on those 292 domains. The general-complex scope uses only the previously accepted Galois-to-pseudo-unitary reduction.

The portable replay scripts are

```
r1154_quartic_necessary_filters.py
r1154_quartic_coordinate_propagation.py
r1154_quartic_reconcile.py.
```

They bind their inputs by SHA-256 and give byte-identical normal and optimized-Python receipts. The last script only reconciles the already verified height-six certificate indices; it does not rerun that campaign.

4.4. Separate replay of the center-product refinement. The source-bound proofs, complete surviving matrix tuples, rejected pattern indices and independent arithmetic reconstructions are in `evidence/rank8_center_followup_20261021/`. Run

```
python evidence/rank8_center_followup_20261021/verify.py --root .
```

in normal Python and with `-O`. This separately checked refinement preserves the original continuation ledger and its 25 additional ring exclusions. At this stage both rings were unresolved. These necessary data alone do not constitute fusion categories; the later construction of `R8_3_721` is Theorem 3.16. The remaining center patterns for `R8_3_1154` are refined below.

4.5. Pointed-associator rigidity for `R8_3_65`. This refinement removes categorical branches of the connected-algebra search, while leaving the fusion ring unresolved.

Proposition 4.3. *In any complex categorification of `R8_3_65`, the pointed subcategory has nonzero class $k \in H^3(C_4, \mathbb{C}^\times) \cong \mathbb{Z}/4$. Neither of the two rank-two based module actions can be categorified. Either rank-four action requires $k = 2$; if $k = 2$, the subgroup algebra $\mathbf{1} \oplus g^2$ has a Morita dual with the same Grothendieck ring `R8_3_65` and pointed class 2.*

Proof. If $k = 0$, the actual subgroup algebra for $H = \langle g^2 \rangle$ exists. Its four simple right modules arise from the four free right H -orbits. Each has two left H -linearizations, since the possible projective multiplier is trivializable by $H^2(C_2, \mathbb{C}^\times) = 0$. The bimodule dual has four invertibles forming $\widehat{H} \times (C_4/H) \cong V_4$, and a free orbit of four simples of dimension $d = (11 + 5\sqrt{5})/2$, where

$d^2 = 1 + 11d$. This argument uses a subgroup algebra, without assuming a central or Tannakian lift.

For a free noninvertible orbit over an abelian group G of order four, write $\rho g = \phi(g)\rho$ and $\rho^2 = u + (\sum_h n_h h)\rho$, with $\sum_h n_h = 11$. Associativity forces $\phi^2 = 1$ and invariance of n_h under $\{g\phi(g)^{-1}\}$. If $\phi \neq 1$, this subgroup has order two and would make the coefficient sum even. Thus $\phi = 1$. Duality and the odd coefficient sum permit a translate with $\rho^* = \rho$, $\rho^2 = 1 + N\rho$ and $N = N^*$. For V_4 the complete possibilities are the 27 unordered partitions of 11 into four parts. For a character parameter $m = \chi(N)$,

$$\text{Norm}(f) = 16(m^2 + 4), \quad \text{Norm}(\text{FPdim}(\mathcal{D})) = 2000.$$

The two-FP-value Galois/pseudo-unitary reduction applies to this dual. Integrality of its distinguished central dimension D/f , taking norms in the quadratic compositum, implies $125/(m^2 + 4) \in \mathbb{Z}$. Hence $|m| = 1$ or 11 . Only the partitions $(0, 0, 0, 11)$, $(0, 0, 5, 6)$ and $(2, 3, 3, 3)$ remain. They are excluded, respectively, by the general rank-two classification, the repository's unrestricted rank-four dimension bound, and the exact identification with the already excluded R8_3_1453. The accepted rank-four and R8_3_1453 results are explicit inputs to this deduction. Thus $k \neq 0$.

In a rank-two action each invertible fixes every module simple, so its scalar module associators would trivialize k . In a rank-four action the stabilizers are C_2 , so restriction of k to $H^3(C_2, \mathbb{C}^\times)$ is zero. The restriction map is reduction modulo two, forcing $k = 2$. Conversely, for $k = 2$ the subgroup algebra exists. Its dual has an order-four invertible group and the same free noninvertible orbit. The complete V_4 comparison above and the corresponding 21 selfdual C_4 coefficient vectors leave only the ring R8_3_65. The rank-four Morita module on the dual side forces its pointed class to be 2 as well. $\square$

The categorical cuts reduce the four quadratic rings' aggregate necessary lists to 144 algebra objects, 178 induction factors and 30 full-action classes. For R8_3_65 the remaining action ranks are 4, 4, 8, 8. The conditional subgroup dual supplies neither an associator for this ring nor an equivalence between the dual and the original category. The complete finite coefficient domains, theorem dependencies and source hashes are recorded in `evidence/rank8_next_20261021/quadratic/POINTED_COCYCLE.md` and `pointed_dual.py` in that directory, with an independent categorical review in `evidence/rank8_next_20261021/semion_fermion/construction/R65_ADDENDUM_REVIEW.md`.

4.6. Three remaining center patterns for R8_3_1154. The following arguments concern necessary induction and fusion data in the pseudo-unitary Drinfeld center. They do not construct a center, a fusion category, or a categorification of the source ring. In particular, R8_3_1154 remains unresolved. The previously established reductions to this pseudo-unitary center problem are used with their original scope.

Write $A_0 = \mathbf{1}, A_1, \dots, A_4$ for the five distinguished central simples, in the fixed character order of the retained induction frame, and write M for the forgetful multiplicity matrix. All five distinguished objects are self-dual. Consequently their fusion matrices are real symmetric, and scalar products of their fusion-word vectors are the exact vacuum Verlinde moments. Number the ten pair words by 11, 12, 13, 14, 22, 23, 24, 33, 34, 44.

Proposition 4.4 (Joint fourth-moment fibers). *Of the 292 restriction patterns left by the earlier coordinate bounds, exactly 224 admit the required joint pair-product fiber. Up to permutation of identical restriction columns, these patterns have exactly 2904 nonnegative integral pair-product completions.*

Proof. The previously computed minimum-norm solution X_0 to the forgetful pair-product equations has rank-one residual Gram matrix

$$G - X_0^T X_0 = \alpha h^T h, \quad h = (1, 0, 1, -1, 0, 0, 0, 1, -1, 1).$$

Every completion therefore has the form $X = X_0 + th$, with $Mt = 0$ and $t^\top t = \alpha$. The allowed integral first coordinate of each row gives a finite interval. The common rational denominator is 205. For each restriction type, enumerate all histograms in this interval with the prescribed number of copies; then impose the linear restriction equations and the exact squared norm jointly. Nonnegative excess costs relative to the nearest allowed coordinate give a finite, exhaustive transfer computation. Every retained histogram is checked against all seven non-unit restriction coordinates and all ten pair-product integrality and positivity conditions. An independent all-seven-coordinate transfer count reproduces the 68 empty fibers, 224 nonempty fibers, and 2904 histograms. $\square$

Proposition 4.5 (A forced distinguished action). *The action of A_2 excludes 216 of these 224 patterns, and 2830 of the 2904 pair-product completions. The eight remaining patterns are*

$$3993, 3994, 6970, 6971, 8765, 8766, 8771, 8772,$$

with 74 pair-product completions in total.

Proof. Adjoin the eight rows of M to the vacuum, the four distinguished simple vectors, and the ten pair-word vectors. Their span V has rank nine. It is invariant under N_{A_2} . On pair words this follows either from the exact fifth/sixth-moment Schur complement or from the vanishing of the triple residual coefficient on every word containing 2; on the forgetful rows it is the induction projection formula. This invariance was independently checked using the full 23-generator Schur identity.

The characteristic polynomial on V is

$$x^4(x+1)(x-3)(x+5)(x^2-270x+225).$$

The complete modular eigenvalue multiset shows that the complementary action is either zero or minus an orthogonal projector of rank three. In the first case the full matrix is forced, so any nonintegral entry excludes the completion. Equivalently, its remaining trace of the square is zero, which forces the symmetric complementary action to vanish.

In the second case write $N_{A_2} = F - E$, where F is its fixed action on V . There are four copies of high restriction type 10, and $F_{ii} = 67/4$ on each. Integrality and $0 \leq E_{ii} \leq 1$ force $E_{ii} = 3/4$ at these four entries. They exhaust $\text{Tr}(E) = 3$, so every other row and column of E vanishes. Orthogonality to the nonzero constant restriction row on this block gives

$$E = I_4 - \frac{1}{4}J_4$$

there. Thus this case also has a uniquely forced full A_2 matrix. Checking its entries gives the stated counts. This reasoning and the complete 2904-histogram domain were independently reproduced. $\square$

Proposition 4.6 (Single-support induction). *The four patterns 3993, 6970, 8765, 8771 are impossible. The remaining four patterns have 37 pair-product completions.*

Proof. Let $g = X_3$ be the nontrivial invertible source simple. All five low restriction columns have zero g coordinate. In each of the four named patterns, the only occupied high type with a nonzero g coordinate is one copy of type 15, whose g coordinate is two. Hence $I(g) = 2Z$ for a single central simple Z . But

$$\text{Tr}(\theta_{I(g)}) = \text{Dim}(\mathcal{C})\nu_1(g) = 0,$$

whereas the left side equals $2\theta_Z \text{FPdim}(Z) \neq 0$. This uses no twist-order guess or numerical feasibility test. $\square$

Proposition 4.7 (The last pattern in the first moment group). *Pattern 3994 is impossible. Exactly three necessary restriction patterns remain: 6971, 8766, 8772, with 36 pair-product completions.*

Proof. Pattern 3994 has one pair-product completion. Adjoining the forgetful rows to the known words leaves a rank-one triple residual. The four copies of type 10 have the same complete distinguished joint spectral tuple $(1, -1, -1, 0)$, so every vacuum fusion word is constant on their forced difference subspace. Exhausting the resulting finite integral triple fiber gives exactly one histogram.

The vacuum-word span now has rank ten. Exact moments through degree eight verify that it is invariant under all four distinguished actions; it is the full cyclic span, because there are exactly ten distinct joint spectral tuples and every vacuum modular weight is positive. On the unique high-type-0 simple, the fixed cyclic part of N_{A_4} has *diagonal entry* -2 . Its complementary spectrum consists of six zeros and one eigenvalue -2 . The complementary operator is therefore negative semidefinite, so the actual diagonal entry is at most -2 . This contradicts nonnegativity of a fusion matrix. The unique triple fiber, degree-eight identities, complete complementary spectrum, and negative diagonal were independently checked. $\square$

Necessary center constraint	Patterns left	Pair completions left
Earlier coordinate bounds	292	—
Joint pair-product fiber	224	2 904
Forced A_2 action	8	74
Single-support induction trace	4	37
Full cyclic negative diagonal	3	36

The three survivors are necessary center-data patterns, not three categorifications. Further finite triple-fiber and bounded-action tests are recorded separately as ongoing necessary-condition work. They do not alter the unresolved status of the source ring.

Separate exact evidence. The accepted construction and scoped refinements, with normal and optimized-Python receipts and independent reviews, are recorded in

`evidence/rank8_next_20261021/`.

Their publication overlay preserves the earlier ledgers and source-bound proof files.

5. SPHERICALIZATION, BRAIDED MORITA OBSTRUCTIONS, AND MODULE RESTRICTIONS

This section removes the pivotality hypothesis from four previously unresolved rank-eight reductions and develops restrictions on possible Morita constructions. A spherical structure on the original category is distinguished throughout from a pseudo-unitary structure obtained after Galois conjugation. None of the necessary module actions below supplies module associators or a new categorification.

5.1. A signed-quotient criterion for spherical structures.

Lemma 5.1 (Spherical section). *Let $\mathcal{C}$ be a complex fusion category with fusion tensor N . Choose dual-compatible lifts X_i^+ of its simple objects in the canonical sphericalization $\tilde{\mathcal{C}}$, and let T be the signed quotient by the central sign object $\varepsilon = -1$. If T is dual-compatible sign-gauge equivalent to N , then $\mathcal{C}$ admits a spherical structure.*

Proof. The sphericalization has two lifts $X_i^\pm$, with $X_i^- = \varepsilon X_i^+$ and $\dim(\varepsilon) = -1$; see [74, Section 5.1]. Dual-compatible lifts always exist. For a nonselfdual pair choose one lift and its dual. For a selfdual simple, the dual of a lift lies over the same source object and cannot be the opposite lift: sphericity gives $\dim(Y^*) = \dim(Y)$, whereas $\dim(\varepsilon Y) = -\dim(Y)$, and a simple spherical dimension is nonzero. This also proves the assertion in [74, Section 5.1(a)] that both lifts are selfdual.

Write a_{ij}^k and b_{ij}^k for the multiplicities of X_k^+ and X_k^- in $X_i^+ X_j^+$. Then

$$a_{ij}^k + b_{ij}^k = N_{ij}^k, \quad T_{ij}^k = a_{ij}^k - b_{ij}^k.$$

Changing lifts by signs g_i , with $g_0 = 1$ and $g_i = g_i^*$, changes T_{ij}^k to $g_i g_j g_k T_{ij}^k$. After the stated gauge change, $T = N$ and every b_{ij}^k vanishes. The chosen lifts therefore form a spherical fusion

subcategory. The forgetful functor restricts to a tensor equivalence from this subcategory to $\mathcal{C}$, proving the claim; compare [74, Section 5.1(c)]. $\square$

For use in a complete finite search, the signed tensor necessarily satisfies

$$|T_{ij}^k| \leq N_{ij}^k, \quad T_{ij}^k \equiv N_{ij}^k \pmod{2}, \quad (2)$$

together with unit, duality, cyclic trace symmetry, dual reversal and all ordered associativity identities. Commutativity of T must not be assumed from commutativity of N .

Lemma 5.2 (Two valid normalizations). *The following reductions preserve every categorically possible signed quotient.*

- (1) *In the full domain (2), select signed variables with odd original coefficient whose sign characters are linearly independent on the group of dual-compatible basis signs. One may require all selected variables to be positive, retaining at least one representative of every sign-gauge class.*
- (2) *For a selfdual source simple one may choose its lift to have second Frobenius–Schur indicator +1. With this choice for every selfdual simple, one has*

$$T_{j,j^*}^i \equiv N_{j,j^*}^i \pmod{4} \quad \text{whenever } i = i^*. \quad (3)$$

The first normalization is not asserted to preserve the second one.

Proof. For the first assertion, an odd signed variable cannot vanish. Its sign transforms by a character of the elementary abelian group of dual-compatible signs. Independence of the selected characters makes their joint evaluation map surjective onto all sign patterns. An appropriate basis-sign change thus makes the selected variables positive simultaneously.

For the second assertion, both lifts of a selfdual object are selfdual by Lemma 5.1. Their pivotal isomorphisms differ by a sign. The defining second-indicator rotation uses the inverse pivotal isomorphism once, so changing this sign changes the indicator’s sign; see [94, Definition 2.14 and Lemma 2.16]. A selfdual simple has indicator ± 1 , so precisely one lift has indicator +1. Its opposite has indicator -1 . The Fuchs–Runkel–Schweigert parity theorem, [94, Theorem 3.9], says that a selfdual simple of indicator -1 occurs with even multiplicity in Y^*Y in a pivotal fusion category. Applying this inside $\tilde{\mathcal{C}}$ gives

$$b_{j,j^*}^i = \frac{N_{j,j^*}^i - T_{j,j^*}^i}{2} \in 2\mathbb{Z}.$$

Replacing j by j^* reconciles the two ordered products without a commutativity assumption. This proves (3). An arbitrary subsequent change of lift can reverse the indicator, which explains the final qualification. $\square$

Theorem 5.3 (Four general sphericalization reductions). *Every complex fusion categorification of any of*

$$R8_3_982, \quad R8_3_1768, \quad R8_3_1795, \quad R8_3_3472$$

admits a spherical structure and has a pseudo-unitary Galois conjugate.

Proof. We enumerate all signed variables modulo cyclic trace symmetry and dual reversal, imposing every ordered associativity equation. For the first three rings we use the full parity domain and only part (1) of Lemma 5.2. The resulting complete counts are

ring	variables	sign fixes	normalized	unrestricted	gauge classes
$R8_3_982$	42	5	3	96	3
$R8_3_1768$	59	6	49	3136	28
$R8_3_1795$	59	6	73	4672	42

Surjectivity of the sign normalization is checked against the full dual-compatible sign group. Applying that entire group to the normalized solutions constructs the unrestricted sets explicitly; the

displayed counts do not assume a free group action. Each row has exactly one gauge class of the original tensor.

To exclude every other class, consider its complete dimension characters. A possible sphericalization character d must satisfy $d_i \neq 0$ and $d_i = d_i^*$. Set $\delta = \sum_i d_i d_i^*$. The two central sign sectors of $K_0(\tilde{\mathcal{C}}) \otimes \mathbb{C}$ are the original algebra and the signed algebra. A formal codegree f from either sector becomes $2f$ in the sphericalization, whose dimension is 2δ . Thus central induction requires δ/f to be an algebraic integer for every formal codegree of both sectors, by [74, Theorem 2.13 and Corollary 2.14].

For completeness, the exact norm test used here is the following. If p, q are the monic minimal polynomials of δ, f , of degrees a, b , integrality requires

$$b v_\ell(|p(0)|) \geq a v_\ell(|q(0)|) \quad \text{for every rational prime } \ell. \quad (4)$$

Indeed, take norms in a common number field and cancel its degree. Every nongauge signed tensor in these three searches has a commutative semisimple algebra, verified after enumeration. A separating regular matrix with squarefree characteristic polynomial produces every character orbit, and all character multiplication identities are checked exactly. The regular Casimir factors provide all formal codegrees. Every character is excluded either by a zero simple dimension, by unequal dimensions on a dual pair, or by a prime violating (4). The receipt records all character coordinates, global-dimension polynomials and norm witnesses.

For R8_3_3472 all simples are selfdual. Here we use precisely the indicator normalization in part (2) of Lemma 5.2, with no further positive-variable sign gauge. The complete search has 84 signed variables, visits 298 865 973 native nodes, and returns exactly one tensor, namely N . In particular no character exclusion is needed for this signed search. The independent check reconstructs the modulo-four domains and verifies the full ordered tensor identities. An earlier unrestricted search timed out; that search supplies no conclusion and is not used here.

In all four cases Lemma 5.1 gives a spherical structure on the original category. The complete ordinary character test leaves exactly the FP Galois orbit. More explicitly, for R8_3_982 the four characters agreeing on duals form that orbit; for R8_3_1768 and R8_3_3472 every other character has a zero simple dimension. For R8_3_1795 the other possible dimensions are 20, 10, 4: the dimension-4 characters have zero entries, and for the others the quotient by the FP codegree has norm respectively $1/9$ or $1/36$. These statements are checked independently in the full exact character calculation. The spherical dimension character is therefore a Galois conjugate of the FP character. A number-field model and Galois conjugation [21, Theorem 2.30 and the following remark] give a spherical category with positive FP dimensions, hence a pseudo-unitary category. $\square$

The source-bound complete replay is `verify_general_spherical_gates.py` in `evidence/rank8_research_20261022/pivotality/`; its receipt is `general_spherical_gates.json` and its detailed proof is `PROOF.md` in the same directory. The native generator substitutes no singleton-domain variables and uses no domain-dependent sign simplifications. Domains affect only the variable order, the enumeration loops, and an absolute-value overflow bound; the verifier replaces every loop and checks that the bound does not increase. Independent tensor reconstruction, sign-gauge expansion and an optimized full replay are recorded in `evidence/rank8_research_20261022/r928_r941/PIVOTALITY_REVIEW.md`.

5.2. General braided exclusions and braided Morita representatives.

Proposition 5.4 (Minimal tensor image). *Let $\mathcal{C}$ be a fusion category whose global FP dimension is nonintegral, and suppose every proper fusion subcategory of $\mathcal{C}$ is integral. If $\mathcal{C}$ is categorically Morita equivalent to a braided fusion category, then $\mathcal{C}$ itself admits a braiding.*

Proof. Let $\mathcal{B}$ be a braided Morita representative. Its braiding defines a fully faithful tensor functor $\mathcal{B} \rightarrow Z(\mathcal{B})$. Compose with $Z(\mathcal{B}) \simeq Z(\mathcal{C})$ and the forgetful functor to obtain a tensor functor $F : \mathcal{B} \rightarrow$

$\mathcal{C}$. Let $\mathcal{E}$ be its tensor image, closed under direct summands. If $\mathcal{E}$ were proper, its integrality and preservation of FP dimension by tensor functors would give integer FP dimension for every simple of $\mathcal{B}$. Thus $\text{FPdim}(\mathcal{B})$ would be an integer, contradicting Morita invariance $\text{FPdim}(\mathcal{B}) = \text{FPdim}(\mathcal{C})$. Hence F is dominant. A dominant tensor functor between fusion categories of equal FP dimension is an equivalence. Transport its braiding to $\mathcal{C}$. The cited standard facts are [22, Proposition 6.3.4, Corollary 7.16.7 and Theorem 8.12.3]. $\square$

Theorem 5.5 (No braided Morita representatives). *No complex categorification of any of*

$$\begin{aligned} &R8_3_65, R8_3_928, R8_3_941, R8_3_982, \\ &R8_3_1154, R8_3_1768, R8_3_1795, R8_3_3472 \end{aligned}$$

is Morita equivalent to a braided fusion category.

Proof. First exclude a braiding on the source itself. The rings $R8_3_928$, $R8_3_941$ and $R8_3_1154$ are noncommutative, immediately excluding braidings. The ring $R8_3_65$ is an irrational generalized near-group ring: its invertibles form C_4 and act transitively on four noninvertibles of dimension $(11 + 5\sqrt{5})/2$. Neither possibility in [96, Theorem 1], whose noninvertible dimensions are $(1 + \sqrt{5})/2$ and $1 + \sqrt{2}$, has this dimension. That theorem does not assume pivotality.

For the other four rings, Theorem 5.3 allows reduction to a pseudo-unitary Galois conjugate. A braiding transports under the same field automorphism because its naturality and hexagon identities are algebraic. It suffices to exclude pseudo-unitary braided categorifications.

For $R8_3_1768$ the simple dimension $5 + \sqrt{37}$ has minimal polynomial $x^2 - 10x - 12$, and for $R8_3_3472$ the simple dimension $6 + 5\sqrt{2}$ has polynomial $x^2 - 12x - 14$. Squared norms of simples in a braided fusion category are d-numbers [24, Theorem 1.8]. In the pseudo-unitary setting they are squared FP dimensions; an algebraic integer is a d-number if and only if its square is, by [24, Corollary 2.9]. For a quadratic polynomial $x^2 - tx + n$, the d-number criterion [24, Lemma 2.7(v)] requires $n \mid t^2$. Here $12 \nmid 100$ and $14 \nmid 144$, respectively.

For $R8_3_982$, the complete earlier induction-frame and indicator analysis forces a unique central invertible G forgetting to the source invertible $g = X_6$, with $G^2 = \mathbf{1}$ and twist $\pm i$. The completeness is essential: the two excluded frames have exact finite-domain certificates, while both surviving frames force the same central lift and the indicator $\nu_2(g) = -1$. A braiding on the source supplies a central lift of g , necessarily G . Its pointed C_2 subcategory is therefore semionic and nondegenerate. The nondegenerate-subcategory decomposition [4, Theorem 3.13] would give $\mathcal{C} \simeq \text{Semion} \boxtimes \mathcal{D}$. Its adjoint subcategory is proper, being contained in $\text{Vec} \boxtimes \mathcal{D}$. In the canonical ring, closure of the supports of every XX^* instead gives the full ring. This is a contradiction. The complete frame and lift certificates are `evidence/rank8_next_20261021/r982/FRAME01_RESULT.md` and `SEMIION_ACTION.md` in that directory.

It remains to treat $R8_3_1795$. Its adjoint subring is the whole ring, and its only proper nontrivial based subring is $\{\mathbf{1}, g\}$, where $g = X_1$ has order two. In a braided category invertibles centralize the adjoint category, so g is transparent. The whole category is not symmetric, since its FP dimensions are nonintegral. Thus its Müger center is precisely $\text{Rep}(C_2)$ or sVec . Set $D = 540 + 240\sqrt{5}$. Multiplication by g pairs the simples as $(0, 1), (2, 7), (3, 6), (4, 5)$. The four characters with $\chi(g) = 1$ have formal codegrees $D, \bar{D}, 10, 10$. The pair quotient halves those codegrees.

If the center is Tannakian, de-equivariantization is nondegenerate. Each pair gives one simple because the free module on X has endomorphism space $\text{Hom}(X, (\mathbf{1} \oplus g)X)$ of dimension one. The modularization has rank four, dimension $D/2$, and simple dimensions d_i for one representative of each pair. The positive spherical structure descends; see [4, Proposition 4.56(iii) and Section 6.3]. The modular codegree identity is therefore $(D/2)/d_i^2$. If the center is sVec , the reduced S-matrix on one representative per pair has square $(D/2)I$ in this positive, selfdual situation. Its normalized rows give all quotient characters, and the same row-norm argument gives the same codegrees; see [103, Lemma 3.1 and Proposition 3.3]. Thus in either case a full-ring codegree 10 forces $d_i^2 =$

$D/10 = 54 + 24\sqrt{5}$. The actual squared dimensions are $1, 81 + 36\sqrt{5}, 94 + 42\sqrt{5}$, none of which has that value. This excludes the final braided case.

For all eight rings, complete enumeration of unit-containing dual-closed subsets shows that every proper based subring is pointed. The global FP dimension is irrational, as checked by isolating the largest regular Casimir eigenvalue in an irreducible factor of degree greater than one. Proposition 5.4 now excludes every braided Morita representative. $\square$

The braided numerical and character calculations are replayed by `site/verify_braided_scope.py`. The last subring and dimension checks have the independent source-bound replay `evidence/rank8_research_20261022/braided_morita/verify.py`. The theorem concerns braided Morita representatives; it does not exclude a Morita representative which is itself unbraided.

5.3. The two equal-dimension rings have different centers.

Proposition 5.6. *No complex categorifications of $R8_3_1154$ and $R8_3_1795$ are categorically Morita equivalent.*

Proof. Both have FP dimension $D = 540 + 240\sqrt{5}$. In the pseudo-unitary case, the eight scalar characters of $R8_3_1795$ have formal codegrees

$$4, 20, 10, 10, 10, 10, D, \overline{D}.$$

The injective assignment in the induced-unit theorem [74, Theorem 2.13] gives four *distinct* central simples of dimension $D/10 = 54 + 24\sqrt{5}$. For $R8_3_1154$, the exhaustive restrictions of Section 4.6 leave the following complete patterns:

pattern	center rank	$\#\{Z : \text{FPdim}(Z) = 54 + 24\sqrt{5}\}$
6971	23	0
8766	21	0
8772	29	0.

The unique allowed dimension type of this size is high-column type 2, whose multiplicity vanishes in all three patterns. Equivalent centers preserve FP dimensions, so these inventories cannot be equivalent.

For general complex categorifications, apply the established Galois/pseudo-unitary reduction for $R8_3_1154$ to the entire putative Morita context. It preserves both fusion rings and Morita equivalence. Global dimension and FP dimension are Morita invariants [21, Theorem 2.15 and Corollary 8.14]; the transformed partner is consequently pseudo-unitary as well. The preceding contradiction applies. This proof does not require the new separate sphericalization result for $R8_3_1795$. $\square$

The direct exact comparison is `evidence/rank8_research_20261022/r1154_r1795/morita_obstruction.py`. It verifies every character identity against the source tensor and every displayed dimension multiplicity, while retaining the earlier complete center enumerations as explicit dependencies. This proposition rules out a shared Morita construction for the pair, without excluding either ring individually.

5.4. All module ranks and based module types for $R8_3_982$.

Theorem 5.7. *For any complex categorification $\mathcal{C}$ of $R8_3_982$, every indecomposable semisimple module category has rank eight. Its based module is the regular one or one of two explicitly recorded, mutually dual, nonregular types. The complete necessary list of underlying connected separable algebra objects contains eight objects. Every nonregular module type has a simple object with internal end $1 \oplus X_5$, of FP dimension $3 + \sqrt{2} + \sqrt{3} + \sqrt{6}$.*

Proof. The preceding central semion analysis gives a nontrivial pointed associator on $\langle g \rangle$, where $g = X_6$. This applies first in the pseudo-unitary Galois conjugate supplied by Theorem 5.3. The two classes in $H^3(C_2, \mathbb{C}^\times)$ have representatives valued in $\{\pm 1\}$, so nontriviality is Galois invariant and applies to $\mathcal{C}$. The element g acts freely on the simple objects of every module category. Indeed a fixed module simple would be a rank-one module for the pointed C_2 category, and its scalar module pentagon would trivialize that associator. Equivalently, use the pointed-module classification in [5, Example 2.1].

For a simple module object m , let $A = \underline{\text{End}}(m)$. Its coefficient at g is $\dim \text{Hom}(gm, m) = 0$. The earlier complete 31-object algebra filter therefore leaves exactly eleven objects. The source permutation of g is $(0\ 6)(1\ 2)(3\ 4)(5\ 7)$. Fix one of these candidates $A = \sum_i a_i X_i$, and define the induction matrix

$$B_{i,n} = \dim \text{Hom}(X_i m, n).$$

Adjunction gives

$$(BB^\top)_{ij} = \sum_k a_k N_{j,k}^i. \quad (5)$$

Every module simple occurs in some $X_i m$, so every column is nonzero. The columns at m, gm are a and its g -translate. All others have zero coordinates at rows 0, 6 and occur in pairs (v, gv) .

Remove the distinguished pair and order the six remaining rows as $(1, 2; 3, 4; 5, 7)$. For one representative of each residual column pair, let u and w be its three within-pair sums and differences. If H is the residual matrix in (5), set

$$P_{ab} = H_{2a,2b} + H_{2a,2b+1} = \sum u_a u_b, \quad M_{ab} = H_{2a,2b} - H_{2a,2b+1} = \sum w_a w_b.$$

The column conditions are exactly $u \geq 0$, $|w_a| \leq u_a$, and $u_a \equiv w_a \pmod{2}$; they reconstruct the nonnegative columns $(u+w)/2, (u-w)/2$. Each u is nonzero.

Seven candidates have $\det P = 1$. By Cauchy–Binet the sum of squares of all 3×3 minors of an integral factor U is one. Precisely one minor is nonzero. Its three columns form an integral basis. Replacing any of them by another column has determinant zero, so that additional column must be zero. Hence there are precisely three residual column pairs, and the module rank is eight.

The other four candidates have $\det P = 4$ and checked unimodular isometries to $\text{diag}(1, 2, 2)$. Any integral embedding of this lattice, up to signed coordinate permutation, has a norm-one row on one coordinate and two orthogonal norm-two rows outside it. The latter each have two entries ± 1 ; their supports either coincide or are disjoint. These are the only possibilities, giving three or five nonzero columns. Transforming back gives complete factors U . In every five-column case its second row has five odd entries, whereas $M_{11} = 1$ in zero-based indexing. The parity condition would force five nonzero entries in the corresponding row of w , contradicting its squared norm one. Only three-column factors remain, again giving module rank eight.

In each case $\det M = \det P \in \{1, 4\}$, so the residual Gram matrix has rank six. The distinguished columns have independent coordinates at rows 0, 6, giving full Gram rank eight. Thus B is square and invertible. Each source action is forced by $B^{-1}N_i B$. Exhaust all integral three-column factors of P and M within their exact squared-norm bounds, and require these actions to be nonnegative

integral. The complete results, indexed by the prior algebra list, are

index	$(a_0, \dots, a_7)$	outcome
0	$(1, 0, 0, 0, 0, 0, 0, 0)$	regular
2	$(1, 0, 0, 0, 0, 1, 0, 0)$	two nonregular types
5	$(1, 0, 0, 1, 1, 1, 0, 0)$	two nonregular types
9	$(1, 1, 1, 1, 1, 1, 0, 0)$	two nonregular types
10	$(1, 1, 1, 1, 1, 1, 0, 1)$	forced action entry -1
12	$(1, 1, 1, 1, 1, 2, 0, 0)$	regular
14	$(1, 1, 1, 2, 2, 1, 0, 1)$	forced action entry -1
16	$(1, 1, 1, 2, 2, 2, 0, 0)$	regular
19	$(1, 2, 2, 3, 3, 2, 0, 2)$	regular
21	$(1, 2, 2, 3, 3, 3, 0, 1)$	no compatible factor
23	$(1, 3, 3, 4, 4, 3, 0, 2)$	two nonregular types.

All surviving actions satisfy every source multiplication, dual transpose, connectivity and free- g identity. Exact simultaneous permutation checks give exactly three types, the two nonregular ones interchanged by source duality. Their internal-end vectors are exactly the four at indices 2, 5, 9, 23, each twice. Thus both contain $\mathbf{1} \oplus X_5$. A module category of regular type has a simple with internal end $\mathbf{1}$, and is equivalent to the regular module category. This proves all assertions. $\square$

The complete small-lattice factors, forced negative entries, full actions and intertwining permutations are recorded and replayed by `evidence/rank8_research_20261022/r982/module_types.py`. The proof depends on the previous exhaustive algebra filter through its source-bound receipt, and on no chosen bound for module rank. Whether $\mathbf{1} \oplus X_5$ admits the required multiplication and associativity remains open; the theorem reduces every nonregular Morita route to this concrete algebra problem.

5.5. Rank-two Morita-dual spectra for R8_3_3472.

Proposition 5.8. *If the recorded rank-two based module of R8_3_3472 whose X_1 action is $\begin{pmatrix} 5 & 7 \\ 7 & 7 \end{pmatrix}$ is realized by a module category, its Morita dual has one of the ten FP dimension spectra in the table below. This is a necessary condition, without an existence assertion for either module category or any spectrum.*

Proof. Put $r = \sqrt{2}$, $d = 7 + 5r$ and $D = 700 + 490r$. After reversing the two module simples, the source action of X_1 is

$$\begin{pmatrix} 7 & 7 \\ 7 & 5 \end{pmatrix}.$$

An action matrix of a simple Y in the dual category commutes with this matrix. Solving the commutator equations over $\mathbb{Z}$ gives

$$\begin{pmatrix} s+t & 7t \\ 7t & s-t \end{pmatrix},$$

where $t \geq 0$ and $s \geq t$. The common positive eigenline is the module FP-dimension line, and hence $\text{FPdim}(Y) = s + 5tr$.

Morita invariance gives $\text{FPdim}(\mathcal{B}) = D$ for the dual. The regular FP-dimension virtual object acts with rank one on an indecomposable module; see [22, Exercise 7.16.8]. Thus its eigenvalue on the other line vanishes, giving

$$\sum_Y (s_Y + 5t_Y r)^2 = 700 + 490r, \quad \sum_Y (s_Y + 5t_Y r)(s_Y - 5t_Y r) = 0.$$

Equivalently,

$$\sum s_Y^2 = 350, \quad \sum t_Y^2 = 7, \quad \sum s_Y t_Y = 49.$$

Writing $k_Y = s_Y - 7t_Y$ reduces these to $\sum k_Y^2 = 7$ and $\sum t_Y k_Y = 0$.

The integral-dimensional simples, exactly those with $t = 0$, form a fusion subcategory because all radical coefficients are nonnegative. If its dimension is h , the last square sum gives $1 \leq h \leq 7$. Algebraic-integrality of D/h [22, Theorem 7.17.6] requires integer trace $1400/h$ and norm $9800/h^2$, leaving $h \in \{1, 2, 5, 7\}$. This subcategory is pointed. Only the cases $h = 5, 7$ need comment: a nonpointed dimension-five category would have dimensions 1, 2, contradicting $4 = 1 + 2n$ for the square of its noninvertible. For dimension seven the only nonpointed possibility is 1, 1, 1, 2. Its three invertibles fix the dimension-two simple, whose product with its dual would require $4 = 3 + 2n$, again impossible.

Let $g = h$ be the number of invertibles. The nonintegral dimensions are $td + k$, and their multiplicities $n_{t,k}$ satisfy the complete finite conditions

$$t \in \{1, 2\}, \quad k \in \{-2, -1, 0, 1, 2\}, \quad \sum n_{t,k} t^2 = 7, \quad \sum n_{t,k} k^2 = 7 - g, \quad \sum n_{t,k} tk = 0.$$

Exact enumeration gives eleven spectra, all with nonnegative matrices in this commutant cone. For $g = 7$ the two are $1^7, d^7$ and $1^7, d^3, (2d)^1$. Since $d^2 = 1 + 14d$, the product XX^* for a d -dimensional simple contains exactly one invertible: comparison of the constant terms in $\mathbb{Q}(d)$ proves this assertion. Its stabilizer in the order-seven invertible group is therefore trivial. The number of d -dimensional simples is divisible by seven, excluding the second spectrum. $\square$

In the following table exponents denote multiplicities, and the nonintegral part is written using $d = 7 + 5\sqrt{2}$.

invertibles	rank	nonintegral dimensions
1	5	$(d-1)^2, d+2, 2d$
1	8	$(d-1)^2, d^4, d+2$
1	8	$(d-1)^3, d, (d+1)^3$
1	5	$d-2, (d+1)^2, 2d$
1	8	$d-2, d^4, (d+1)^2$
2	6	$d^2, d+2, 2d-1$
2	6	$d-2, d^2, 2d+1$
5	9	$d-1, d, d+1, 2d$
5	12	$d-1, d^5, d+1$
7	14	d^7

The notation in this table refers to lists, not powers of algebraic numbers. The replay `evidence/rank8_research_20261022/r1768_r3472/r3472_rank_two_dual_spectra.py` checks the stated source action matrix, the complete square-sum enumeration, matrix nonnegativity and the orbit exclusion. Repeated simple objects with identical action matrices are allowed; no commutativity or faithfulness assumption on the dual fusion ring is used. The other recorded rank-two action has a larger integral commutant and is outside the scope of this proposition.

5.6. Pointed induction and an exclusion of R8_3_65. The next argument covers every pointed associator class directly. In particular it does not use the earlier class-zero subgroup-dual argument or its lower-rank and R8_3_1453 dependencies.

Lemma 5.9 (Pointed induction and integral Fourier fibers). *Let $\mathcal{C}$ be pseudo-unitary with its positive spherical structure, let $D = \text{FPdim}(\mathcal{C})$, and let g be an invertible simple of order n . Write*

$$I(g) = \bigoplus_Z m_Z Z, \quad z_Z = \text{FPdim}(Z) > 0,$$

where I is adjoint to the central forgetful functor. Then every constituent satisfies $\theta_Z^n = \nu_n(g)$. For any choice $\alpha^n = \nu_n(g)$, each twist fiber has weight

$$\sum_{\theta_Z = \alpha e^{2\pi i a/n}} m_Z z_Z = D/n \quad (a \in \mathbb{Z}/n). \quad (6)$$

Suppose further that all noninvertible source simples have the same irrational FP dimension d , and inductions of distinct invertibles have disjoint simple supports. Write $D/n = A + Bd$ with $A, B \in \mathbb{Q}$. If $F(Z) = m_Z g + \sum_X b_{Z,X} X$ and $b_Z = \sum_X b_{Z,X}$, then in each fiber

$$\sum m_Z^2 = A, \quad \sum m_Z b_Z = B. \quad (7)$$

Proof. The pointed restriction gives $|\nu_n(g)| = 1$ and $\nu_j(g) = 0$ for $1 \leq j < n$. Central induction gives

$$\sum_Z m_Z z_Z \theta_Z^j = D \nu_j(g), \quad \sum_Z m_Z z_Z = D;$$

see [59, Theorem 4.1 and the calculation preceding Lemma 9.3]. At $j = n$, equality in the triangle inequality, with every nonzero weight positive, forces $\theta_Z^n = \nu_n(g)$ for each constituent. The n possible twists are therefore $\alpha e^{2\pi i a/n}$. The first $n - 1$ indicator identities and the dimension identity form the discrete Fourier transform of their fiber weights. Its inverse gives (6).

By adjunction, a constituent of $I(g)$ contains g in its forgotten image with multiplicity m_Z . Disjointness of the invertible inductions excludes every other invertible from that image. Thus $z_Z = m_Z + b_Z d$ with $b_Z \in \mathbb{Z}_{\geq 0}$. Comparing coefficients of $1, d$ in (6) proves (7). No bound on twist orders or unitarity assumption is used. $\square$

Theorem 5.10. *The fusion ring $R\mathcal{S}_3\text{-}65$ has no categorification by a complex fusion category.*

Proof. Its source invertibles form C_4 , generated by $g = X_1$ with $g^2 = X_7$ and $g^3 = X_2$. The other four simples, X_3, X_4, X_5, X_6 , have dimension

$$d = (11 + 5\sqrt{5})/2, \quad d^2 = 1 + 11d, \quad D = 8 + 44d.$$

The published two-dimension reduction [25, Lemma 3.1] implies that every hypothetical complex categorification has a pseudo-unitary Galois conjugate. We work in that conjugate.

The exact induction Gram matrix has diagonal 8 at the four pointed positions and zero between distinct pointed positions. Thus their inductions have disjoint supports. For either generator g or g^3 , Lemma 5.9 gives, in each of its four twist fibers,

$$\sum m_Z^2 = 2, \quad \sum m_Z b_Z = 11.$$

There are exactly two constituents per fiber, each with $m_Z = 1$, and each satisfies $b_Z \leq 11$. In particular $I(g)$ and $I(g^3)$ are multiplicity free with eight constituents each. The certificate below uses only the weaker necessary column restrictions $m_Z = 1$ and $b_Z \leq 11$.

Let $k \in \mathbb{Z}/4$ be the pointed associator class, with cyclic representative

$$\omega_k(a, b, c) = \exp\left(\frac{2\pi i k a}{4} \left\lfloor \frac{b + c}{4} \right\rfloor\right).$$

The pointed indicators are $\nu_4(g) = \nu_4(g^3) = i^k$ and $\nu_2(g^2) = (-1)^k$. The phase assertion of Lemma 5.9 therefore imposes the following exact normalized relative traces of $(\theta, \theta^2, \theta^3, \theta^4)$ over $K = \mathbb{Q}(\sqrt{5})$:

k	constituent over g or g^3	constituent over g^2
0	$(1, 1, 1, 1), (-1, 1, -1, 1), (0, -1, 0, 1)$	$(1, 1, 1, 1), (-1, 1, -1, 1)$
1, 3	$(0, 0, 0, 0)$	$(0, -1, 0, 1)$
2	$(0, 0, 0, -1)$	$(1, 1, 1, 1), (-1, 1, -1, 1).$

For example, odd k forces primitive order-16 generator twists, and $k = 2$ forces primitive order-8 generator twists. These trace evaluations use the linear disjointness of $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\zeta_{16})$. They respect duality, since central duals have equal twists and g, g^3 have equal fourth indicator.

We impose these necessary restrictions on the established exact central-induction cone with full-character-field S-symmetry and the real part of the CM-field STS identities. It has 283 rational equality rows. Only forbidden pointed columns are removed; all other center columns and all

auxiliary rotation columns remain. For $k = 0, 1, 3$ we also use the proved Fourier restrictions $m_Z = 1, b_Z \leq 11$. The complete second-indicator possibilities, in source order $1, \dots, 7$, are $(0, 0, 0, 0, 1, e, f)$ with $e, f \in \{\pm 1\}$, and the pointed class fixes $f = (-1)^k$. Thus the following six branches are exhaustive:

k	e	allowed columns	certificate
0	-1	2413	pointed_phase_k0_fourier.json, 0
0	+1	2413	pointed_phase_k0_fourier.json, 1
1, 3	-1	1992	pointed_phase_k1_fourier.json, 0
1, 3	+1	1992	pointed_phase_k1_fourier.json, 1
2	-1	2222	pointed_phase_k2.json, 0
2	+1	2222	pointed_phase_k2.json, 1.

In every row an exact rational covector y satisfies $y \cdot c \geq 0$ for *every* allowed column c , but $y \cdot b < 0$ for that branch's right-hand side. Hence the necessary system $\sum_c x_c c = b$, $x_c \geq 0$, has no solution. The certificates check these inequalities both by the factorized exact oracle and by explicit sparse-column arithmetic. A numerical linear program proposes the covectors but contributes no proof step. Changed tensors, missing branches or changed domain metadata fail verification.

We justify the finiteness of the twist-trace domain to distinguish this argument from a bounded-order search. The Cauchy theorem for spherical fusion categories [61], with $\text{Norm}(D) = 2000$, restricts the prime divisors of the central twist order to 2, 5. For order $2^a 5^b$, the normalized relative traces at powers 1, 2, 3, 4 all vanish if $a \geq 4$ or $b \geq 2$. This follows respectively from the remaining primitive 2-power trace, or the trace down the 5-power tower. The finite box $a \leq 3, b \leq 1$ is represented by divisors of 40. These traces, together with the zero tuple for all higher supported orders, give exactly the thirteen trace types used in the cone. The metadata value 40 is therefore a compression bound, not a bound on actual twists. The order-three rotation eigenvalues on $\text{Hom}(\mathbf{1}, X^3)$ belong to a different auxiliary domain; their columns are retained and are not subjected to this prime-support restriction.

Every $k \in \mathbb{Z}/4$ is excluded. This contradicts the existence of the pseudo-unitary Galois conjugate and proves the general complex exclusion. $\square$

The exact implementation and theoretical dependency record are `evidence/rank8_research_20261022/r65/pointed_phase_cone.py` and `PROOF.md` in the same directory. The inherited cone is specified by `PROOF.md` and `ATOMIC_MODULAR.md` in `evidence/rank8_frontier_20261021/global_induction/`, and by `GLOBAL_CONVEX_INDUCTION.md` in `evidence/rank8_m3_20261021/criteria/`. The earlier feasible witness for the weaker odd-class relaxation is retained only to show that the Fourier-integrality step is essential; it is not a categorification or a survivor of the strengthened conditions.

6. ORDER-TWO FOURIER FIBERS AND THE FINAL R8_3_1795 OBSTRUCTION

The pointed-induction argument admits a finite integer refinement when several different nonintegral dimensions occur. We apply it to the two order-two pointed subcategories of `R8_3_1768` and `R8_3_1795`. The former acquires a necessary nontrivial pointed associator; the latter is excluded. We also record the completed subgroup-dual calculation for the latter ring as an intermediate necessary-condition result, not as a construction.

6.1. Integer packing of the two twist fibers. Let g be an invertible object of order two in a pseudo-unitary category, let D be its global FP dimension, and write $I(g) = \bigoplus_Z m_Z Z$. Put $z_Z = \text{FPdim}(Z)$ and $\epsilon = \nu_2(g) \in \{1, -1\}$. Lemma 5.9, or directly the central-induction trace formula of [59, Theorem 4.1], gives

$$\theta_Z^2 = \epsilon \quad (m_Z > 0), \quad \sum_{\theta_Z = \alpha} m_Z z_Z = \sum_{\theta_Z = -\alpha} m_Z z_Z = D/2, \quad \alpha^2 = \epsilon. \quad (8)$$

Indeed the second-indicator identity is an equality in the triangle inequality, and the first indicator of $g \neq \mathbf{1}$ vanishes. For $\epsilon = 1$ the first four powers of the two possible twists have relative traces $(1, 1, 1, 1)$ and $(-1, 1, -1, 1)$ over the real FP field. For $\epsilon = -1$, both i and $-i$ give $(0, -1, 0, 1)$.

Lemma 6.1 (Finite Fourier-fiber packing). *Suppose all simple FP dimensions belong to $\mathbb{Z} + \mathbb{Z}\sqrt{q}$, with nonnegative coefficients in this basis, and write $D/2 = T_0 + T_1\sqrt{q}$, with $T_0, T_1 \in \mathbb{Z}_{\geq 0}$. Let $G = \dim \text{End}(I(g))$. For a possible forgotten central simple $v = (v_i)$ containing g , put*

$$m = v_g > 0, \quad \sum_i v_i d_i = z_0 + z_1\sqrt{q}, \quad (c, x, y) = (m^2, mz_0, mz_1).$$

Let $\mathcal{A}$ be any finite set of such triples containing every categorically possible one. Define $S_a = \emptyset$ for $a < 0$, $S_0 = \{(0, 0)\}$, and, for $1 \leq a \leq G$,

$$S_a = \left(\bigcup_{(c, x, y) \in \mathcal{A}} ((x, y) + S_{a-c}) \right) \cap ([0, T_0] \times [0, T_1]). \quad (9)$$

A triple (c, x, y) can occur only if, for some a with $c \leq a \leq G$,

$$(T_0, T_1) \in S_a \cap S_{G-a}, \quad (T_0 - x, T_1 - y) \in S_{a-c}. \quad (10)$$

Deleting triples that fail this condition, and repeating until the finite set stabilizes, preserves every categorically possible triple.

Proof. Adjunction identifies m_Z with the coefficient of g in $F(Z)$. Thus the two fibers contribute integers a and $G - a$ to $\sum_Z m_Z^2 = G$. Equation (8) identifies the other two coordinate sums of each fiber with (T_0, T_1) . Every summand has $c = m_Z^2 \geq 1$. Nonnegative dimension coordinates ensure that every partial sum stays within the indicated rectangle. Induction on a shows that (9) enumerates exactly all nonnegative integer packings of the available triples with Gram contribution a and dimension coordinates in that rectangle. Removing one occurrence of a given triple from its actual fiber proves (10). The other fiber witnesses the complementary Gram value. After any deletion step, both fibers of an actual category remain witnesses for the next step. Induction proves the last claim.

Distinct central simples with equal forgotten objects are allowed to contribute repeated copies of the same triple. No bound on such repetitions, beyond the Gram and dimension bounds, is imposed. The lemma is therefore a necessary condition even when restriction vectors do not distinguish central simples. $\square$

For the two applications, the source tensor gives $g = X_1$, $g^2 = \mathbf{1}$, $\langle I(\mathbf{1}), I(g) \rangle = 0$, and $G = 8$. In source order the FP dimensions are

	$(d_0, \dots, d_7)$	$D/2$
R8_3_1768	$(1, 1, 5 + r, 6 + r, 7 + r, 7 + r, 6 + r, 5 + r), \quad r = \sqrt{37}$	$222 + 36r$
R8_3_1795	$(1, 1, 6 + 3r, 7 + 3r, 7 + 3r, 7 + 3r, 7 + 3r, 6 + 3r), \quad r = \sqrt{5}$	$270 + 120r$

The disjointness from $I(\mathbf{1})$ means that pointed restrictions occur only among the high columns of the established central-induction domain. The finite packing filter retains 243 of the 255 pointed high columns for R8_3_1768, and 1187 of the 1476 for R8_3_1795. It imposes no condition on columns not containing g , or on the auxiliary rotation columns. In particular the latter are not subjected to central-twist prime-support restrictions.

The filtered columns enter the complete inherited real S/STS and indicator cone, with 241 rational equality rows in either case. The all-order relative-trace compression, rather than a new twist-order cutoff, is unchanged. The pre-existing domain has 31 trace types in each case; its compression parameters are 2664 and 360, respectively. The packing lemma forgets the other Gram rows and correlations between different columns, so surviving packings or feasible cone points do not assert existence of a center.

6.2. Complete certificates and their categorical consequences.

Theorem 6.2. *The following statements hold for general complex fusion categories.*

- (1) *Every categorification of $R8_3_1768$ has the nontrivial associator class on its pointed C_2 subcategory. In particular the object $1 + g$ cannot be a connected separable subgroup algebra.*
- (2) *The ring $R8_3_1795$ has no categorification.*

Proof. By Theorem 5.3, any hypothetical categorification has a pseudo-unitary Galois conjugate. We first work in that positive spherical form. The complete admissible second-indicator assignments in source order $1, \dots, 7$ are

$$(\epsilon, 1, 1, a, 1, b, c) \quad \text{for } R8_3_1768, \quad (\epsilon, a, b, c, 1, 1, 1) \quad \text{for } R8_3_1795, \quad a, b, c \in \{1, -1\}.$$

These are the exhaustive indicator domains of the inherited cone; the source tensor and the indicator parity constraints are rechecked by the certificate verifier. Lemma 6.1 and (8) give the following exact domains and results:

ring	ϵ	allowed columns	branches checked	result
$R8_3_1768$	$+1$	10711	8	all infeasible
$R8_3_1768$	-1	10468	1	feasible relaxation
$R8_3_1795$	$+1$	59261	8	all infeasible
$R8_3_1795$	-1	58074	8	all infeasible.

For every infeasible branch the saved exact rational covector y satisfies

$$y \cdot c \geq 0 \quad \text{for every permitted column } c, \quad y \cdot b < 0,$$

where b is that branch's right-hand side. Consequently the necessary system $\sum_c x_c c = b$, $x_c \geq 0$, is impossible. Every permitted sparse column is checked directly, as well as by the factored scoring oracle. Previously valid covectors for the larger domain are rechecked on this domain. A floating-point linear program is used only to propose covectors; the proof consists entirely of the displayed rational inequalities. Full branch coverage is checked before an entire parity class is declared impossible.

For $R8_3_1795$ the two rows exhaust all sixteen branches, contradicting the pseudo-unitary Galois form. This proves the second assertion without using the subgroup-dual calculation below. For $R8_3_1768$, the canonical pointed indicator $\nu_2(g) = +1$ is impossible. The two classes in $H^3(C_2, \mathbb{C}^\times) = C_2$ give $\nu_2(g) = +1$ and -1 , respectively, so the pointed associator must be nontrivial. Both cohomology classes are fixed by Galois conjugation; this conclusion therefore holds in the original category as well. A connected separable algebra supported on the whole subgroup would trivialize that associator, proving the algebra obstruction. The odd assignment $(-1, 1, 1, 1, 1, 1, 1)$ for $R8_3_1768$ has an exact nonnegative rational cone witness. It supplies no associators or model. The other seven odd assignments were not searched in this refinement; they are not needed for the stated restriction. $\square$

Corollary 6.3 (Free pointed action on modules). *For any hypothetical categorification of $R8_3_1768$, the nontrivial invertible g acts freely on the simple objects of every module category. Every indecomposable module category consequently has even rank at least four.*

Proof. If g fixed a simple module object, the isomorphisms identifying its translates would give a scalar two-cochain whose coboundary is the restriction of the pointed associator to C_2 . This would trivialize the nontrivial class forced by Theorem 6.2. Thus all g -orbits have size two. In rank two its action matrix would be the transposition matrix. Every other action matrix commutes with it, since the source fusion ring is commutative, and so has the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a, b \in \mathbb{Z}_{\geq 0}$. The positive module dimension vector is fixed by g and hence has equal coordinates. The FP dimension of the corresponding source simple would therefore be $a + b$, an integer. The source has irrational simple dimensions, giving a contradiction. $\square$

Applying Corollary 6.3 to the complete previous induction-factor domain reduces 61 factors on 49 possible algebra objects to 21 factors on 21 objects. The surviving module ranks and factor counts are $(4, 4), (6, 6), (8, 9), (12, 2)$. Here the finite test is exact: a restriction row and its g -translate must have equal multiplicity, while a row fixed by translation must occur an even number of times. These conditions are necessary and sufficient for a free involution on that row multiset; rank-two factors are then excluded by the corollary. Among nine previously fully determined action classes, three survive, of ranks 8, 6, 6, accounting for six of the retained factors. The other fifteen factors have undetermined full actions and remain in the necessary domain. Thus this is not a classification of module categories. The complete factor and action checks, including $B^\top B = N_A$ and all source fusion products, are recorded in `pointed_followup/r1768_modules/PR00F.md`, `verify.py`, and `verification.json` below the research evidence directory.

The complete terminal records are `R8_3_1768.json` and `R8_3_1795.json` in `evidence/rank8_research_20261022/pointed_followup/`. Their producer and exact replayer is `c2_fibres.py`; it reconstructs the frozen cone and checks equality of the complete source metadata, including domain hashes, before applying the new filter. The independent audits in `evidence/rank8_research_20261022/r1768_r3472/` reconstruct the source Gram matrix, independently enumerate the packing closure, reconstruct the full phase masks, and pair every explicit column with every separating covector. The full R1795 audit is `R1795_COMPLETE_EXCLUSION_AUDIT.md`; its even-domain and odd-domain replayers independently check all sixteen branches. The odd-only snapshot `R1795_pointed_associator_obstruction.json` is retained to bind the logically earlier subgroup-dual investigation.

6.3. The completed subgroup-dual route for R8_3_1795. The odd-class certificates alone force a hypothetical `R8_3_1795` category to have trivial pointed associator and hence the actual separable algebra $A = \mathbf{1} + g$. Before using the final even-class certificates, its Morita dual can be determined completely at the level of fusion rings. We record this intermediate argument because it closes a natural model route and illustrates a separate finite-cover criterion. It is not a claim that either category exists.

Proposition 6.4 (Necessary subgroup-dual fusion ring). *Using only the odd-class obstruction above, a hypothetical complex categorification $\mathcal{C}$ of `R8_3_1795` has subgroup algebra $A = \mathbf{1} + g$, and the Morita dual $\mathcal{B} = A\text{-bimod}(\mathcal{C})$ has the same fusion ring `R8_3_1795`. This does not assert a tensor equivalence $\mathcal{B} \simeq \mathcal{C}$.*

Proof. The free g -orbits of source simples are

$$(0, 1), \quad (2, 7), \quad (3, 6), \quad (4, 5).$$

Choose representatives 0, 2, 3, 4. The induced right modules $X_i A$ are simple because $\text{End}_A(X_i A) = \text{Hom}(X_i, X_i A)$ is one-dimensional. They exhaust the four simple right modules, since any simple module occurs in an induced object. The source ring is commutative, so left tensoring by g fixes each module isomorphism class. An A -bimodule structure is a left C_2 -linearization. Its scalar projective multiplier is trivializable since $H^2(C_2, \mathbb{C}^\times) = 0$. Each right module therefore has exactly two linearizations, differing by the nontrivial character. These are the eight simple objects of $\mathcal{B}$. The usual bimodule dimension normalization gives

$$\text{FPdim}_{\mathcal{B}}(Y_i) = \frac{\text{FPdim}_{\mathcal{C}}(X_i A)}{\text{FPdim}_{\mathcal{C}}(A)} = d_i;$$

see [22, Chapter 7 and Exercise 7.16.9(i)].

Let h be the nontrivial character bimodule over A . It is an order-two invertible acting freely on the pairs of linearizations. It is central in the fusion ring: the underlying object of each pair is $X_i \oplus gX_i$, and both left and right g -actions exchange these two distinct simple summands. Conjugation by the diagonal map $(1, -1)$ changes both action signs. Changing only the left sign is therefore isomorphic

to changing only the right sign. This proves equality of the left and right actions on isomorphism classes; no half-braiding is asserted.

The action on the four right modules satisfies

$$(X_j A) \otimes_A Y_i \cong X_j \otimes Y_i \cong X_j X_i A$$

as right modules. Thus the pair quotient $h = 1$ has multiplication matrices Q_i equal to the source pair quotient. These four matrices are linearly independent, as their unit columns form the standard basis. Writing its nonunit basis as a, b, c , its products are

$$\begin{aligned} a^2 &= 1 + 4a + 4b + 4c, & ab &= 4a + 5b + 4c, & ac &= 4a + 4b + 5c, \\ b^2 &= 1 + 5a + 4b + 5c, & bc &= 4a + 5b + 5c, & c^2 &= 1 + 5a + 5b + 4c. \end{aligned}$$

The respective positive dimensions are $6 + 3\sqrt{5}, 7 + 3\sqrt{5}, 7 + 3\sqrt{5}$.

Selfduality of the bimodule simples must be proved. Source duality preserves each pair. In the signed quotient $h = -1$, write $a_i^* = e_i a_i$ with $e_i \in \{1, -1\}$, and let τ be the coefficient-of-unit trace. Cyclicity and invariance under the duality anti-involution, without any commutativity assumption, give

$$\tau(a_j^2 a_i) = \tau((a_j^2 a_i)^*) = e_i e_j^2 \tau(a_i a_j^2) = e_i \tau(a_j^2 a_i).$$

If $e_i = -1$, this trace, and hence the signed coefficient T_{jj}^i , is zero. But $T_{jj}^i \equiv Q_{jj}^i \pmod{2}$. The three odd coefficients $Q_{22}^1 = Q_{33}^2 = Q_{22}^3 = 5$ force all $e_i = 1$. All eight simples are selfdual, so the duality anti-involution fixes the basis and the fusion ring is commutative.

There are now ten signed variables, one per unordered triple from $\{1, 2, 3\}$, with domains

$$-Q_{ij}^k, -Q_{ij}^k + 2, \dots, Q_{ij}^k.$$

The unit and coefficient-of-unit entries are fixed. Frobenius reciprocity makes the remaining tensor symmetric in its three indices. Changing the chosen lift of a pair changes its basis sign; the odd variables $T_{122}, T_{223}, T_{233}$ transform by the independent signs of bases 1, 3, 2. They may all be required positive, retaining exactly one representative per sign-gauge orbit. The complete finite box has 2531250 assignments. Associativity leaves 45 signed tensors. Every one expands to an eight-dimensional tensor by

$$N_{(i,e),(j,f)}^{(k,u)} = \frac{Q_{ij}^k + (-1)^{e+f+u} T_{ij}^k}{2}, \quad e, f, u \in \{0, 1\}. \quad (11)$$

Nonnegativity, the unit, duality, and all ordered associativity identities are checked again on each full tensor.

For the remaining exclusions, apply the Galois reduction to the whole Morita context. Morita invariance of global and FP dimension makes $\mathcal{B}$ pseudo-unitary as well; see [21, Theorem 2.15 and Corollary 8.14]. Its dimension is $D = 540 + 240\sqrt{5}$, with minimal polynomial $x^2 - 1080x + 3600$. If a formal codegree f has monic minimal polynomial p of degree s , the algebraic integrality of D/f implies

$$s v_\ell(3600) \geq 2 v_\ell(|p(0)|) \quad \text{for every rational prime } \ell.$$

This follows by taking norms in a common number field, or from (4). The exact Casimir factorizations exclude 41 covers this way.

Two of the four remaining covers have a formal codegree $60 + 24\sqrt{5}$. Central induction would then supply a simple object of dimension $D/f = 5 + 2\sqrt{5}$ by [74, Theorem 2.13]. Every object forgotten to $\mathcal{B}$ has radical coefficient divisible by three, because all eight simple dimensions do. This is impossible. Another cover has $T = Q$; its even-labelled simples form a rank-four fusion subcategory with dimension $270 + 120\sqrt{5}$. This contradicts the established general rank-four bound $24 + 12\sqrt{3}$, specifically Corollary `finite:cor:sharp-bounds` of the companion manuscript `manuscripts/rank4/papers/companion_sections/14_finite_certification.tex`. The sole remaining cover is identified with `R8_3_1795` by an explicit based permutation, checked on all 512 coefficients. $\square$

The complete cover enumeration and witnesses are `enumerate_covers.py` and `covers.json` in `evidence/rank8_research_20261022/pointed_followup/r1795_morita/`. The independent `independent_review.py` traverses the same entire box using direct commutation of symmetric multiplication matrices, without importing the producer's six associativity polynomials. It then checks every ordered identity of the full tensor and computes the full eight-dimensional regular Casimir, reproducing the $41 + 2 + 1$ exclusions and the unique exact source match. Its source-bound receipt and categorical audit are `independent_review.json` and `INDEPENDENT_REVIEW.md`. No associator, module associator, or new model is supplied by this calculation.

6.4. Dimension-block screening at the seven-ring intermediate stage. The following independent screen was applied to the seven rings remaining before Theorem 6.2. It does not exclude any of them; its equality cases yield further exact restrictions on possible centers of the six surviving rings.

Proposition 6.5 (Saturated dimension-block restrictions). *Work in a pseudo-unitary Galois form of a hypothetical categorification. For $R8_3_928$, $R8_3_941$, and $R8_3_982$, every twist-one central simple Z outside the FP dimension orbit satisfies*

$$\frac{\text{FPdim}(Z)}{D} \in \mathbb{Q}(\sqrt{2}).$$

Equivalently, if $\text{FPdim}(Z) = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, then

$$a + b = 3c, \quad a + 2b = 6d. \quad (12)$$

For $R8_3_1768$ every such additional twist-one central simple has $\text{FPdim}(Z)/D \in \mathbb{Q}$. Writing its dimension as $a + b\sqrt{37}$, this is equivalent to

$$6a = 37b. \quad (13)$$

These restrictions exclude none of the already forced constituents of $I(\mathbf{1})$.

Proof. Let $K = \mathbb{Q}(d_X) = \mathbb{Q}(D)$, let $G = \text{Gal}(K/\mathbb{Q})$, and put

$$h_\chi = \sum_{\sigma \in G} \frac{\chi(\sigma)}{\sigma(D)}.$$

The dimension-block theorem gives $f_\chi |h_\chi|^2 \leq 1$, where f_χ is the conductor of the character χ ; see [104, Lemmas 2.1 and 4.1, Theorem 5.1]. The dimensional grading order is one here because $K = \mathbb{Q}(D)$. We also use the equality case of the norm argument in that theorem. Let w_χ be the signed sum of the center's dimension-orbit simples, and let P project onto the twist-one coordinates. The coordinates of $PS_{\text{normalized}}w_\chi$ on the dimension orbit already contribute $|G| |h_\chi|^2$ to its squared norm, whereas the full norm satisfies

$$\|PS_{\text{normalized}}w_\chi\|^2 \leq |G|/f_\chi.$$

If $f_\chi |h_\chi|^2 = 1$, the operator bound in the cited proof leaves no norm for coordinates outside that orbit. The center row formula therefore gives

$$\theta_Z = 1, \quad Z \text{ outside the dimension orbit} \implies \sum_{\sigma \in G} \chi(\sigma) \sigma\left(\frac{\text{FPdim}(Z)}{D}\right) = 0 \quad (14)$$

for each saturated character; compare [104, Section 6, equation (30)].

The exact values for all seven input rings are as follows. The final column lists the nontrivial characters' conductor-weighted squares; trivial-character squares are also at most one.

ring indices	D	h_{triv}	$f_\chi h_\chi ^2$
928, 941, 982	$144 + 96\sqrt{2} + 80\sqrt{3} + 56\sqrt{6}$	1/2	$0, 1, 1$ ($f_\chi = 8, 12, 24$)
1154, 1795	$540 + 240\sqrt{5}$	3/10	$4/9$ ($f_\chi = 5$)
1768	$444 + 72\sqrt{37}$	1/6	1 ($f_\chi = 37$)
3472	$700 + 490\sqrt{2}$	1/7	$4/25$ ($f_\chi = 8$).

For the quartic field,

$$D^{-1} = \frac{1}{8} - \frac{\sqrt{3}}{24} - \frac{\sqrt{6}}{48}.$$

The two saturated characters are the $\sqrt{3}$ and $\sqrt{6}$ Fourier components, so (14) says that $\text{FPdim}(Z)/D$ lies in $\mathbb{Q}(\sqrt{2})$. Multiplication by the displayed inverse gives exactly (12). For R8_3_1768 the nontrivial Fourier sum is $-1/\sqrt{37}$. Its saturation forces a rational quotient, equivalent to (13).

Finally, the complete formal-codegree calculation checks the forced $I(1)$ constituents outside the dimension orbit. For 928, 941 there is one remaining two-dimensional irreducible representation, with formal codegree 4. For 982 the remaining codegrees are $16 - 8\sqrt{2}$ and $16 + 8\sqrt{2}$, each twice. For 1768 they are 6 four times and 12 twice. Since the corresponding center simple has dimension ratio $1/f_E$ by [74, Theorem 2.13], all these ratios belong to the required subfield. Thus no forced constituent violates the new restrictions. $\square$

The source-bound exact verifier `verify.py` and receipt `verification.json` in `evidence/rank8_research_20261022/pointed_followup/dimension_block/` reconstruct every dimension character and regular Casimir directly from the seven canonical tensors. They check the character conductors, Fourier sums, field equations, and complete forced-constituent lists. No refined cone search with these extra equations is claimed here. The equations concern only additional twist-one simples outside the specified dimension orbit, and only the pseudo-unitary Galois form; they do not assert original pseudo-unitarity or unitarity.

7. A KNOWN EXCEPTIONAL REALIZATION AND FURTHER CENTER RESTRICTIONS

This continuation combines a realization from a known exceptional $SU(3)$ module category with necessary restrictions on the remaining centers and Morita constructions. A finite witness for selected center equations is distinguished from a categorical construction throughout. The exact source identities, proofs and replay programs are collected in `evidence/rank8_creative_20261022/`.

7.1. R982 is the neutral component of the known E_{21} category.

Theorem 7.1 (Exceptional $SU(3)$ realization). *The ring R8_3_982 is categorified by the triality-zero component of the known unitary $E_{21} = \mathcal{E}^{(24)}$ chiral fusion category associated to the conformal embedding $SU(3)_{21} \subset (E_7)_1$. It therefore left the unresolved list at this historical checkpoint. The exact-multiplicity-three counts at that checkpoint were 3506 general exclusions, 16 unitary models and 5 unresolved cases; cumulatively through multiplicity three they were 4402, 93 and 5, respectively. The later exclusion of R8_3_3472 in Theorem 8.1 gives the current totals in Theorem 3.1.*

Proof. Let $\mathcal{C} = \text{Rep}(SU(3)_{21})$ and let A be the connected commutative algebra supplied by the local conformal embedding. Its vacuum branching object is

$$\begin{aligned} A = & (0, 0) + (4, 4) + (6, 6) + (10, 10) + (21, 0) + (0, 21) \\ & + (13, 4) + (4, 13) + (10, 1) + (1, 10) + (9, 6) + (6, 9). \end{aligned}$$

The category $\mathcal{E} = \mathcal{C}_A$ of all right modules, with relative tensor product, is a unitary fusion category. This categorical existence is classical: Böckenhauer–Evans, *Modular Invariants, Graphs and*

α -Induction for Nets of Subfactors III, Section 6.4 ([hep-th/9812110](#)), construct the twenty-four irreducible chiral sectors inside a unitary endomorphism category and give their fundamental fusion graph in Figure 13. This is the E_{21} graph, not merely an abstract nonnegative matrix representation. The algebra/module description is also covered by Davydov–Müger–Nikshych–Ostrik, *The Witt group of non-degenerate braided fusion categories*, Sections 3.3 and 6.1 ([arXiv:1009.2117](#)). The local-module category has rank two; here we use all modules.

Every summand of A has triality $a - b = 0 \pmod{3}$. Decomposing the underlying object of an A -module by triality decomposes its module structure, so each simple module is homogeneous. Relative tensor products add degrees and duals negate them. Free induction is nonzero on each of the three source degrees, so this grading is faithful. Its degree-zero part $\mathcal{D} = \mathcal{E}_0$ is therefore a full unitary fusion subcategory.

For exact identification, use the twenty-four vertices and directed fundamental adjacency matrix G of Figure 43 in Coquereaux–Isasi–Schieber, *Notes on TQFT wire models and coherence equations for $SU(3)$ triangular cells* ([arXiv:1007.0721](#)). The matrices G, G^T commute. Evaluation of polynomials in these matrices at the specified unit has successive ranks 1, 3, 6, 10, 15, 21, 24 through total degree six. Thus the unit is cyclic, and the graph determines the full fusion algebra uniquely: a polynomial representing any basis vector at the unit is its multiplication matrix. The exact reconstruction verifies all twenty-four integral nonnegative fusion matrices and their associativity. Categorical existence is supplied by the preceding construction, independently of these ring tests.

The three graph colors agree with the categorical triality, since the unit has color zero and the fundamental graph advances color by one. The neutral component has eight vertices. In the vertex labels of Figure 43, the unit-preserving identification from the canonical source basis is

$$(X_0, X_1, \dots, X_7) \mapsto (1_1, 6_1, 6_2, 7_1, 7_2, 3_1, 1_2, 3_2).$$

Every one of the 512 neutral structure constants agrees under this map with the exact canonical tensor of `R8_3_982`. The neutral multiplicity is three. This proves the required based-ring identity, not just agreement of dimensions or spectra. $\square$

As an independent scalar check, the faithful grading and the local-module category $\text{Rep}((E_7)_1)$ of dimension two give

$$\text{FPdim}(\mathcal{D}) = \frac{\sqrt{2\text{FPdim}(\mathcal{C})}}{3} = \frac{\sqrt{6}}{\sin^2(\pi/24)\sin(\pi/12)} = 144 + 96\sqrt{2} + 80\sqrt{3} + 56\sqrt{6}.$$

The source graph and every canonical tensor entry are bound in `r982/e21_model.json`; the categorical audit is `unified/E21_MODEL_AUDIT.md`. Neither braiding of $\mathcal{E}$ nor braiding of its neutral component is asserted. The chiral systems have a relative braiding, while their local intersection is a different subcategory.

The construction also realizes a previously necessary algebra object. For the simple fundamental object $V \in \mathcal{E}_1$, the exact tensor identity is $V \otimes V^* = 1 + X_5$. Its canonical endomorphism algebra is connected and separable, and its multiplication stays in $\mathcal{D}$. Thus the earlier surviving object $1 + X_5$ does carry an algebra structure in this model. The degree-one and degree-two components give invertible $\mathcal{D}$ -bimodule categories with dual equivalent to $\mathcal{D}$. Their exact action tensors match the two nonregular based-module types from the earlier complete classification; thus all three necessary based-module types are realized. Their nonregularity distinguishes both from the identity bimodule, and the faithful C_3 grading supplies an order-three subgroup of $\text{BrPic}(\mathcal{D})$. The graded-component module identifications and tensor-product equivalences follow from Etingof–Nikshych–Ostrik, *Fusion categories and homotopy theory*, Theorem 6.1 ([arXiv:0909.3140](#)). Neither uniqueness of module associators nor the whole Brauer–Picard group is claimed.

7.2. Saturated dimension characters control every twist fibre. Let ρ be the canonical normalized linear modular representation of the Drinfeld center of a spherical fusion category. Its

congruence property makes it a representation of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for a suitable N . Write s, t for the standard modular generators, and let P project onto the t -fixed space.

Lemma 7.2 (Equality in the local conductor bound). *Let π be a unitary representation of $\mathrm{SL}_2(\mathbb{Z}/p^a\mathbb{Z})$, and let v be fixed by the upper unipotent subgroup and transform by the character α under the diagonal torus. If*

$$\|P\pi(s)v\| = \mathrm{cond}(\alpha)^{-1/2}\|v\|,$$

then the subrepresentation generated by v factors through $\mathrm{SL}_2(\mathbb{Z}/\mathrm{cond}(\alpha)\mathbb{Z})$. For trivial α , this means that the generated representation is trivial.

Proof. Apply the local norm bound of [104, Lemma 4.1] to each irreducible summand supporting v . Equality holds on every such summand. An irreducible with a nonzero unipotent-fixed vector embeds into the permutation representation on primitive columns (x, y) modulo p^a . For a nontrivial character of conductor p^c , the full maximal singular space of the compression $P\pi(s)P$ is spanned by

$$F_\alpha(x, y) = \alpha(y)[y \text{ is a unit}], \quad H_\alpha(x, y) = \bar{\alpha}(x)[y \equiv 0 \pmod{p^c}].$$

Indeed the large-cell line and the small-cell space are interchanged by the compression; the small-to-large map has rank one, norm $p^{-c/2}$, and adjoint image the displayed H_α line. The two image spaces are orthogonal. Thus there are no additional maximal singular vectors. Both displayed functions inflate from primitive columns modulo p^c . That inflation subspace is invariant, so irreducibility forces every supporting summand into it.

For trivial α , equality of norms says that $\pi(s)v$ is unipotent-fixed. Thus v is fixed both by the upper and lower elementary unipotent subgroups, which generate the local special linear group. $\square$

The Chinese remainder decomposition and the tensor-product decomposition of irreducible representations imply the corresponding global result: equality for a torus character χ forces the cyclic representation generated by its vector to have level dividing $f = \mathrm{cond}(\chi)$. This removes both excess prime powers and primes not dividing f ; no twist-order cutoff is imposed.

Theorem 7.3 (Saturated conductor support). *Use the dimension-block notation $K = \mathbb{Q}(\dim X : X \text{ simple})$, $D = \dim \mathcal{C}$, and*

$$w_\chi = \sum_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})} \chi(\sigma) e_{A_\sigma}, \quad h_\chi = \sum_{\sigma} \frac{\chi(\sigma)}{\sigma(D)}.$$

If the individual bound $f|h_\chi|^2 = 1$ is saturated, every central simple Z satisfies

$$\theta_Z^f \neq 1 \implies c_\chi(Z) := \sum_{\sigma} \chi(\sigma) \sigma\left(\frac{\dim Z}{D}\right) = 0.$$

Proof. The dimension-block identities [104, Lemma 3.1] give $\|w_\chi\|^2 = |\mathrm{Gal}(K/\mathbb{Q})|$ and the matrix coefficient $|\mathrm{Gal}(K/\mathbb{Q})| h_\chi$. Saturation forces equality both in Cauchy–Schwarz and in the sharp compression bound. Lemma 7.2 and its Chinese remainder consequence therefore give $\rho(t)^f \rho(s) w_\chi = \rho(s) w_\chi$. The Z coordinate of $\rho(s) w_\chi$ is $c_\chi(Z)$, whereas $\rho(t)$ is diagonal with entry θ_Z . $\square$

There are useful signed refinements. For a real quadratic character modulo a prime $p \equiv 1 \pmod{4}$, write $h_\chi = \epsilon/\sqrt{p}$ with $\epsilon \in \{1, -1\}$. Only twist 1 and primitive twists $\exp(2\pi i k/p)$ with $\left(\frac{k}{p}\right) = \epsilon$ can have $c_\chi(Z) \neq 0$. The exact squared masses in these fibres are

$$\sum_{\theta_Z=1} |c_\chi(Z)|^2 = \frac{\|w_\chi\|^2}{p}, \quad \sum_{\theta_Z=e^{2\pi i k/p}} |c_\chi(Z)|^2 = \frac{2\|w_\chi\|^2}{p}. \quad (15)$$

To prove both assertions, in the primitive-column model use the compression eigenvector $F + \epsilon\sqrt{p}H$. On each large-cell unipotent orbit its s -image is $\chi(y)(\chi(t) + \epsilon\sqrt{p}[t = 0])$. The nonzero-frequency

Fourier coefficients are $\sqrt{p}(\chi(k) + \epsilon)$. The total squared norm is $2p(p-1)$, the zero-frequency squared norm is $2(p-1)$, and each allowed nonzero-frequency squared norm is $4(p-1)$. Equivariant isometric embeddings and orthogonal decomposition transfer these ratios to the center representation.

For the quartic field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ shared by R928, R941 and R982, put

$$D = 144 + 96\sqrt{2} + 80\sqrt{3} + 56\sqrt{6}, \quad \dim Z = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6},$$

and define

$$L_{12} = 3c - a - b, \quad L_{24} = 6d - a - 2b.$$

The two saturated coordinates are $L_{12}/\sqrt{12}$ and $L_{24}/\sqrt{24}$. Their full signed spectral supports are obtained by tensoring the local equality vectors at conductors 4, 3 and 8, 3. At conductor 12, the allowed exponents are

$$\{0, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \pmod{12};$$

at conductor 24, they are all exponents except $1, 5, 19, 23 \pmod{24}$. These restrictions apply only when the corresponding mask is nonzero. The common fibre capacities are

$$\sum_{\theta_Z=1} L_{12}^2 = \sum_{\theta_Z=-1} L_{12}^2 = 4, \quad \sum_{\theta_Z=\eta} L_{24}^2 = 4 \quad (\eta \in \{1, -1, i, -i\}), \quad (16)$$

and each simple satisfies $|L_{12}|, |L_{24}| \leq 4$. The local spectral distributions and their zero mixed projector pairings are checked in the exact cyclotomic fields at orders 3, 4, 8. The two global endpoint spectra then determine every possible distribution by a convex combination. This proves the capacities and bounds without assuming that the two endpoint lines generate distinct irreducible representations.

Combining these masks with the previously complete phase domains preserves all ten R928 and all fourteen R941 pattern allocations; each has an exact rational feasible witness. For R928 the stronger oriented $\mathbb{Q}(\zeta_{24})$ cone includes imaginary distinguished *STS* rows, both complex components of the central-*g* *STS* equations against the distinguished objects, and every individual spectral fibre, with the two endpoint-mixture parameters. All ten allocations still pass over nonnegative rational variables.

An integer refinement reduces this R928 domain to four patterns. Each orbit of the central semion G has exactly one positive-charge representative, since G has charge -1 . Associate to that orbit the six normalized relative twist traces over $\mathbb{Q}(\zeta_{24})$. Relative Galois averaging changes the trace atom but leaves its coefficient one; therefore each of the 670 phase-atom coefficients counts actual orbits and must be a nonnegative integer. This applies also to the two unrestricted restriction types, whose complete all-order trace domain includes the zero tail. Rotation auxiliaries and both endpoint-mixture parameters remain arbitrary nonnegative real variables.

After one-signed zero-equation deductions, complete binary branching on $x \leq k$ or $x \geq k+1$ for an integer phase variable gives exact infeasibility certificates for patterns 44, 58, 196, 243, 245, 248. The respective trees have 13, 13, 11, 13, 31, 11 nodes and 49 rational Farkas leaves in total. Each leaf contradicts only necessary equations, so the complete branch trees exclude these six patterns without relying on a numerical solver's status. Patterns 40, 49, 186, 191 have explicit nonnegative integer phase counts and exact rational auxiliary witnesses satisfying every equation of this relaxation. The replay is `r928_r941/integer_followup/verify.py` in the new evidence directory. These four witnesses do not supply the missing modular rows, full center fusion rules or associators; R928 itself remains unresolved.

For R941 an independent degree-two fusion-algebra check has span rank fifteen and verifies 900 physical/spectral Gram identities per allocation. Of its 75 generator-times-basis products, 55 have zero projection residual with forced nonnegative integral coordinates; the remaining 20 have nonzero residual, leaving closure open in all fourteen allocations. These tests provide restrictions and witnesses, not a new exclusion or a model for R928 or R941.

7.3. The irrational dimension-ratio sector of R1768. Work in the pseudo-unitary Galois form supplied by the earlier general sphericalization theorem. Put $q = \sqrt{37}$, $u = 6 + q$ and $D = 444 + 72q$. The source dimensions in canonical order are

$$(1, 1, 5 + q, 6 + q, 7 + q, 7 + q, 6 + q, 5 + q).$$

The saturated quadratic character has $h = -1/\sqrt{37}$, and

$$c_\chi(Z) = \frac{\sqrt{37}}{222}(37b_Z - 6a_Z) \quad \text{for } \dim Z = a_Z + b_Z\sqrt{37}.$$

Equation (15) gives capacity 72 at twist one and capacity 144 at each of the eighteen nonresidue primitive 37th twists, measured by the sum of $(37b_Z - 6a_Z)^2$. The unit and its Galois partner of dimension $\alpha = 73 + 12q$ already exhaust the twist-one capacity. Every other simple with irrational $\dim Z/D$ therefore has one of those eighteen primitive twists.

Proposition 7.4 (Complete irrational-sector restriction). *Every pseudo-unitary Galois form of a categorification of R1768 has all six noninvertible second indicators equal to +1. Put $z = (0, 0, 1, 1, 1, 1, 1, 1)$. Its center simples with irrational $\dim Z/D$, other than the unit and its distinguished Galois partner, have exactly one of the following forms:*

- (1) 72 simples of dimension $6u$, each with restriction z , four at every allowed twist;
- (2) 18 simples of dimension $12u$, each with restriction $2z$, one at every allowed twist.

The distinguished dimension- α object has the unique restriction $(1, 0, 2, 2, 2, 2, 2, 2)$.

Proof. In the previously complete high-column domain, the nonrational-ratio dimensions avoiding the pointed object g are ku or $k\alpha$. The latter cannot have primitive-37 twists: modular Galois symmetry sends its dimension to k , and an integer-dimensional central object forgets entirely to $1, g$. Such an object has twist 1 or $\pm i$, respectively, by induction and the already forced nontrivial pointed associator. Galois square action preserves twist order.

For dimension ku , the Galois sign is negative. All six rational source-character rows of the normalized S -matrix therefore negate their values on the restriction vector, while both FP-character values are unchanged. Character orthogonality gives the exact complement

$$v \mapsto \hat{v} = (0, 0, k/3 - v_2, \dots, k/3 - v_7).$$

Applying integrality and nonnegativity of this complement to the complete dimension domain leaves just

$$v = (0, 0, 0, 0, 1, 0, 1, 1), \quad w = (0, 0, 1, 1, 0, 1, 0, 0), \quad 2v, \quad 2w, \quad z, \quad 2z.$$

The dimensions of v, w are $3u$, those of $2v, 2w, z$ are $6u$, and that of $2z$ is $12u$.

Galois covariance across all eighteen twists forces aggregate multiplicities $9a, 9a, 9b, 9b, 18c, 18d$, respectively. The fibre capacities imply $a + 4b + 4c + 16d = 16$. Thus their entire Gram contribution is one of the five matrices

$$G_{37} = 9U(vv^\top + ww^\top) + 18Vzz^\top, \quad U + 4V = 16, \quad V = 0, 1, 2, 3, 4.$$

This aggregate statement also covers Galois orbits larger than eighteen. The common primitive trace ray fixes all retained indicator, rotation and atomic modular rows as well. Subtract each of these five contributions from the exact previously complete cone and retain its full all-order trace domain for the rational-ratio columns.

There are eight remaining second-indicator assignments, hence forty branches. The exact certificates reject thirty-nine. The only surviving branch has $V = 4, U = 0$ and indicators $(-1, 1, 1, 1, 1, 1, 1)$ on $(g, X_2, \dots, X_7)$. The nonnegative integer capacity equation then has just $(a, b, c, d) = (0, 0, 4, 0), (0, 0, 0, 1)$, proving the two stated possibilities.

Finally the Galois image of the unit has positive sign. All rational character values of its restriction are one, and its two FP values are α, α' . Orthogonality gives uniquely

$$e_0 + \frac{\alpha - 1}{D} \mathbf{d} + \frac{\alpha' - 1}{D'} \mathbf{d}' = e_0 + \frac{\mathbf{d} - \mathbf{d}'}{\sqrt{37}} = (1, 0, 2, 2, 2, 2, 2, 2).$$

□

Even after this unique restriction is imposed, the surviving cone has an exact rational witness. This is not an integral center construction. Its remaining rational sector has dimensions $tD/12$ with positive integral t and $\sum t^2 = 70$; removing its known $I(1)$ constituents leaves $\sum t^2 = 52$ for the high rational sector. The complete forty-branch certificate and both rational witnesses are replayed by `r1768/verify.py` in the new evidence directory. Rational-ratio simples may also have primitive 37th twists; the preceding counts do not count those objects, and their all-order domain is retained. R1768 remains unresolved.

7.4. A forced four-object packet in the R1154 center. For R1154 put $D = 540 + 240\sqrt{5}$ and use the positive spherical normalization. In all three remaining center patterns, the four simple constituents of $I(g)$ have dimension $D/4$. Label them so that their twists are $(i, i, -i, -i)$, as forced by $\sum \theta_j = 0$ and $\sum \theta_j^2 = -4$.

Proposition 7.5 (Semionic packet reconstruction). *The constituents satisfy $Z_0^* = Z_1$, $Z_2^* = Z_3$, and their normalized S block is*

$$B = \begin{pmatrix} -\frac{1}{4} + \frac{i}{2} & -\frac{1}{4} - \frac{i}{2} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} - \frac{i}{2} & -\frac{1}{4} + \frac{i}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} - \frac{i}{2} & -\frac{1}{4} + \frac{i}{2} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} + \frac{i}{2} & -\frac{1}{4} - \frac{i}{2} \end{pmatrix}.$$

Their four full normalized S columns and all four fusion matrices are uniquely determined by the previously retained restriction data. Every one of the twenty-five pair histograms passes the resulting exact integrality and nonnegativity tests.

Proof. Write $s = S/D$, let Q project onto the four packet coordinates, and put

$$\begin{aligned} u &= \frac{1}{2}(e_{Z_0} + e_{Z_1} + e_{Z_2} + e_{Z_3}), \\ v &= \frac{1}{2}(e_{Z_0} + e_{Z_1} - e_{Z_2} - e_{Z_3}), \\ w &= \frac{1}{2}(e_{A_0} + e_{A_1} - e_{A_2} - e_{A_3}). \end{aligned}$$

Here $A_0 = 1, A_1, \dots, A_4$ are the distinguished twist-one objects. For each retained histogram the spectral projector of the forced A_2 matrix onto eigenvalue -1 is exactly

$$P = \frac{N_2(N_2 - 3)(N_2 + 5)(N_2^2 - 270N_2 + 225)}{7936} = ww^\top + Q - uu^\top.$$

Verlinde diagonalization gives $P = sQs^*$, whence $su = w$ and $s(\text{im } Q \cap u^\perp) = \text{im } Q \cap u^\perp$. Since s^2 is duality, $sw = u$; applying $sTs = T^{-1}sT^{-1}$ to w , using $Tu = iv$, gives $sv = -v$.

On the remaining two-dimensional space with orthonormal basis $(e_{Z_0} - e_{Z_1})/\sqrt{2}, (e_{Z_2} - e_{Z_3})/\sqrt{2}$, the complete symmetric-unitary solutions of the modular relation compatible with charge conjugation are

$$A = \text{diag}(i, -i), \quad A = \frac{i}{2} \begin{pmatrix} -1 & \pm\sqrt{3} \\ \pm\sqrt{3} & 1 \end{pmatrix}.$$

Each has $A^2 = -I$, forcing the two dual pairs. In either mixed solution the (Z_2, Z_3) entry of $N_{Z_0} - N_{Z_1}$ is $\pm\sqrt{3}$, contradicting integrality. The diagonal solution gives the displayed block. Outside the packet every one of these four normalized s columns equals $w/2$, so the full columns are known.

Let M be the restriction matrix and V the matrix of these four full normalized S columns. The projection formula gives $J = N_{I(g)} = M^\top N_g M$. Set

$$\begin{aligned} D_+ &= V \operatorname{diag}(4(B_{0j} - B_{1j}))V^*, \\ D_- &= V \operatorname{diag}(4(B_{2j} - B_{3j}))V^*, \\ H &= V \operatorname{diag}(-4, -4, 4, 4)V^*. \end{aligned}$$

Then

$$N_{Z_0}, N_{Z_1} = \frac{J + H \pm 2D_+}{4}, \quad N_{Z_2}, N_{Z_3} = \frac{J - H \pm 2D_-}{4}.$$

The exact replay checks all entries, vacuum rows, duality transpose, commutation with each other and N_2 , and the sum identity in all twenty-five histograms. $\square$

This packet is not asserted to be a fusion subcategory. The remainder of the center and the source associators are still unknown. A separate necessary-data witness for pattern 6971, pair 1, triple 0 extends all seventy words of degree at most four and passes 4900 Gram and 560 restriction checks. It is one verified witness, not an exhaustive quartic enumeration or a categorification.

All twenty-five packet branches also have full tensor support: for each packet object Z , the vacuum row of $1 + N_Z + N_Z^\top + (N_Z + N_Z^\top)^2$ is strictly positive. Thus Z and Z^* generate every center label in at most two steps. Nevertheless the known forced fusion operators leave a common $(n-9)$ -dimensional scalar eigenspace, and their joint complex commutant has dimension $9 + (n-9)^2$. This identifies remaining spectral freedom without constructing the missing fusion matrices.

A different construction test eliminates a tempting dimension match. There is an exact identity

$$\operatorname{FPdim}(SU(3)_7 \boxtimes \operatorname{Fib}) = 5(540 + 240\sqrt{5}).$$

In the full seventy-two-object product, every simple dimension has nonnegative $\sqrt{5}$ coefficient, and this coefficient vanishes precisely for the three invertibles. Hence a rational-dimensional connected separable algebra is supported on those invertibles. Each invertible occurs with multiplicity at most one: the regular right algebra-module A is simple because $\operatorname{End}_A(A) = \operatorname{Hom}(1, A) = \mathbb{C}$; for invertible g , the module gA is also simple, so free-module adjunction and Schur's lemma give $\dim \operatorname{Hom}(g, A) \leq 1$. Therefore no connected separable algebra has dimension five, excluding this direct connected-étale module-category route to R1154. The argument generalizes whenever a rational-linear field functional vanishes on 1, is nonnegative on all simple FP dimensions, and is strictly positive on every noninvertible. It does not exclude longer constructions or R1154 itself. The exact dimension calculation is recorded in `r1154/quantum_subgroup_route.json`.

7.5. The complete based-ring restriction on the R982 Morita class.

Proposition 7.6 (Morita partners preserve the R982 fusion ring). *Every fusion category in the Morita class of a complex categorification of R982 has based fusion ring R982. This has no bound on the rank or multiplicity of a possible Morita partner and does not assert equivalence of its tensor categories.*

Proof. Work first in a pseudo-unitary Galois form and conjugate any module associators along with the category. The previous complete calculation gives three necessary indecomposable based-module types. Each is isomorphic over $\mathbb{Q}$ to the regular representation, although two are not regular as based modules.

Here is the relevant general full-center argument. Put $A_C = I_C(1)$ and transport $A_M = I_{C_M}^*(1)$ into $Z(C)$ using the module-induced center equivalence. The module trace exists in the pseudo-unitary case. The full-center and character identifications of Shimizu, *Further Results on the Structure of (Co)Ends in Finite Tensor Categories*, Theorems 5.4, 5.11 and Corollary 5.13 ([arXiv:1801.02493](https://arxiv.org/abs/1801.02493)), identify the module representation with $\operatorname{Hom}(A_M, A_C)$ over $\operatorname{End}(A_C)$. If a central simple has multiplicity e_j in A_C , regularity forces its multiplicity in A_M also to be e_j . Thus A_C

is a summand of A_M . Their common FP dimension $\text{FPdim}(C)$ eliminates a complement. Therefore $A_M \simeq A_C$ as objects. Since the source Grothendieck algebra is commutative of dimension eight, the dual also has exactly eight simples and is commutative. Equality of algebra structures on the Lagrangian objects is not assumed.

Write M_i, P_j for the left and right action matrices of the Morita bimodule, in row-action convention. Internal-Hom associativity gives

$$\sum_i (M_i)_{ab} (M_i)_{cd} = \sum_j (P_j)_{ac} (P_j)_{bd}.$$

Let U, V have columns the flattened M_i, P_j . For every one of the three left module types, the exact Gram identity is $VV^\top = UU^\top$, with U of rank eight. Hence $V = UQ$ for an orthogonal eight-by-eight matrix Q . The row lattice of U is $\mathbb{Z}^8$ for the regular type and $D_8 = \{x \in \mathbb{Z}^8 : \sum_i x_i \equiv 0 \pmod{2}\}$ for each other type. Explicit row minors of determinant one or two certify these equalities.

In the regular case integrality of V makes Q an integral orthogonal matrix. In the D_8 cases, it makes $\mathbb{Z}^8 Q^\top$ an odd integral unimodular index-two overlattice of D_8 . The three possible nonzero discriminant cosets are represented by e_1 , $h = (1/2, \dots, 1/2)$ and $h - e_1$. The latter two give even lattices, so only the $\mathbb{Z}^8$ overlattice is possible. Thus Q is a signed permutation in every case. Nonnegativity of V removes negative signs. After relabelling, the right actions are exactly the left actions. Their linear independence recovers every source multiplication coefficient from matrix products, proving the based-ring identity. Galois conjugation leaves all these integers unchanged. $\square$

The matrix replay verifies all 4096 bimodule Gram identities for each type and the integral row-lattice certificates. The lattice argument itself does not construct module associators. The exceptional realization in Theorem 7.1 separately supplies an algebra structure on $1 + X_5$ and both nonregular based-module types; uniqueness of their algebra or module structures is not claimed.

7.6. A Galois Gram bound and the two R3472 module branches.

Lemma 7.7 (Integral-dimension Galois Gram bound). *Let R be a fusion ring with Frobenius–Perron dimensions d_i , put $K = \mathbb{Q}(d_i : i)$ and $D = \sum_i d_i^2$, and let $h = \sum_{d_i \in \mathbb{Z}} d_i^2$. Then*

$$h \leq \frac{1}{\text{Tr}_{K/\mathbb{Q}}(D^{-1})}.$$

If equality holds, every nonintegral d_i satisfies $\text{Tr}_{K/\mathbb{Q}}(d_i/D) = 0$. Neither categorification nor closure of the integer-dimensional basis elements is required.

Proof. Every conjugate of d_i^2 is an eigenvalue of the positive semidefinite integer matrix $N_i N_i^* = N_i N_i^\top$, so K is totally real. For distinct embeddings $\sigma, \tau : K \rightarrow \mathbb{R}$, the dimension characters $\sigma(d_i)$ and $\tau(d_i)$ are distinct. Evaluating the τ -dimension character on the regular element $\sum_i \sigma(d_i) X_i$, which satisfies $X_j \sum_i \sigma(d_i) X_i = \sigma(d_j) \sum_i \sigma(d_i) X_i$, gives

$$\sum_i \sigma(d_i) \tau(d_i) = 0.$$

The Gram matrix of the conjugate dimension rows is therefore $\text{diag}(\sigma(D))$. Its integer-dimensional columns contribute $h \mathbf{1} \mathbf{1}^\top$. The residual Gram matrix is positive semidefinite. Evaluate its quadratic form at $v_\sigma = 1/\sigma(D)$ and put $S = \text{Tr}_{K/\mathbb{Q}}(D^{-1}) > 0$ to obtain $S - hS^2 \geq 0$. If equality holds, the same quadratic form is a sum of squares, one for each nonintegral dimension; each square is $\text{Tr}_{K/\mathbb{Q}}(d_i/D)^2$ and hence vanishes. $\square$

The next assertions are conditional on the indicated based module lifting to an indecomposable module category. No existence of either lift is asserted. Put $r = \sqrt{2}$ and $D = 700 + 490r$. Up to

interchanging the two module simples, the two source action matrices are

$$\begin{pmatrix} 7 & 7 \\ 7 & 5 \end{pmatrix}, \quad \begin{pmatrix} 11 & 5 \\ 5 & 1 \end{pmatrix}.$$

Their full integral commutants and positive dimension eigenvalues are

matrix	dimension	
$\begin{pmatrix} s+t & 7t \\ 7t & s-t \end{pmatrix}$	$s + 5tr$	$t \geq 0, \quad s \geq t, \quad s, t \in \mathbb{Z}.$
$\begin{pmatrix} s+t & t \\ t & s-t \end{pmatrix}$	$s + tr$	

Morita invariance gives global dimension D for either dual category [22, Corollary 7.16.7]. The regular dimension element acts with rank one on the module, as in [22, Exercise 7.16.8].

Lemma 7.7 gives $h \leq 7$. Here the integral simples form a fusion subcategory, since all radical coefficients are nonnegative. By categorical Lagrange integrality [22, Theorem 7.17.6], D/h is integral, hence $1400/h, 9800/h^2 \in \mathbb{Z}$ and $h \in \{1, 2, 5, 7\}$. The subcategory is pointed: the only nonpointed integral profiles to consider are 1, 2 for $h = 5$ and 1, 1, 1, 2 for $h = 7$; the square of the unique dimension-two simple would give $4 = 1 + 2a$ or $4 = 3 + 2a$, respectively. Thus write $g = h$ for its cyclic group order.

Proposition 7.8 (Three further first-branch exclusions). *For the first branch, the ten previously retained dual dimension spectra reduce to seven. The following three spectra cannot be fusion rings:*

g	<i>nonintegral dimensions with multiplicity</i>
1	$(6 + 5r)^2, (9 + 5r)^1, (14 + 10r)^1$
1	$(5 + 5r)^1, (8 + 5r)^2, (14 + 10r)^1$
2	$(7 + 5r)^2, (9 + 5r)^1, (13 + 10r)^1.$

Superscripts in this table denote multiplicities, not powers.

Proof. In each row the repeated pair is either self-dual or exchanged by duality; all other nonintegral dimensions occur once. For each of these six possibilities introduce nonnegative integers $m_{ijk} = [\mathbf{1} : X_i X_j X_k]$, identifying cyclic rotations and the reverse-dual triple (k^*, j^*, i^*) . Unit triples and cyclic-group products are fixed. Equating rational and radical coefficients in every product dimension yields a finite nonnegative integer linear system. No commutativity or associativity assumption is used.

The exact verifier exhausts all six systems using interval bounds, row gcd congruences and complete finite branching. Their search-tree sizes are 72, 22, 39, 11, 380, 79, and every tree is empty. Every variable has a finite upper bound from a positive coefficient in a dimension equation; there is no node or time cutoff. The original R3472 tensor is checked as a positive control for the equation builder. Thus the six systems supply exact incompatibility certificates for the three spectra. The receipt retains the seven other spectra only as necessary possibilities, without claiming fusion multiplications. $\square$

For the second branch the regular-element identities give

$$\sum s_i^2 = 350, \quad \sum t_i^2 = 175, \quad \sum s_i t_i = 245.$$

After removing the g invertibles and putting $u_i = s_i - t_i$, the complete arithmetic domain is

$$\sum u_i^2 = 35 - g, \quad \sum u_i t_i = 70, \quad \sum t_i^2 = 175, \quad u_i \geq 0, \quad t_i \geq 1.$$

In particular $u_i \leq 5$ and $t_i \leq 13$. A finite multiplicity recursion over these types, completed by square partitions for $u = 0$, counts each dimension multiset once. The exact counts are

g	1	2	5	7	total
spectra	272348	124228	3900	2	400478.

When $g = 7$, equality in Lemma 7.7 gives $5s_i = 7t_i$. Writing $\delta = 7 + 5\sqrt{2}$, the two possibilities are $1^7, \delta^7$ and $1^7, \delta^3, (2\delta)^1$. Since $\delta^2 = 1 + 14\delta$, the stabilizer of a δ -dimensional simple is trivial and its orbit under C_7 has size seven. The second possibility is impossible. Thus 400477 arithmetic spectra survive; full multiplication tests for this entire domain are not supplied.

The exact programs, source identities and receipts are in `evidence/rank8_creative_20261022/r3472/`. Normal and optimized Python replay produce identical results. These refinements are conditional Morita restrictions. R3472 was unresolved at this checkpoint; its later general exclusion is Theorem 8.1.

7.7. A common cyclic-parent obstruction for R1768 and R3472.

Lemma 7.9 (Cyclic stabilizer divisibility). *Let H be a cyclic subgroup of invertibles in a complex fusion category $\mathcal{C}$, and let X be a simple object with $Xh \simeq X$ for all $h \in H$. Suppose H lifts to a Tannakian pointed subcategory of $Z(\mathcal{C})$ whose forgetful image in $\mathcal{C}$ is fully faithful. Then $\text{FPdim}(X)/|H|$ is an algebraic integer.*

Proof. Let $A = \bigoplus_{h \in H} h$ be the regular algebra and $\mathcal{D} = \mathcal{C}_A$ the ordinary de-equivariantization [22, Theorem 8.23.3]. Module structures supported on copies of X are the representations of a twisted group algebra of H . Every scalar two-cocycle on a cyclic group over $\mathbb{C}$ is a coboundary: rescale a projective generator with $T^{|H|} = \lambda$ by a root of λ . Hence there are $|H|$ simple A -modules with underlying object X . Their dimensions in $\mathcal{D}$ are $\text{FPdim}_{\mathcal{C}}(X)/\text{FPdim}_{\mathcal{C}}(A)$. For completeness, this normalization follows because the positive dimension vectors before and after forgetting satisfy the same indecomposable $\mathcal{C}$ -module eigenvector equations; evaluating at the unit A of $\mathcal{D}$ fixes the constant to $|H|$. Dimensions of fusion-category simples are algebraic integers. $\square$

Corollary 7.10 (Cyclic quotient square divisibility). *An ordinary de-equivariantization of a category with near-group rules $\text{NG}(C_n, n)$ by a group of order h necessarily has $h^2 \mid n$.*

Proof. Every simple in the fully faithful image of the de-equivariantizing representation category has integral dimension, hence is invertible. Its group of simple objects is therefore cyclic. The unique noninvertible of the near-group category is fixed by this subgroup. Its dimension d satisfies $d^2 = n + nd$ and has quadratic norm $-n$. Lemma 7.9 makes d/h integral, so its norm $-n/h^2$ belongs to $\mathbb{Z}$. The discriminant $n^2 + 4n$ lies strictly between $(n+1)^2$ and $(n+2)^2$, so this is a genuine quadratic norm. $\square$

Set $\delta = g + \sqrt{g^2 + 1}$ and $D_g = g(1 + \delta^2)$. The hypothetical cyclic near-group parent $\text{NG}(C_{4g^2}, 4g^2)$ has noninvertible dimension $2g\delta$ and global dimension $4gD_g$. Dividing its global dimension by $4g$ would give the desired multiplicity-two generalized-near-group scalar profile, but $(4g)^2 \nmid 4g^2$. Equivalently, the quotient would contain a simple of dimension $\delta/2$, whose norm is $-1/4$. For $g = 6$ and $g = 7$ this excludes, respectively, the direct ordinary quotient routes

$$\text{NG}(C_{144}, 144) \longrightarrow D = 444 + 72\sqrt{37}, \quad \text{NG}(C_{196}, 196) \longrightarrow D = 700 + 490\sqrt{2}.$$

No assertion that these parents exist is required. The obstruction does not exclude noncyclic parents, other Morita representatives or generalized-Haagerup parents. In particular the scalar possibilities of a C_{12} or C_{14} generalized-Haagerup parent followed by a C_2 quotient are not settled here. These are restrictions on construction routes for R1768 and R3472, not general exclusions of those rings or constructions of their associators.

8. GENERAL COMPLEX EXCLUSION OF R8_3_3472

The remaining case $R = \text{R8_3_3472}$ is excluded by a complete central-induction and rotation argument. The earlier feasible center relaxations and conditional Morita restrictions are retained below with their historical scope. They are not the terminal obstruction.

Theorem 8.1. *The fusion ring $R8_3_3472$ has no fusion categorification over $\mathbb{C}$. In particular, the conclusion does not require a pivotal, spherical, braided, unitary, or pseudo-unitary structure as an initial hypothesis.*

Proof. Theorem 5.3 reduces a hypothetical complex categorification to a pseudo-unitary Galois conjugate with the same fusion rules. We recall that reduction, then exclude every possible center restriction frame. There are 372 labelled low frames; exact span and phase certificates leave three equivalent frames. Their twists belong to a complete domain of 424 Galois orbit types. The necessary rotation equations for these types have an exact integer contradiction, proved by 1648 rational bound deductions and a final separating inequality. The details and replay scope follow. $\square$

8.1. Reduction from arbitrary complex categorification. All eight source simples are selfdual. In the canonical sphericalization of an arbitrary categorification, both lifts of each simple are selfdual, and changing the lift reverses its second indicator [74, Section 5.1]. Choose the lift of indicator +1. The odd-multiplicity theorem [94, Theorem 3.9] makes the multiplicity of every negative lift in a positive-lift square even. Consequently the signed quotient tensor satisfies

$$|T_{ij}^k| \leq N_{ij}^k, \quad T_{ij}^k \equiv N_{ij}^k \pmod{2}, \quad T_{jj}^i \equiv N_{jj}^i \pmod{4}.$$

The complete search, with the unit, duality, cyclic trace symmetry, dual reversal and all ordered associativity equations, has 84 signed variables and 298865973 nodes. Its only solution is $T = N$. No further sign gauge or commutativity assumption is added. The positive lifts therefore form a spherical section, proving that the original category admits a spherical structure.

Of the eight ordinary characters of R , the six outside the FP Galois orbit have zero values on simple basis elements. They cannot be spherical dimension characters. The remaining pair is the FP character and its $\sqrt{2}$ -conjugate. Extend the required embedding to an automorphism of $\mathbb{C}$ and transport the associator, rigidity and spherical structure simultaneously. All coherence identities are preserved, and the resulting spherical dimensions are positive FP dimensions. Thus the transported category is pseudo-unitary. This also avoids any separate assumption that a chosen pivotal structure descends to a number field. The original gate, source hashes and complete replay remain in `evidence/rank8_research_20261022/pivotality/`; the new proof package contains an independent audit and fresh successful replay of this step.

8.2. The complete distinguished restriction frames. Put

$$r = \sqrt{2}, \quad \delta = 7 + 5r, \quad \pi = \delta^2 = 99 + 70r, \quad \beta = 1 + \pi = 100 + 70r, \quad \gamma = 7\delta = 49 + 35r, \quad D = 7\beta.$$

The source dimensions, in canonical order, are $(1, \delta - 1, \delta - 1, \delta, \delta - 1, \delta + 1, \delta + 1, \delta + 1)$. Let $V = X_1 + \dots + X_7$. We use $F : Z(\mathcal{C}) \rightarrow \mathcal{C}$ and its adjoint I , and write v_Z for the nonnegative integral restriction column of a central simple Z . The source formal codegrees are $D, D' = 700 - 490r$, and six copies of 7. Central induction gives eight distinct twist-one summands A_E of $I(\mathbf{1})$, with dimensions $1, \pi, \beta, \dots, \beta$ [74, Theorem 2.13].

The restriction of the distinguished object of dimension π is forced:

$$F(A_{D'}) = \mathbf{1} + 2V. \tag{17}$$

To avoid an identification assumption, first let P be the image of the vacuum under a modular Galois automorphism that flips r . The Galois vacuum row gives $\dim(P) = D/D' = \pi$ and, by the induction character formula [98, Lemma 4.3 and Corollary 4.5],

$$\chi_E(F(P)) = \begin{cases} \pi & E = \text{FP}, \\ \pi^{-1} & E = \text{FP}', \\ 1 & f_E = 7. \end{cases}$$

The element $1 + 2V$ has exactly these eight character values. Character inversion proves $F(P) = 1 + 2V$. Its unit multiplicity is one, so P belongs to $I(1)$ and is the distinguished $A_{D'}$. This proves (17) without presuming that no other central object has dimension π .

The induction Gram matrix is the tensor-derived integer matrix $B = \sum_i N_i N_i^\top = \sum_Z v_Z v_Z^\top$. Set $u = e_0$, $p = (1, 2, \dots, 2)^\top$ and $Q = B - uu^\top - pp^\top$. A distinguished β column v has $v_0 = 1$ and $\sum_{i>0} v_i = 14$. With $A = v_1 + v_2 + v_4$ and $C = v_5 + v_6 + v_7$, its dimension equation gives $C = A + 1$ and $v_3 = 13 - 2A$, where $0 \leq A \leq 6$. These equations give exactly 2142 possible columns. The necessary condition $Q - vv^\top \succeq 0$ retains 96. For the six labelled columns, impose distinguished S -symmetry, the pairwise Gram inequalities, their prescribed sum $Q_{0,*}$, and positivity of the residual Gram matrix. Exact exhaustive enumeration leaves 372 labelled frames. Repeated restriction columns are allowed throughout. A separate enumeration and exact positive-semidefinite test reproduce this complete list.

The high columns, those with $v_0 = 0$, are reconstructed from the source rather than taken from a previously feasible witness. The complete induction bounds, modular eigenvalue integrality and Galois spectral support tests leave 90 restrictions of dimension $\eta = \beta/2$, 78 of dimension β , and one each of dimensions $c\gamma$ for $c = 1, 2, 5, 10$; the latter columns are cV . Let a, b count the high η, β simples, and put $w = \sum c^2$ over the other high simples. Comparing global dimension and orthogonality of the dimension character with its r -conjugate gives

$$50a + 200b + 49w = 8402, \quad 50a + 200b - 49w = -1202.$$

Hence $a + 4b = 72$ and $w = 98$, excluding $c = 10$. The resulting 171 high restriction possibilities form a complete necessary domain. The source-derived domain replay checks each integrality and missing-Galois-conjugate rejection exactly.

8.3. Reduction of all frames to one orbit. Let $X = (\chi_E(X_j))$ be the full character table and let L have columns v_{A_E} in the same character order. The distinguished modular STS identities and induction give

$$\sum_Z \theta_Z v_Z v_Z^\top = LX. \quad (18)$$

For each frame, the high columns must lie in the support of its residual positive-semidefinite Gram matrix. Exact linear annihilators show that (18) cannot lie in the span of the supported rank-one matrices for 276 frames. Nine verified based-ring automorphisms partition the remaining 96 frames into twelve orbits. Eleven exact rational separating certificates exclude eleven orbits. The only frames left are 42, 117, 304, one orbit under these automorphisms.

These phase certificates do not assume a maximum twist order. Apply normalized relative trace to $K = \mathbb{Q}(r, \zeta_7 + \zeta_7^{-1})$ in (18). The 33 exact trace values of roots whose orders divide 280, together with their convex hull, contain the normalized relative trace of every root of unity. Cyclotomic tower transitivity and the factor $\mu(m)/\varphi(m)$ for the part of the order coprime to 14 prove the coverage. Introducing nonnegative real masses on these atoms is therefore a necessary relaxation for arbitrary twists. Each separating covector is checked against every supported column and atom, as well as the exact right-hand side. The frame-coverage replay verifies all 276 span certificates, the nine automorphisms, all eleven phase certificates, and the three surviving labels.

8.4. Rigidity of the surviving frame. Use frame 42, and identify high columns with their seven nonunit coordinates. Its residual Gram matrix has null-vector $(1, 1, -6, 1, 1, 1, 1)^\top$. Thus every high column satisfies $\sum_{i>0} v_i = 7v_3$. An η column would give $v_3 = 1$, $A + C = 6$ and $C - A = 1$, impossible over the integers. Hence there are 18 high β simples and $\sum c^2 = 98$ for the remaining objects. For a high β column, $v_3 = 2$, $A = 5$, and $C = 7$. Writing $z = v - 2V$ gives $\|z\|^2 \geq 2$. The residual Gram trace is 1226 and forces $\sum \|z\|^2 = 36$, so equality holds for all 18 columns. Consequently the only possibilities are

$$2V - e_a + e_c, \quad a \in \{1, 2, 4\}, \quad c \in \{5, 6, 7\}.$$

The cross entries of the Gram matrix force each of these nine columns to occur twice. The remaining columns are cV , $c \in \{1, 2, 5\}$, with dimensions $c\gamma$.

The ten associated rank-one matrices are linearly independent. Solving (18) makes the twist sum on each two-object β column equal to $2\cos(2\pi k/7)$, where, in the displayed row and column orders, the k -matrix is

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

A pair of unit complex numbers with this sum is $\{\zeta_7^k, \zeta_7^{-k}\}$. Thus the 18 high β twists are fixed, while the six distinguished β twists are one. For the other simples define

$$P_n = \sum_{\dim Z = c\gamma} c^2 \theta_Z^n.$$

The same STS computation and the second and third indicators give

$$P_0 = 98, \quad P_1 = -2 - 10r, \quad P_2 = -2, \quad P_3 = -2 + 10r. \quad (19)$$

For the second indicators, odd-square witnesses for objects 1, 2, 3, 4, 5, 6, 7 are respectively squares of objects 4, 1, 1, 2, 6, 7, 5; Theorem 3.9 of [94] gives $\nu_2 = 1$. The third indicators are all two by Proposition 9.10 below. The central indicator formula [59] now becomes, for $i > 0$,

$$\nu_n(X_i) = b_i(n) + \frac{r}{20}(P_n + 2), \quad b_i(n) = \begin{cases} 1 & 7 \nmid n, \\ 6 & i \in \{1, 2, 4\}, 7 \mid n, \\ 7 & i = 3, 7 \mid n, \\ 8 & i \in \{5, 6, 7\}, 7 \mid n. \end{cases} \quad (20)$$

8.5. A complete finite twist domain. Cauchy's theorem [61, Theorem 3.9] and $DD' = 9800$ restrict exponent primes to 2, 5, 7. Modular Galois symmetry sends a twist θ to θ^{a^2} for every unit a modulo the exponent. It preserves each dimension $c\gamma$: under the r -flip, $D/\sigma(D) = \pi$ and $\sigma(\gamma) = -\gamma/\pi$, with the Galois sign correcting the minus sign. Therefore the weighted multiplicities in P_n are constant on unit-square orbits.

For an order $q = 2^a 5^b 7^c$, the orbit size is the product of

$$s_2(a) = 2^{\max(a-3, 0)}, \quad s_5(b) = \begin{cases} 1 & b = 0, \\ 2 \cdot 5^{b-1} & b > 0, \end{cases} \quad s_7(c) = \begin{cases} 1 & c = 0, \\ 3 \cdot 7^{c-1} & c > 0. \end{cases}$$

Since the entire weighted mass is 98, every occurring orbit has size at most 98. It follows that $a \leq 9$, $b \leq 3$, $c \leq 2$, with the product bound still imposed. There are exactly 63 possible orders and 424 unit-square orbit types. This is a proved finite domain, not a conductor cutoff. For each type $\mathcal{O}$ introduce its nonnegative integer weighted multiplicity $x_{\mathcal{O}}$; a categorical realization would have $x_{\mathcal{O}} = \sum_c c^2 m_{c, \mathcal{O}}$. Allowing every nonnegative integer $x_{\mathcal{O}}$ only weakens the necessary system. Then

$$P_n = \sum_{\mathcal{O}} x_{\mathcal{O}} \sum_{\theta \in \mathcal{O}} \theta^n, \quad \sum_{\mathcal{O}} |\mathcal{O}| x_{\mathcal{O}} = 98.$$

The power sums are evaluated by exact local Gauss periods. They lie in $\mathbb{Q}(\zeta_8, \sqrt{5}, \sqrt{-7})$; computation in $\mathbb{Q}(\zeta_{1120})$ is only a common coordinate representation, not a restriction that twists have order dividing 1120. Independent reduction modulo cyclotomic polynomials checks the local formulas and all 424 orbit types.

8.6. Rotation constraints and the exact terminal contradiction. Let E_n be pivotal cyclic rotation on $\text{Hom}(\mathbf{1}, X_i^{\otimes n})$. It has order dividing n . For $1 \leq k < n$, set $g = \gcd(n, k)$ and $m = n/g$. The rotation identity [109, Theorem 5.1 and its proof] gives

$$\text{Tr}(E_n^k) = \sigma_{k/g}(\nu_m(X_i^{\otimes g})), \quad \sigma_{k/g}(\zeta_m) = \zeta_m^{k/g}.$$

The tensor determines $X_i^{\otimes g}$ exactly, and the indicator is additive. Thus (20) computes each trace. The Fourier multiplicities

$$\frac{1}{n} \sum_{k=0}^{n-1} \zeta_n^{-tk} \text{Tr}(E_n^k) \quad (0 \leq t < n)$$

are nonnegative integers. Impose the necessary integrality congruences from their constant coefficient in the chosen cyclotomic power basis, for $i = 1, 3, 5$ and $n \in \{4, 5, 7, 8, 10, 16, 20\}$. Together with (19), this gives 362 integer equations in the 424 orbit variables and 148 integer quotient variables. All variable bounds follow from the mass 98 and these congruences.

There is a further exact necessary condition: $\nu_n(X_i) \in \mathbb{Q}(\zeta_n)$. If a is coprime to the exponent, $a \equiv 1 \pmod{n}$, and $\chi(a)$ is the sign of the action on r , (20) implies

$$P_n - \chi(a)P_{na} = 2(\chi(a) - 1).$$

Their coordinate equations add 266 rows, giving a necessary system

$$Ax = b, \quad \ell \leq x \leq u, \quad x \in \mathbb{Z}^{572}, \quad A \in \mathbb{Z}^{628 \times 572}. \quad (21)$$

Every coefficient was independently reconstructed with arbitrary-precision arithmetic from the tensor, exact power sums and rotation identities.

The terminal proof uses only rational arithmetic and integer rounding. For any rational row covector y and coordinate vector e_j , the identity $Ax = b$ gives the valid lower bound

$$x_j \geq yb + \sum_k \min((e_j - yA)_k \ell_k, (e_j - yA)_k u_k).$$

Rounding its right-hand side upward is valid because x_j is an integer; applying the same argument to $-x_j$ gives upper bounds. The certificate records $380 + 486 + 782 = 1648$ such sequential deductions. For the resulting box $[\ell^*, u^*]$, its final rational covector y satisfies

$$yb - \sum_j \max((yA)_j \ell_j^*, (yA)_j u_j^*) = \frac{2717541156573712541191977062303}{359103652930042901916159999600} > 0.$$

This contradicts $Ax = b$. It excludes frame 42 and its two equivalents, so no low frame can be the center of a pseudo-unitary categorification. The general complex reduction proves Theorem 8.1.

8.7. Proof record, replay and historical scope. The canonical proof package is

`evidence/r3472_exclusion_20261022/.`

It binds the canonical source tensor, complete domain and frame audits, source-derived rotation system, rational cuts and final separator. The complete replay command is

`python3 evidence/r3472_exclusion_20261022/replay.py.`

For the self-contained terminal arithmetic certificate alone, run

`python3 evidence/r3472_exclusion_20261022/terminal/check.py.`

The terminal checker uses Python's standard library and exact fractions; no optimizer or floating-point feasibility verdict is used as proof. An independently implemented checker replays the same contradiction. A separate exact SCIP certificate is also accepted by the unmodified VIPR checker, with its input independently matched to the audited system. These arithmetic checks certify the supplied system; the source derivation, complete frame coverage and general sphericalization reduction are distinct parts of the full proof and are all included in the record.

The exact-M3 totals are therefore 3507 general complex exclusions, 16 unitary models and four unresolved rings: R8_3_928, R8_3_941, R8_3_1154 and R8_3_1768. Cumulatively through multiplicity three the totals are 4403 exclusions, 93 models and four unresolved rings. The earlier structural arguments below remain useful records of the investigation. Their feasible necessary-condition witnesses and failed construction routes were not nonexistence proofs; the exclusion above is independent of the existence of either proposed rank-two module category.

9. HISTORICAL MORITA AND CONSTRUCTION RESTRICTIONS FOR R8_3_3472

This earlier continuation strengthened necessary conditions for R8_3_3472 while its classification was still unresolved. Theorem 8.1 now settles that question. The statements below retain their original conditional proof domains. The all-Morita arguments do not assume that a rank-two based module lifts. The two surviving seven-invertible fusion rings are candidate ring data, not constructed categories. The parent obstruction below is specific to the explicit Radchenko–Wheeler family and ordinary de-equivariantization.

9.1. The hypothetical center is simple. Write $R = \text{R8_3_3472}$ and $r = \sqrt{2}$. Its positive dimension character, in canonical order, is

$$(1, 6 + 5r, 6 + 5r, 7 + 5r, 6 + 5r, 8 + 5r, 8 + 5r, 8 + 5r), \quad D = 700 + 490r.$$

An exhaustive check of the 128 unit-containing subsets shows that the only based fusion subrings are $\mathbb{Z}1$ and R . Independently, every nonunit fusion matrix has entrywise-positive square. The regular Casimir has characteristic polynomial

$$(x - 7)^6(x^2 - 1400x + 9800).$$

Consequently, in a pseudo-unitary categorification, the distinguished summands of $I(\mathbf{1})$ have dimensions

$$1, \quad 99 + 70r, \quad \underbrace{100 + 70r, \dots, 100 + 70r}_{6 \text{ copies}}.$$

Proposition 9.1. *If R has a complex categorification $\mathcal{C}$, then $Z(\mathcal{C})$ is simple. Every nonunit central simple has nonintegral Frobenius–Perron dimension. In particular its center has no nontrivial pointed, integral, or Tannakian fusion subcategory.*

Proof. Use the previously established general sphericalization and Galois reduction for R . If $\mathcal{C}$ were braided, its braiding would transport to its pseudo-unitary Galois conjugate. In a braided category, the squared norm of a simple is a d -number. In the pseudo-unitary conjugate this is its squared Frobenius–Perron dimension, and an algebraic integer is a d -number if and only if its square is one. The dimension $6 + 5r$ has trace 12 and norm -14 ; the quadratic d -number divisibility condition fails because $14 \nmid 144$. Thus no categorification of R is braided.

Liu–Palcoux–Ren [108, Theorem 5.1] state that a nonpointed simple complex fusion category has simple center if and only if it admits no braiding. The theorem does not assume integrality, despite the title of the paper. The verified based simplicity of R therefore proves the first assertion.

For the second assertion let Z be a central simple of integral FP dimension, and write $F(Z) = \sum_i n_i X_i$. The coefficient of r in its FP dimension is $5 \sum_{i>0} n_i$. It follows that $F(Z)$ is a sum of units. Every object in the fusion subcategory generated by Z consequently also forgets to a sum of units. If $Z \neq \mathbf{1}$, simplicity would make this the entire center, contrary to the dominance of the forgetful functor onto the nontrivial category $\mathcal{C}$. Hence $Z = \mathbf{1}$. $\square$

These assertions are restrictions on a hypothetical category, not an exclusion of R . In particular the integral prime-power central-simple obstruction in Remark 5.15 of the cited paper cannot be applied here: its hypotheses require an integral modular category. The center here has irrational global FP dimension D^2 .

The exact ring checks are recorded in

`evidence/r3472_focus_20261022/center/verify_structure.py`.

They recheck the tensor, based subrings, positive dimension character, formal codegrees and primitive fusion matrices. They do not replace the earlier complete signed-quotient proof of the spherical reduction.

9.2. Exact rational and integer center witnesses. The previously published strengthened center system for `R8_3_3472` has 3347 equations and 13242 variables. Its constraints comprise the induction Gram matrix, distinguished S -symmetry, cubic Verlinde equations, Galois spectral balance, Frobenius–Perron bounds, and unit-product constraints. The certificate

`evidence/r3472_focus_20261022/center/rational_witness.json`

gives an exact nonnegative rational solution satisfying every propagated box bound. It has 234 nonzero entries, all integral except one: the high-column multiplicity for restriction vector $(0, 10, 10, 10, 10, 10, 10, 10)$ equals $49/50$.

The accompanying verifier reconstructs the unchanged published system from the original census, checks its source inventory and matrix hashes, replays the box-bound derivation, and verifies all equations and bounds by rational arithmetic. There is also an exact integer solution to this same system. Let $u = (0, 1, 1, 1, 1, 1, 1, 1)$. The complete column for $10u$ equals 100 times the column for u . Replacing weight $49/50$ on $10u$ by weight 98 on u preserves all equations and satisfies all propagated bounds. Every other coordinate was already integral. The certificate

`evidence/r3472_focus_20261022/center/integer_witness.json`

records the resulting 234 nonzero integer entries. The same verifier checks the full column identity and all equations and bounds for both witnesses in one reconstruction.

Thus this particular necessary relaxation cannot exclude R , even by integer infeasibility. These certificates do not incorporate twist-phase equations and do not provide a modular datum or categorification. At this historical checkpoint categorifiability remained unresolved; Theorem 8.1 supplies the later exclusion.

9.3. Dimension descent and pointed-orbit restrictions for every Morita partner. The following argument does not assume that either rank-two based module is categorifiable. It applies to every Morita partner of every complex categorification of `R8_3_3472`.

Lemma 9.2 (Dimension-field descent). *Let $\mathcal{B}$ be a complex fusion category such that its Drinfeld center has no nontrivial invertible objects. Then*

$$\mathbb{Q}(\text{FPdim}(Y) : Y \in \text{Irr}(\mathcal{B})) = \mathbb{Q}(\text{FPdim}(\mathcal{B})).$$

Proof. Write $d_Y = \text{FPdim}(Y)$, $D = \sum_Y d_Y^2$ and $K = \mathbb{Q}(d_Y : Y)$. Every conjugate of d_Y^2 is an eigenvalue of the positive semidefinite integer matrix $N_Y N_Y^* = N_Y N_Y^T$; consequently K is totally real. Every embedding $\sigma : K \rightarrow \mathbb{R}$ satisfies $|\sigma(d_Y)| \leq d_Y$, since d_Y is the spectral radius of the nonnegative fusion matrix N_Y . If $\sigma(D) = D$, equality in

$$D = \sum_Y \sigma(d_Y)^2 \leq \sum_Y d_Y^2 = D$$

gives $\sigma(d_Y) = \varepsilon_Y d_Y$, with $\varepsilon_Y \in \{1, -1\}$. Applying σ to the dimension equation for XY , equality in the triangle inequality gives $\varepsilon_Z = \varepsilon_X \varepsilon_Y$ whenever $N_{XY}^Z > 0$. The scalars ε_Y therefore define a tensor automorphism of the identity functor, equivalently a central object whose underlying object is the unit. The hypothesis forces this automorphism to be trivial. Thus every embedding of K fixing $\mathbb{Q}(D)$ is the identity, and separability gives $K = \mathbb{Q}(D)$. $\square$

Proposition 9.3 (All-Morita pointed restriction). *Let $\mathcal{C}$ be any complex categorification of `R8_3_3472` and let $\mathcal{B}$ be Morita equivalent to $\mathcal{C}$. Put $r = \sqrt{2}$ and $\delta = 7 + 5r$. Then every simple of $\mathcal{B}$ has dimension $a + br$, with $a, b \in \mathbb{Z}_{\geq 0}$, and*

$$\sum a^2 = 350, \quad \sum b^2 = 175, \quad \sum ab = 245. \quad (\dagger)$$

The integral-dimensional subcategory is pointed, and its order is either 1 or 7. In the latter case the complete dimension profile is $1^7, \delta^7$.

Proof. The source tensor has only one invertible basis element, the unit, and $N_{ii}^i = 2$ for each of its seven nonunit basis elements. An invertible object of $Z(\mathcal{C})$ must therefore forget to the unit. Its half-braiding is a tensor automorphism of the identity, with nonzero scalars λ_i satisfying $\lambda_i^2 = \lambda_i$ because X_i occurs in X_i^2 . All λ_i are one. Thus $Z(\mathcal{C})$ has no nontrivial invertibles, and neither does $Z(\mathcal{B})$ by Morita invariance.

Morita invariance also gives $D = \text{FPdim}(\mathcal{B}) = 700 + 490r$. The preceding lemma puts every d_Y in $\mathbb{Q}(r)$, whose ring of integers is $\mathbb{Z}[r]$. Write $d_Y = a_Y + b_Y r$. Positivity of d_Y and the bound $|a_Y - b_Y r| \leq a_Y + b_Y r$ imply $a_Y, b_Y \geq 0$. Orthogonality of the two distinct dimension characters gives $\sum (a_Y^2 - 2b_Y^2) = 0$. Combining this with $\sum d_Y^2 = D$ proves $(\dagger)$.

The integral-dimensional simples form a fusion subcategory because all radical coefficients are nonnegative. Let h be its dimension. The integral-dimension Galois Gram bound gives $h \leq 7$, since $\text{Tr}_{\mathbb{Q}(r)/\mathbb{Q}}(D^{-1}) = 1/7$. Categorical Lagrange integrality of D/h gives $1400/h, 9800/h^2 \in \mathbb{Z}$, whence $h \in \{1, 2, 5, 7\}$. This subcategory is pointed. The only nonpointed integral profiles at these dimensions are 1, 2 at dimension five and 1, 1, 1, 2 at dimension seven. Their unique dimension-two simple is self-dual and fixed by every invertible, giving respectively the impossible dimension equations $4 = 1 + 2m$ and $4 = 3 + 2m$. Write $g = h$ for the number of invertibles.

We use strong Lagrange divisibility: if $\mathcal{D} \subseteq \mathcal{B}$ is a fusion subcategory and $\mathcal{M}$ is an indecomposable $\mathcal{D}$ -module component of $\mathcal{B}$, then $\text{FPdim}(\mathcal{M})/\text{FPdim}(\mathcal{D})$ is an algebraic integer [105, Proposition 5.7]. For the pointed subcategory and a left orbit $\mathcal{O}$ of simples of common dimension d , this says $|\mathcal{O}|d^2/g$ is integral. If $g = 2$, a fixed simple of dimension $a + br$ must have $d^2/2$ integral, so a is even. Free orbits contribute an even integer to $\sum ab$, as do fixed simples. This contradicts $\sum ab = 245$.

Suppose $g = 5$. A fixed simple has $d^2/5$ integral, so $a^2 + 2b^2 \equiv 2ab \equiv 0 \pmod{5}$, implying $5 \mid a, b$. From $(\dagger)$, after removing the invertibles,

$$\sum (5a - 7b)^2 = 50.$$

Together with $a^2 \leq 345$, $b^2 \leq 175$ this leaves only $(a, b) = (15, 10)$ for a fixed nonintegral simple. There can be at most one. After removing it, the remaining nonintegral dimensions would satisfy $\sum a^2 = 120$, $\sum b^2 = 75$, $\sum ab = 95$, contrary to $95^2 > 120 \cdot 75$. Hence every nonintegral orbit has length five. For orbit representatives, the moment equations become

$$\sum a^2 = 69, \quad \sum b^2 = 35, \quad \sum ab = 49, \quad \sum (5a - 7b)^2 = 10.$$

The complete possible types are $(1, 1), (2, 1), (3, 2), (4, 3), (5, 4), (6, 4), (7, 5)$. Their nonnegative multiplicities give exactly the following four lists of nonintegral orbit-representative dimensions:

A	$2 + r, 4 + 3r, 7 + 5r$
B	$1 + r, (4 + 3r)^2, 6 + 4r$
C	$1 + r, (3 + 2r)^4, (4 + 3r)^2$
D	$(1 + r)^2, (3 + 2r)^2, 7 + 5r$

Superscripts denote multiplicities of free orbits. This finite seven-type enumeration is replayed with exact integer arithmetic. In A, the product XX^* for $\text{FPdim}(X) = 2 + r$ contains exactly one invertible, since its stabilizer is trivial. Its nonintegral part would have dimension $5 + 4r$, which is no nonnegative combination of the three available dimensions (their rational parts are even or at least seven). Thus A is impossible.

For B and C, put $u = 1 + r$. There is one free orbit of u -dimensional simples, necessarily stable under duality. The right orbit of X^* is free, since its stabilizer is dual to the left stabilizer of X , and therefore fills this same five-element orbit. Hence for any two u -dimensional simples X, Y there is exactly one $g \in C_5$ with $Y \simeq X^*g$. Their product contains exactly one invertible and has

nonintegral part $u^2 - 1 = 2u$. No larger available dimension fits that remainder. They therefore generate, with the five invertibles, a fusion subcategory of profile $1^5, u^5$.

In D, there are two u -orbits, two u^2 -orbits and one u^3 -orbit. Right multiplication by C_5 preserves each left orbit: it acts on the left orbits of any fixed dimension, and there are fewer than five of these. Hence identifying the five invertibles with the unit gives a based fusion-ring quotient on the six left orbits, of dimensions $1, u, u, u^2, u^2, u^3$. Here are the details that ensure this is a based quotient, rather than merely a dimension quotient. For orbit representatives x, y, z , set $\overline{N}_{[x][y]}^{[z]} = \sum_{g \in C_5} N_{xy}^{gz}$. Left translation of x translates the product, and replacing y by gy does the same because xg belongs to the left orbit of x . The coefficients are therefore independent of representatives and inherit associativity and duality. Their unit coefficient counts g with $y \simeq x^*g$, so it is one precisely for dual orbits. Finally, Frobenius reciprocity follows from

$$\sum_g N_{xy}^{gz} = \sum_g N_{y, z^*g^{-1}}^{x^*} = \sum_g N_{y, z^*}^{x^*g} = \overline{N}_{[y][z^*]}^{[x^*]},$$

since right multiplication runs through the same free left orbit. Write A, B for the two u -dimensional basis elements of this quotient. A product with a unit term has the form $1 +$ two u -terms; without a unit term it is a single u^2 -dimensional basis element. If $A^* = B$, the squares A^2, B^2 have no u -terms. Cyclic Frobenius reciprocity then gives $[AB : A] = [BB : B] = 0$ and $[AB : B] = [AA : A] = 0$, contradicting $\text{FPdim}(AB - 1) = 2u$. Thus both are self-dual. Now AB is one u^2 -dimensional basis element, so reciprocity gives $[A^2 : B] = [AB : A] = 0$ and similarly $[B^2 : A] = 0$. Consequently $A^2 = 1 + 2A$ and $B^2 = 1 + 2B$. Lifting either orbit back to $\mathcal{B}$ again gives a fusion subcategory of profile $1^5, u^5$. Lemma 9.4 below excludes this profile and finishes the order-five case.

Finally, if $g = 7$, equality in the Galois Gram bound forces $5a = 7b$ for every nonintegral dimension. Thus those dimensions are $k\delta$, with positive integers k satisfying $\sum k^2 = 7$. The possibilities are δ^7 and $\delta^3, (2\delta)^1$. Since $\delta^2 = 1 + 14\delta$, a δ -dimensional simple has trivial stabilizer among the seven invertibles. Its orbit has seven members, excluding the second possibility. $\square$

Lemma 9.4 (The order-five silver-ratio profile). *There is no complex fusion category with dimension profile $1^5, (1 + \sqrt{2})^5$ whose invertibles act freely on the noninvertible simples.*

Proof. Let g generate the invertibles and choose a noninvertible X . The left action is free and transitive. The right action is too: the coefficients of invertibles in XX^* and X^*X are each forced by $\text{FPdim}(X)^2 = 1 + 2\text{FPdim}(X)$ to sum to one. Thus $Xg = g^a X$ for some $a \in (\mathbb{Z}/5\mathbb{Z})^\times$. Since the dual remains in the same free orbit, dualizing the relation gives $a^2 = 1$, so $a = \pm 1$. One can choose $X = X^*$: for $a = 1$ shift its index by half its duality translation, and for $a = -1$ the involution forces that translation to vanish. Write

$$X^2 = 1 + \sum_{j \in \mathbb{Z}/5\mathbb{Z}} m_j g^j X, \quad m_j \in \mathbb{Z}_{\geq 0}, \quad \sum m_j = 2.$$

If $a = -1$, comparing X^2g with $X(Xg)$ forces $m_j = m_{j+2}$, which would make all five m_j equal, an impossibility. If $a = 1$, duality gives $m_j = m_{-j}$. There are two possibilities up to changing the generator: $m_0 = 2$, or $m_1 = m_{-1} = 1$. The first contains the rank-two fusion subcategory $X^2 = 1 + 2X$, excluded by the rank-two classification [23].

In the second case the ring is commutative. Choose a primitive fifth root ζ and a one-dimensional character with $\chi(g) = \zeta$, $\chi(X) = x$, where $x^2 - (\zeta + \zeta^{-1})x - 1 = 0$. Its formal codegree is $f = 5(1 + x^2)$. Since $t = \zeta + \zeta^{-1}$ satisfies $t^2 + t - 1 = 0$, eliminating x, t gives

$$f^4 - 55f^3 + 1000f^2 - 7250f + 18125 = 0.$$

The polynomial is irreducible: modulo two it is $z^4 + z^3 + 1$, which has neither a root nor the factor $z^2 + z + 1$. The formal-codegree d -number criterion [24, Theorem 1.2 and Lemma 2.7] would require $18125 \mid 55^4$, whereas the remainder is 15625. This excludes the second case without a pivotality assumption. $\square$

Corollary 9.5. *The first rank-two branch of Proposition 7.8 retains only four of its seven dimension spectra: its three one-invertible spectra and the seven-invertible spectrum $1^7, (7 + 5\sqrt{2})^7$.*

Proof. The all-Morita proposition removes the remaining order-two spectrum and both order-five spectra. This is still a conditional restriction on the rank-two construction route: neither its lift nor a categorification of the source ring is asserted. $\square$

The verifier in `evidence/r3472_focus_20261022/morita/verify.py` checks the source tensor, all arithmetic bounds, the complete four-profile enumeration and the irreducible formal-codegree obstruction. The general categorification status of `R8_3_3472` remains open.

9.4. The seven-invertible Morita candidates. Put $r = \sqrt{2}$, $\delta = 7 + 5r$, and $D = 700 + 490r$. The following statement concerns the seven-invertible alternative in the preceding Morita restriction. It does not assert that this alternative occurs.

Proposition 9.6. *Every seven-invertible Morita partner of a categorification of `R8_3_3472` has one of the following two based rings. Their basis is $\{g_i, X_i : i \in \mathbb{Z}/7\mathbb{Z}\}$, with*

$$g_i g_j = g_{i+j}, \quad g_i X_j = X_{i+j}, \quad X_i g_j = X_{i+j}, \quad X_i X_j = g_{i+j} + 2 \sum_k X_k, \quad a \in \{1, -1\}.$$

Both displayed multiplications are associative fusion rings. No associator or Morita identification with the source ring is established here.

Proof. The preceding dimension restriction gives $1^7, \delta^7$. Since $\delta^2 = 1 + 14\delta$, reciprocity makes both stabilizers of every noninvertible trivial. Write $X_i = g^i X$ and $Xg = g^a X$, where a is a unit modulo seven. The unique invertible summand of X^2 is g^b . Comparing the invertible parts in $X^2 g = g^{a^2} X^2$ gives $a^2 = 1$. Reciprocity gives $X^* = g^{-b} X$, and applying duality twice gives $(a - 1)b = 0$. Thus $b = 0$ if $a = -1$, and replacing X by $g^c X$ makes $b = 0$ if $a = 1$. In particular X is self-dual.

For $a = -1$, the identity $X^2 g = g X^2$ makes the noninvertible coefficients invariant under translation by two. They sum to fourteen, so all are two. For $a = 1$, write $X^2 = 1 + m(g)X$ with $m_j = m_{-j} \geq 0$ and $\sum_j m_j = 14$. Pass to the pseudo-unitary Galois conjugate supplied by the source analysis; this does not change any fusion coefficient. All simple dimensions in its common center belong to $\mathbb{Q}(r)$, by forgetting to the source category. The formal-codegree induction formula $\text{FPdim}(A_E) = D/f_E$ therefore gives $f_E \in \mathbb{Q}(r)$ for every irreducible representation E .

Choose a primitive seventh root ζ . The characters above $g \mapsto \zeta$ have $X \mapsto x$, where $x^2 - tx - 1 = 0$ and $t = m(\zeta)$ is real. Their formal codegrees are $7(1 + x^2)$. Consequently

$$t^2 = \frac{(x^2 - 1)^2}{x^2} \in \mathbb{Q}(r) \cap \mathbb{Q}(\zeta + \zeta^{-1}) = \mathbb{Q}.$$

The latter field has odd degree three, hence $t \in \mathbb{Q}$. The polynomial Φ_7 now forces $m_0 = A$ and $m_j = B$ for $j \neq 0$, with $A + 6B = 14$. The possibilities are $(A, B) = (14, 0), (8, 1), (2, 2)$. The first contains the impossible rank-two categorification $X^2 = 1 + 14X$. The second has formal codegrees $(371 \pm 49\sqrt{53})/2$, outside $\mathbb{Q}(r)$. Hence $A = B = 2$. The complete tensors, dualities, dimensions, reciprocity identities and all 14^4 associativity equations for each ring are checked in `evidence/r3472_focus_20261022/constructions/verify_constructions.py`. $\square$

9.5. An obstruction to all direct Radchenko–Wheeler parent routes. We use the explicit near-group categories of Radchenko–Wheeler, *Real quadratic fields and finite quantum dilogarithms* *I*, [107, Sections 4.3 and 5.4]. The scope below is their construction from a matrix $\gamma \in \text{SL}_2(\mathbb{Z})$; no classification of all categorifications of the parent fusion ring is asserted.

Lemma 9.7. *For a Radchenko–Wheeler matrix input with $\text{tr}(\gamma) = N + 2$, put*

$$G = \mathbb{Z}^2 / \mathbb{Z}^2(\gamma - I), \quad \chi(u, v) = \exp\left(\frac{2\pi i}{N} \omega(u(\gamma - I), v)\right).$$

The central invertibles embed into G , and their self-braidings agree with $q(g) = \chi(g, g)^{-1}$ on their image.

Proof. The strict algebraic construction gives $\alpha_h \rho = \rho$, $\rho \alpha_g = \text{Ad}(U_g) \rho$, and $\alpha_h(U_g) = \chi(h, g) U_g$. The half-braiding of a central lift of α_g with ρ is a nonzero scalar times U_g , since ρ is simple. Its hexagon at $\alpha_h \rho = \rho$ forces its half-braiding with α_h to be $\chi(h, g)^{-1}$. This uses neither a star operation nor unitarity. The forgetful kernel on central invertibles consists of tensor automorphisms of the identity. It is trivial because the universal grading is trivial: ρ and every α_g occur in ρ^2 . $\square$

Lemma 9.8. *For $N = 4s^2$ with s odd, the bosonic part of the radical of (G, q) has order at most two.*

Proof. The polarization is χ^{-2} , with radical $G[2]$. The two-primary part of the two-generated group G has order four. If it is cyclic, $|G[2]| = 2$. Otherwise $\gamma - I = 2A$, where

$$\text{tr } A = 2s^2 \equiv 2 \pmod{4}, \quad \det A = -s^2 \equiv 3 \pmod{4}.$$

The two rows of A represent a basis of $G[2]$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the corresponding diagonal bicharacter values are $(-1)^b$ and $(-1)^c$. Triviality of $q|_{G[2]}$ would make b, c even. Then a, d are odd and their sum is two modulo four, so ad is one modulo four. This contradicts $\det A \equiv 3 \pmod{4}$. Thus q is a nontrivial character on the four-element radical, and its kernel has order two. $\square$

Theorem 9.9. *Let s be odd and positive, set $\delta_s = s + \sqrt{s^2 + 1}$ and $D_s = s(1 + \delta_s^2)$. No Radchenko–Wheeler parent with group order $4s^2$, tensored with an arbitrary pointed fusion category, admits an ordinary de-equivariantization of dimension D_s . In particular, no such direct route, including noncyclic parents and pointed tensor factors, supplies a Morita representative for R8_3_3472.*

Proof. Let the pointed factor $\mathcal{P}$ have dimension m . The parent has dimension $4sD_s$, so the required Tannakian group would have order $h = 4sm$. Its fully faithful forgetful image is pointed: simple representations have integer dimensions, whereas all noninvertible dimensions in the parent tensor product are irrational. Thus its simples form an isotropic subgroup H of order h in the central invertibles.

Put $L = \text{Inv}(Z(\mathcal{P}))$, $k = |L|$, and let the radical of its self-braiding form have order t . The centralizer dimension formula in the nondegenerate category $Z(\mathcal{P})$ gives $kt \leq m^2$. Lemma 9.7 embeds H , preserving its quadratic form, into $G \times L$. Its intersection with the radical has order at most $2t$: in the cyclic two-primary case the full radical has that order, and in the other case the radical quadratic form is nontrivial already on $G[2]$, so its kernel has that order. For any isotropic subgroup of a finite premetric group A ,

$$|H|^2 \leq |A| |H \cap \text{rad}(A)|,$$

because $H \subseteq H^\perp$ and $|H^\perp| = |A| |H \cap \text{rad}(A)| / |H|$. It follows that $h^2 \leq 4s^2 k(2t) \leq 8s^2 m^2$, contradicting $h^2 = 16s^2 m^2$.

For the stated target, no other near-group parent order can have the required rational dimension scale. The global dimension E_N of $\text{NG}(G, N)$, $|G| = N$, has trace $N(N + 4)$ and norm $N^2(N + 4)$. Hence its scale-invariant trace-squared/norm ratio is $N + 4$. For D the ratio is $1400^2/9800 = 200$, forcing $N = 196$. The case $s = 7$ above therefore covers all Radchenko–Wheeler matrix parents with a possible dimension match. $\square$

This rules out the specified direct ordinary quotients, also followed by a Morita equivalence. It does not address a prior Morita change of the parent, arbitrary near-group categorifications with different input data, generalized-Haagerup parents, or arbitrary chains of constructions. The two rank-fourteen tensors remain unconstructed candidate based rings; R8_3_3472 was unresolved at this checkpoint. Its subsequent general exclusion is Theorem 8.1.

9.6. Maximal third indicators.

Proposition 9.10. *In the canonical positive spherical structure of every pseudo-unitary categorification of `R8_3_3472`, the third Frobenius–Schur indicator of every nonunit simple is 2. Equivalently, rotation acts identically on each two-dimensional space $\text{Hom}(1, X_i^{\otimes 3})$, $i > 0$.*

Proof. Put $\delta = 7 + 5\sqrt{2}$, $D = 700 + 490\sqrt{2}$, and $D' = 700 - 490\sqrt{2}$. The source dimensions are $(1, \delta - 1, \delta - 1, \delta, \delta - 1, \delta + 1, \delta + 1, \delta + 1)$, and $\dim \text{Hom}(1, X_i^{\otimes 3}) = 2$ for all $i > 0$. The spherical Cauchy theorem [61, Theorem 3.9] and $DD' = 9800$ imply that 3 is coprime to $N = \text{FSexp}(\mathcal{C})$. The dimensions lie in $\mathbb{Q}(\zeta_N)$, so $8 \mid N$. For $3\tilde{3} \equiv 1 \pmod{N}$, the automorphism $\sigma(\zeta_N) = \zeta_N^{\tilde{3}}$ therefore flips $\sqrt{2}$. The higher Gauss-sum formula [106, Theorem 4.4, (4.17)] gives

$$\sum_i d_i \nu_3(X_i) = \frac{D}{\sigma(D)} = \frac{D}{D'} = \delta^2 = 1 + 2 \sum_{i>0} d_i.$$

We have $\nu_3(1) = 1$. For $i > 0$, $\nu_3(X_i)$ is the trace of an order-three operator on a two-dimensional space, so $\text{Re } \nu_3(X_i) \leq 2$. All d_i are positive, so the displayed equality forces equality for every summand. Every rotation eigenvalue is consequently 1; the operator is diagonalizable because its cube is the identity. $\square$

The existing height-four full-character-field phase witness already has third-rotation weights $(0, 2)$, for trace types $(-1/2, 1)$, at every nonunit simple. Thus this restriction does not exclude that relaxation. Exact source arithmetic and the cached-coordinate audit are recorded in `evidence/r3472_focus_20261022/independent/verify_third_indicators.py`.

An exact phase extension of the integral center point. The integral center-relaxation point also admits a nonnegative rational extension satisfying all relative indicator and rotation moments through degree four, with the third indicators fixed by Proposition 9.10. Its ten positive high restriction types have multiplicities $2, \dots, 2, 98$, the last belonging to $(0, 1, 1, 1, 1, 1, 1, 1)$. Splitting these counts among the complete 37 joint relative-trace types, and adding three fourth-rotation variables for each nonunit simple, gives 391 variables. The 80 equations consist of ten fixed-mass equations, 42 coordinate equations for indicators of degrees 1, 2, 3, fourteen fourth-rotation dimension/square-trace equations, and fourteen fourth-indicator coordinate equations. An exact rational point has 40 positive entries.

The source-bound verifier

`evidence/r3472_focus_20261022/independent/verify_phase_extension.py`

regenerates the trace domain and checks every equation using rational arithmetic. It binds the restriction labels to the independently verified integral center point, and passes both normally and under `python -O`. The trace compression bound is 1120; orders outside its divisor box have zero joint traces through degree four, so no bound on the categorical FS exponent is assumed. The added constraints are indicator/rotation moments, not the complete complex modular relation *STS*. The phase and rotation weights remain rational; this feasible necessary relaxation supplies neither actual modular data nor a categorification.

9.7. Recovery of the arithmetic-count verifier. The preceding source package omitted the imported counting helper `second_branch_count.py`. Its reconstructed implementation reproduces exactly 272348, 124228, 3900, and 2 second-branch spectra for $h = 1, 2, 5, 7$, respectively. A separate generating-function algorithm independently verifies these counts. The three original mathematical scripts are unchanged. The historical receipt is preserved, and the new receipt differs only in four cache-state diagnostics. Both normal and optimized executions pass; no mathematical assertion was weakened. The repair, original source bindings, and independent replay are in `evidence/r3472_focus_20261022/reproducibility/`. These historical arithmetic spectra are superseded, as necessary Morita data, by the all-Morita restrictions above; none represents an asserted categorical construction.

For the second branch, the new order-two and order-five exclusions remove 124228 and 3900 of the previously retained 400477 arithmetic spectra. Exactly $272349 = 272348 + 1$ dimension multisets remain under these filters: the one-invertible spectra and the sole seven-invertible profile $1^7, \delta^7$. Indeed $\delta = 7 + 5r$ has the old coordinates $(s, t, u) = (7, 5, 2)$, whose seven copies give $\sum u^2 = 28$, $\sum ut = 70$, and $\sum t^2 = 175$. This is a count of necessary arithmetic profiles, not of fusion rings or fusion categories.

9.8. Continuation evidence and reproducibility. The new proofs and construction-family audits are in `evidence/rank8_frontier_20261021/`. The original M3 command above replays the original ledger. The separate continuation command is

```
python tools/verify_rank8_frontier.py
```

Run each command with normal Python and with `-O`. The continuation receipt binds the unchanged original ledger, every accepted new decision, the proof sources and the exact replay logs. Scoped failed construction searches and feasible necessary-condition witnesses retain their stated domains; they are not proofs of nonexistence or categorifiability.

The 22 October research overlay replays separately with

```
python3 tools/verify_rank8_research_20261022.py
```

It accepts the general exclusions of `R8_3_65` and `R8_3_1795`.

The exceptional-model overlay and new necessary-condition checks replay with

```
python3 tools/verify_rank8_creative_20261022.py
```

Run also with `-O`. The flag `-metadata-only` validates recorded receipts; `-historical` additionally replays earlier mathematics from the pinned historical archive. The separate archive is

```
supplements/rank8/release/rank8-certified-creative-20261022.zip.
```

Only `R8_3_982` changed status in that historical overlay, leaving five unresolved cases at that checkpoint.

The subsequent exact center-indicator certificate excludes `R8_3_3472` over $\mathbb{C}$ by Theorem 8.1. Its full source-derived replay is

```
python3 evidence/r3472_exclusion_20261022/replay.py.
```

The self-contained terminal checker is

```
python3 evidence/r3472_exclusion_20261022/terminal/check.py.
```

It verifies 1648 exact integer bound deductions and the final rational separator; the complete package also checks the categorical reduction, domain coverage and construction of the necessary system.

10. HISTORICAL MULTIPLICITY-TWO CRITERIA AND THEIR EXACT APPLICATION

10.1. Short tensor and subring tests. If X is self-dual and $X^2 = 1 + kX$ with $k \geq 2$, its span with 1 is a fusion subring that cannot arise over $\mathbb{C}$ by [12, 23]. When exactly seven basis elements are invertible and one additional object X satisfies

$$gX = Xg = X, \quad X^2 = \sum_{g \in G} g + \kappa X,$$

the rank-eight near-group case with $\kappa = 1$ or 2 is commutative and satisfies $0 < \kappa < |G| - 1$. Schopieray's general Proposition 3.2 [25] excludes it. The machine certificate checks the complete group and all displayed equations, not just the value of κ .

The zero-spectrum test is the finite nine-index condition of [28, Theorem 2.14] in the form reproduced in [17, Section 3]. It is a necessary condition for an associative tensor lift, with no pivotality assumption. Its C++ scanner searches the nine indices on the full tensor and the Python checker independently recomputes every coefficient, positivity condition and vanishing contraction. The native arrays are sized for rank eight, and the checker has no rank-six restriction.

Every duality-closed, multiplication-closed based subset of a category gives a fusion subcategory. We compare every such proper subring of ranks two through six against the *complete positive list* in the corresponding lower-rank theorem. The lower-rank classifications are unrestricted for ranks two, three and four, and complete through multiplicity 20 for rank five and 2 for rank six [12, 13, 14, 16, 17]; every rank-eight subring in this historical multiplicity-two stage also has multiplicity at most two. A rank-seven subring is not excluded merely because it is absent from an incomplete rank-seven list.

The rank-seven analysis [18] provides four specific, fully checked negative multiplicity-two tensors that occur as rank-eight closed subrings. We transfer their exact signed-sphericalization witnesses by complete seven-dimensional based isomorphism and recheck all four proof trees. This separate stage gives exclusions only for rings actually containing one of those four proved negative subrings; it makes no claim that the rank-seven positive catalogue is exhaustive.

The involution-module theorem of [17, Section 5] says that an invertible involution fixing basis elements forces a finite anti-invariant module representation. Where its stated hypothesis is met, the script searches *all* bounded integer matrices and verifies multiplication, transpose, parity and entry bounds. The FP dimension quotient of a hypothetical fusion subcategory must likewise be an algebraic integer [22, Proposition 8.15]; a rational coefficient in its exact minimal polynomial excludes categorification.

10.2. A rank-eight noncommutative codegree lemma. For any fusion ring, write

$$\Omega = \sum_i b_i b_{i^*}, \quad C_R = \sum_i N_i N_i^T.$$

For an irreducible representation E of $R \otimes \mathbb{C}$ of degree e_E , the central element Ω acts as the scalar $e_E f_E$, where f_E is its ordinary formal codegree. The regular representation contains E with multiplicity e_E , so that scalar has multiplicity e_E^2 in the characteristic polynomial of C_R [24, 17].

Proposition 10.1 (Noncommutative rank-eight codegrees). *Let R be a noncommutative fusion ring of rank eight. Its semisimple complex algebra has Wedderburn decomposition $\mathbb{C}^4 \oplus M_2(\mathbb{C})$. The formal codegree of the unique matrix-block representation is a rational integer. Every irreducible factor of the characteristic polynomial of C_R of degree at least two consists entirely of formal codegrees of one-dimensional representations. Consequently, if any such factor fails Ostrik's d -number divisibility condition, R has no complex fusion categorification.*

Proof. The FP dimension character is one-dimensional, so at least one scalar summand exists. Non-commutativity requires a matrix summand of dimension at least two. Since the sum of the squares of the degrees of simple modules is eight, there can be only one matrix summand and it must have degree two; the four remaining dimensions are one. The rational based algebra is split by some number field. Galois automorphisms permute its irreducible representations preserving their degrees, so the unique two-dimensional representation is Galois-fixed. Its formal codegree is an algebraic integer by [24] and rational by Galois invariance, hence belongs to $\mathbb{Z}$. Its Casimir eigenvalue is therefore rational. Every nonlinear irreducible factor of the integral Casimir polynomial comes from the four one-dimensional representations; on each, the eigenvalue is exactly the formal codegree. Finally, formal codegrees of complex fusion categories are d -numbers [24]. For an irreducible polynomial $q(t) = t^d + a_1 t^{d-1} + \dots + a_d$, this requires $|a_d|^j \mid a_j^d$ for every $j = 1, \dots, d$. Failure of one exact integer divisibility gives the assertion. $\square$

The commutative branch uses the ordinary d -number test on every Casimir factor. The script never treats raw Casimir eigenvalues as formal codegrees of noncommutative representations without Proposition 10.1. We also ran the rational-indivisibility screen of [26, Theorem 4.4], noncommuting field-automorphism witnesses from [17], and finite-field cyclotomic witnesses from [15]. These three screens add no exclusions in the present bound; their zero rows and exact code remain in the stage

data. Schopieray’s greedy formal-codegree obstruction 2, 3, 7 [27] cannot occur before multiplicity five, since its forced rank-four subring has a coefficient five, so it is not an active step here.

10.3. Integral divisibility and the complete signed quotient test. The integral Drinfeld condition [19, Section 2.3] says that, if all simple FP dimensions of a categorifiable ring are integers, every ordinary formal codegree is a positive integer dividing its global FP dimension. For a commutative tensor, the verifier finds its unique positive FP character by an exact integral eigenvector of the strictly positive matrix $\sum_i N_i$. It then compares every exact linear factor of $\det(tI - C_R)$ with $D = \sum_i \text{FPdim}(b_i)^2$. A nonintegral factor, or an integer codegree that does not divide D , is an unconditional complex no-go. This stage includes concrete multiplicity-two failures with $D = 156$ and 110, respectively, at $R8_2_288$ and $R8_2_392$.

The archived rank-eight appendix of [20] prints full tensors and explicit categorification decisions. Its items C.3.2(5) and C.3.2(7) are excluded by the paper’s no-induction-matrix Lemmas 4.3 and 4.2. The appendix-to-census verifier checks all 8^3 entries under a unit-preserving permutation before transferring either published exclusion. We rely on those cited lemmas for the induction obstruction; their original Normaliz computation is not repeated here. The same transfer finds group-theoretical models in items C.3.2(3), C.3.2(6), and C.3.2(11). Item C.3.2(4) matches $R8_1_3$ but supplies no existence or exclusion claim; the explicit twisted group-theoretical construction below now realizes that ring.

Finally, the oriented signed-sphericalization theorem of [17, 18] applies even when the alleged category has no pivotal structure. The canonical sphericalization has two lifts of each simple; quotienting its Grothendieck ring by the negative central invertible yields an ordered signed multiplication tensor. We enumerate every cyclic and duality orbit of signed coefficients within the exact parity bounds, require all ordered associativity equations, then check every possible spherical character by the inherited norm, duality, and dimension-order criteria. A finite search that times out or has an unrefuted branch supplies no exclusion. The attached campaign records each attempted multiplicity-one tensor, incomplete result, and timeout separately; only certificates whose complete signed branches independently regenerate are admitted to the ledger.

10.4. Grading obstructions with rank-one components. The following short arguments give five further exclusions without any pivotality assumption. In a faithful grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, every component is an invertible bimodule category over $\mathcal{D} = \mathcal{C}_e$, and $g \mapsto [\mathcal{C}_g]$ is a homomorphism to $\text{BrPic}(\mathcal{D})$ [7, Theorem 1.3 and Section 6].

Lemma 10.2. *If a pointed category Vec_A^ω has a module category with one simple object, then $[\omega] = 0$.*

Proof. Identify the module category with Vec . Each invertible object acts by the identity functor, up to isomorphism. The scalar module constraints form a nonzero 2-cochain whose coboundary is ω , by the module pentagon. Hence the associator class is trivial. $\square$

Proposition 10.3. *The rings $R8_1_26$, $R8_2_116$, and $R8_2_189$ have no complex fusion categorification.*

Proof. In the original zero-based census labels, $R8_1_26$ has a faithful C_4 grading with components

$$\{0, 3, 4\}, \quad \{1\}, \quad \{5, 6, 7\}, \quad \{2\}.$$

Its neutral group is $A = C_3$. Lemma 10.2 identifies its neutral category with Vec_{C_3} . By [7, Corollary 1.2], its Brauer–Picard group is the orthogonal group of $A \oplus A^*$ with hyperbolic quadratic form. Over $\mathbb{F}_3$ this consists of $I, -I, J, -J$, where J exchanges the two coordinates. Every square is the identity, so $[\mathcal{C}_2] = [\mathcal{C}_1]^2 = 1$. On the other hand, the degree-two invertible object $s = 5$ conjugates $a = 3$ to $a^{-1} = 4$. Its component is the regular left module with right action twisted by inversion. It cannot be equivalent to the regular bimodule: such an equivalence would send s to some $b \in A$ and the relation $as = sa^{-1}$ would imply $ab = ba^{-1}$, which is false in C_3 .

For $R8_2_116$ and $R8_2_189$, the universal grading has five components, of ranks 4, 1, 1, 1, 1, so its group is C_5 . Their neutral groups are respectively C_4 on $\{0, 1, 2, 7\}$ and C_2^2 on $\{0, 5, 6, 7\}$. Again their associator classes are trivial. In the first case the Brauer–Picard group embeds in $\mathrm{GL}_2(\mathbb{Z}/4)$, of order 96; in the second it is $O_4^+(\mathbb{F}_2)$, of order 72. Neither group has an element of order five. Hence the component homomorphism is trivial, forcing every component to be equivalent to the rank-four regular bimodule, contrary to the rank-one components. The verifier enumerates the small orthogonal groups and checks every grading support identity exactly. $\square$

Proposition 10.4. *The rings $R8_2_78$ and $R8_2_168$ have no complex fusion categorification.*

Proof. Each has a faithful grading with a rank-one component and neutral fusion rules $\mathrm{Rep}(S_3)$: $g^2 = 1$, $gx = xg = x$, and $x^2 = 1 + g + x$. A rank-one module category gives a fiber functor on the neutral category $\mathcal{D}$, so $\mathcal{D} \simeq \mathrm{Rep}(H)$ for a semisimple Hopf algebra of dimension six. The dimension-six classification [10] and the rank-three fusion rules force $H \simeq \mathbb{C}[S_3]$. The dual category of its fiber-functor module is $\mathrm{Rep}(H^*) \simeq \mathrm{Vec}_{S_3}$, of rank six [22, Example 7.12.26]. But invertibility of the homogeneous $\mathcal{D}$ -bimodule would identify that dual with $\mathcal{D}$, of rank three, a contradiction. $\square$

10.5. Solvability, equivariantization and module ranks. We give three further general exclusions using only small groups and exact fusion rules. By [3, Theorem 1.6], a fusion category of FP dimension $p^a q^b$ is solvable. The characterization in its Definition 1.2 and Proposition 4.4 expresses it as successive cyclic-prime extensions and equivariantizations. If its adjoint subcategory is the whole category, its last nontrivial step cannot be an extension. Thus it is an actual C_p -equivariantization, with a central $\mathrm{Rep}(C_p)$ whose underlying invertibles embed into the category. We use the orbit/stabilizer description of equivariant simples [6, Section 2]; cyclic stabilizers have only one-dimensional irreducible projective representations.

Proposition 10.5. *The ring $R8_1_29$ has no complex fusion categorification.*

Proof. Its invertibles form C_6 , its other two simples X, Y are self-dual of dimension three, and its adjoint ring is the whole ring. Its dimension is 24, so a putative category $\mathcal{C}$ is $\mathcal{D}^{C_p}$ with $p = 2$ or 3.

For $p = 3$, the order-three character subgroup fixes X, Y and has two free orbits on the six invertibles. The forgetful images of X, Y split into three distinct invertibles each; those of the invertibles give two more. Thus $\mathcal{D} = \mathrm{Vec}_A^\omega$, $|A| = 8$, with two free C_3 -orbits. The only groups of order eight admitting an order-three automorphism are C_2^3 and Q_8 . On C_2^3 the action has a fixed line and an irreducible plane. The three nonzero elements in that plane form a free orbit whose product has coefficient two into the same free orbit. The output stabilizer is trivial, so associator and tensorator choices cannot change this coefficient. On Q_8 , inversion exchanges the orbits $\{i, j, k\}$ and $\{-i, -j, -k\}$; their two induced dimension-three objects are distinct duals. These conclusions contradict multiplicity one and self-duality, respectively.

For $p = 2$, the character involution exchanges X, Y . De-equivariantization has the fusion rules

$$G(\mathcal{D}) = C_3 = \langle a \rangle, \quad a\rho = \rho a = \rho, \quad \rho^2 = 1 + a + a^2 + 2\rho.$$

It has dimension twelve and trivial universal grading. The same last-step argument gives $\mathcal{D} = (\mathrm{Vec}_V^\omega)^{C_3}$, where $V = C_2^2$ and C_3 permutes the three nonzero elements transitively. No assumption on ω is made. A monoidal automorphism of $\mathrm{Id}_{\mathcal{D}}$ has scalar one on ρ , because ρ occurs in ρ^2 , and then on all invertibles. Consequently the map from central invertibles to underlying invertibles is injective, and the central $\mathrm{Rep}(C_3)$ is unique.

The C_2 action on $\mathcal{D}$ fixes every simple: exchanging a, a^2 would produce a dimension-two simple upstairs. Every such tensor autoequivalence preserves the unique central $\mathrm{Rep}(C_3)$. Its restriction is tensor-isomorphic to the identity, since $H^2(C_3, \mathbb{C}^\times) = 0$. Equivariantization/de-equivariantization [4, Theorem 4.18(i) and Proposition 4.19] therefore lifts it to a C_3 -equivariant tensor autoequivalence of Vec_V^ω . Its object permutation centralizes C_3 in $\mathrm{Aut}(V) = S_3$, so lies in C_3 . Composing with the inverse action functor removes that permutation without changing the descended equivalence class.

Identity-on-objects tensor autoequivalences of Vec_V^ω are classified by $H^2(V, \mathbb{C}^\times) = C_2$: the two associators cancel in the tensorator equation. Their equivariant structures have ambiguity $H^1(C_3, \widehat{V}) = 0$, since multiplication by three is invertible on $\widehat{V}$. Both possible classes have explicit representatives. Besides the identity, take

$$J(u, v) = (-1)^{u_1 v_1 + u_1 v_2 + u_2 v_2}, \quad u, v \in \mathbb{F}_2^2.$$

This nontrivial normalized 2-cocycle is C_3 -invariant and satisfies $J^2 = 1$. Thus it commutes even with nontrivial scalar tensorators of the existing C_3 action and descends with identity comparisons. There are no further C_2 coherence choices, because $\text{Aut}_\otimes(\text{Id}_\mathcal{D}) = 1$.

The identity action leaves the coefficient two in ρ^2 . For the J action, all degree-zero channels have $J(u, u) = -1$ for $u \neq 0$; for each nonzero output degree, the two ordered channels have opposite signs. Hence each lift $\rho_\pm$ has square containing the three negative-character invertibles and one copy of each $\rho_\pm$. The unit occurs in $\rho_+ \rho_-$, not in either square. The two lifts are distinct duals, again contradicting the self-duality in $R8_1_29$. This exhausts both prime branches for arbitrary pointed associators. $\square$

Lemma 10.6 (Module-rank divisibility). *For a fusion category $\mathcal{B}$, let $g_\mathcal{B}$ be the greatest common divisor of the ranks of its indecomposable finite semisimple module categories. If $\mathcal{B} \subseteq \mathcal{D}$ is a fusion subcategory, then $g_\mathcal{B} \mid \text{rank}(\mathcal{D})$.*

Proof. Decompose $\mathcal{D}$, with its left tensor action by $\mathcal{B}$, into indecomposable module categories. Their ranks add to $\text{rank}(\mathcal{D})$. This statement uses ranks in the specified module presentation; it makes no assertion that module ranks are Morita invariant. $\square$

Proposition 10.7. *The rings $R8_1_45$ and $R8_1_47$ have no complex fusion categorification.*

Proof. Each ring has simple dimensions $(1, 1, 2, 3, 3, 4, 4, 4)$ and admits labels $1, h, E, P, Q, W_0, W_1, W_2$ with

$$\begin{aligned} h^2 &= 1, & hE &= E, & hP &= Q, & hW_i &= W_i, & P^* &= Q, \\ E^2 &= 1 + h + E, & EP &= EQ = P + Q, & PQ &= 1 + E + P + Q. \end{aligned}$$

The adjoint ring is the whole ring. Dimension 72 and the invertible group C_2 therefore force a putative categorification to be $\mathcal{C} = \mathcal{D}^{C_2}$ with $\text{FPdim}(\mathcal{D}) = 36$. The forgetful images are

$$F(1) = F(h) = 1, \quad F(E) = a + a^2, \quad F(P) = F(Q) = Y, \quad F(W_i) = U_i + V_i.$$

Here a has order three, Y is simple of dimension three, and the six U_i, V_i are distinct simples of dimension two. Thus $\text{rank}(\mathcal{D}) = 10$. The full fusion subcategory $\mathcal{B}$ on $1, a, a^2, Y$ satisfies

$$aY = Ya = Y, \quad Y^* = Y, \quad Y^2 = 1 + a + a^2 + 2Y.$$

The C_2 action exchanges a, a^2 and fixes Y ; its equivariantization inside $\mathcal{C}$ has the displayed nonself-dual P, Q .

As in the preceding proof, solvability and trivial universal grading imply $\mathcal{B} = (\text{Vec}_V^\omega)^{C_3}$, $V = C_2^2$, with the transitive action. The restriction map

$$H^3(V, \mathbb{C}^\times) \longrightarrow \prod_{\substack{L \leq V \\ |L|=2}} H^3(L, \mathbb{C}^\times)$$

is an isomorphism $C_2^3 \rightarrow C_2^3$. In integral-cohomology generators u^2, v^2, uv , the three restrictions of (r, s, t) are $(r, s, r + s + t)$. Since C_3 permutes the factors, there are exactly two invariant classes: zero and the class nontrivial on every subgroup of order two.

Suppose first $[\omega] = 0$. The lift of the object C_3 action is unique in $\text{Aut}_\otimes(\text{Vec}_V) = C_2 \times S_3$, and its coherence is unique because $H^2(C_3, \widehat{V}) = 0$. Therefore $\mathcal{B} \simeq \text{Rep}(A_4)$. Its central $\text{Rep}(C_3)$ is unique, since $\text{Aut}_\otimes(\text{Id}_\mathcal{B}) = 1$, and is preserved by the C_2 action. Sequential equivariantization and de-equivariantization [3, Section 2.6 and Proposition 2.10][4, Proposition 4.19] identify $\mathcal{B}^{C_2}$ with

$(\text{Vec}_V)^{S_3}$. The action on nonzero elements is the usual faithful S_3 permutation action; inversion on the character C_3 forces the group $C_3 \rtimes C_2 = S_3$.

There are exactly two lifts of this object action to $C_2 \times S_3$, given by the trivial or parity homomorphism into $H^2(V, \mathbb{C}^\times) = C_2$. Their coherence ambiguity vanishes: $H^j(S_3, \widehat{V}) = 0$ for $j > 0$, by the normal C_3 subgroup, whose positive cohomology and fixed space on $\widehat{V}$ both vanish. Both lifts yield self-dual dimension-three orbit objects. For completeness choose a stabilizing transposition τ swapping coordinates and its fixed nonzero element $v = (1, 1)$. For the trivial lift, the equivariance scalars $u = \pm 1$ act trivially on the unit channel of $v \otimes v$. For the other lift choose tensorator $\mu(a, b) = (-1)^{a_1 b_2}$. Its square is trivialized by $\eta_a = (-1)^{a_1 a_2}$, so $\eta_v = \mu(v, v) = -1$. The equivariance scalar satisfies $u^2 = \eta_v$ and acts on the unit channel by $u^2/\mu(v, v) = 1$. A τ -invariant character change of η is one on v , so cannot change the conclusion. Frobenius reciprocity for the orbit induction gives a tensor unit in each of the two squares. This contradicts $P^* = Q$.

It remains that $[\omega]$ is nontrivial on every order-two subgroup. The crossed product of Vec_V^ω by C_3 is pointed, and realizes $\mathcal{B}$ as the group-theoretical category

$$\mathcal{B} \simeq \mathcal{C}(A_4, \Omega, K, \psi), \quad K = C_3, \quad [\Omega]|_V = [\omega], \quad [\Omega]|_K = 0.$$

By the classification of pointed module categories and its double-coset rank formula [5, Example 2.1, Theorem 3.1 and Proposition 3.1], indecomposable $\mathcal{B}$ -module categories correspond to subgroups $H \leq A_4$ with $[\Omega]|_H = 0$ and permissible cochains. If $|H|$ were even, Cauchy's theorem would give a subgroup of order two, contradicting the restriction of Ω . Thus H is trivial or has order three. For $H = 1$, the four double cosets in $K \backslash A_4 / H$ have trivial intersections, giving module rank four. For $H = C_3$, conjugate it to K . There are two double cosets, with intersections C_3 and 1; projective cocycles on C_3 are coboundaries, so their contributions are three and one. Every indecomposable $\mathcal{B}$ -module category therefore has rank four, regardless of the permissible cochains. Lemma 10.6 would force $4 \mid \text{rank}(\mathcal{D}) = 10$, a contradiction. $\square$

The portable verifiers check the source-bound tensors, the displayed fusion identities, the complete small cocycle identities, and all relevant A_4 subgroups and double cosets. These finite checks certify the hypotheses; the categorical exhaustion is supplied by the proofs above. Neither argument assumes a braiding or a pivotal structure.

10.6. Rank-one modules and semisimple Hopf algebras.

Proposition 10.8. *The rings $R8_2_41$, $R8_2_86$, $R8_2_162$, $R8_2_140$ and $R8_2_784$ have no complex fusion categorification.*

Proof. For the first three rings, a faithful C_2 grading has a rank-one nonneutral component and a dimension-sixteen neutral category $\mathcal{D}$ of type $(1^4, 2^3)$. A rank-one module supplies a fiber functor, so $\mathcal{D} = \text{Rep}(H)$ for a semisimple Hopf algebra of dimension sixteen. Invertibility of the component further requires $\text{Rep}(H^*) \simeq \mathcal{D}$; in particular both groups of grouplikes have order four. Kashina's classification [44, Theorem 1.2 and Table 1] leaves only $H_{C:\sigma_1}$, whose representation ring is $\text{Rep}(D_{16})$: its invertibles form C_2^2 , and all three dimension-two simples are self-dual. The rings $R8_2_41$, $R8_2_86$ instead have cyclic invertibles, while $R8_2_162$ has a nonselfdual dimension-two pair.

For $R8_2_140$, census indices $\{0, 1, 2, 3, 7\}$ give the neutral Tambara–Yamagami category on C_4 , of dimension eight. Each other simple separately gives a rank-one left module: an invertible acts as one and the dimension-two object acts as two. Thus the neutral category has a fiber functor and is $\text{Rep}(H)$ with $G(H^*) = C_4$. Kashina's prime-power theorem [44, Theorem 1.1] says that a semisimple Hopf algebra of prime-power dimension with cyclic grouplikes is a group algebra. It would give $H^* \simeq \mathbb{C}[C_8]$, which has eight grouplikes, a contradiction. This argument does not require the rank-one modules to be grading components.

For $R8_2_784$, the neutral component has dimension twenty-two, rank seven and type $(1^2, 2^5)$; the other component has a unique simple of dimension $\sqrt{22}$. Its fiber functor reconstructs a Hopf

algebra of dimension $2 \cdot 11$. The Hopf pq classification [10] and the nonpointed neutral rules force $\mathcal{D} = \text{Rep}(D_{22})$. Its fiber-functor dual is $\text{Vec}_{D_{22}}$, of rank twenty-two. The invertible grading component would identify this dual with $\mathcal{D}$, of rank seven. This is impossible. $\square$

10.7. Preserved Tannakian subcategories.

Proposition 10.9. *The rings $R8_2_697$, $R8_2_808$ and $R8_2_181$ have no complex fusion categorification.*

Proof. For $R8_2_697$, dimension sixty-eight, type $(1^4, 4^4)$ and trivial universal grading, with all four invertibles fixing each noninvertible, force a C_2 equivariantization with descent $\mathcal{D}$ of dimension thirty-four and type $(1^2, 2^8)$. Its adjoint is the whole category: a proper adjoint has dimension 1, 2 or 17; a prime-dimensional adjoint is pointed, but these possibilities cannot contain the required dimension-four squares with only two invertibles. Solvability and prime descent now give $\mathcal{D} = (\text{Vec}_{C_{17}}^\omega)^{C_2}$, with inversion. Since $U(\mathcal{D}) = 1$, central lifts of its invertibles are unique. The outer involution therefore preserves its central $\text{Rep}(C_2)$, and the combined equivariantization group is C_2^2 , as its four linear characters are the census invertibles. Every $C_2^2 \rightarrow \text{Aut}(C_{17})$ has image of order at most two. A nonzero orbit has size two and cyclic stabilizer; it therefore supplies dimension-two equivariant simples. The target has only dimension-four noninvertibles.

For $R8_2_808$, dimension $186 = 2 \cdot 3 \cdot 31$ and the integral pqr theorem make any categorification group-theoretical [3, Theorem 9.2]. Groups of order 186 are solvable, so the category is solvable. The adjoint, type $(1^2, 2, 6^5)$, and the character involution fixing all noninvertibles force C_2 descent of type $(1^3, 3^{10})$ and dimension ninety-three. The same prime-dimensional-adjoint argument shows that its universal grading is trivial. Thus it is $(\text{Vec}_{C_{31}}^\omega)^{C_3}$ with nontrivial order-three action. Uniqueness of the central C_3 preserves it under the outer involution. The combined group is S_3 , because its embedded representation subcategory has type $(1^2, 2)$. Every homomorphism $S_3 \rightarrow \text{Aut}(C_{31})$ kills its normal C_3 , contradicting that action.

For the commutative ring $R8_2_181$, the neutral category has dimension twenty-one, type $(1^3, 3^2)$ and trivial universal grading. Solvability makes it $(\text{Vec}_{C_7}^\omega)^{C_3}$, with multiplication by two or four on C_7 . The action on $H^3(C_7, \mathbb{C}^\times) = C_7$ is multiplication by four or two, so its invariant associator class is zero. Coprime action cohomology then identifies the neutral category with $\text{Rep}(C_7 \rtimes C_3)$. Its central C_3 has unique lifts. The three odd simples have free, identical left and right C_3 actions, so the odd bimodule fixes these central lifts pointwise. Relative-center equivariantization [47] and $H^2(C_2, \mathbb{C}^\times) = 0$ give six central invertible lifts upstairs. The unique order-three subgroup has trivial self-braiding, inherited from the neutral $\text{Rep}(C_3)$, and is an embedded Tannakian subcategory. Its de-equivariantization has the Tambara–Yamagami rules on C_7 . The induced C_3 symmetry must preserve its nondegenerate symmetric bicharacter $\chi(a, b) = \exp(2\pi i k a b / 7)$, $k \neq 0$. Multiplication by two or four cannot do so: neither square is one modulo seven. This proves the contradiction without a pivotal or a fermionic-transparency assumption. $\square$

10.8. A dimension-twenty-eight adjoint criterion.

Lemma 10.10. *A nonpointed integral fusion category of dimension twenty-eight has universal grading C_2 and adjoint fusion ring $\text{Rep}(D_{14})$.*

Proof. Jordan–Larson’s theorem [45, Theorem 1.1] applies with $p = 7, q = 2$ and $7 \nmid q + 1$. Write the category as $\mathcal{C}(G, \omega, H, \psi)$, where $|G| = 28$. Sylow theory gives $G = C_7 \rtimes P$, with $P = C_4$ or C_2^2 ; the action is trivial or inversion. The possible subgroup orders are 1, 2, 4, 7, 14, 28. For orders one and seven, or a central subgroup of order two, the cyclic stabilizers give a pointed category. For a noncentral subgroup H of order two, its normal closure is $C_7 \rtimes H = D_{14}$, of index two; support gives a faithful C_2 grading. For $|H| = 14$, H is normal and support again gives such a grading. An abelian $H = C_{14}$ would give a pointed category; otherwise $H = D_{14}$, whose Schur multiplier is

trivial. For $H = G$ one obtains $\text{Rep}(G)$; a nonabelian group of order twenty-eight has center C_2 and quotient D_{14} . In each of these nonpointed cases the neutral fusion ring is $\text{Rep}(D_{14})$.

It remains to justify this ring identification for $|H| = 4$, and also the cocycle reduction used for a reflection complement above. Conjugate H to P . The positive-degree cohomology of C_7 is seven-primary, so the coprime Lyndon–Hochschild–Serre sequence has no mixed positive-degree terms. Since $\omega|_P = d\psi$, the associator class is represented by a P -invariant class on C_7 . Choose a normalized μ_7 representative, average over P , and raise to the inverse of $|P|$ modulo seven. This gives a strictly invariant representative α . On $C_7 \rtimes P$ use

$$\omega_\alpha((n_1, p_1), (n_2, p_2), (n_3, p_3)) = \alpha(n_1, p_1 n_2, p_1 p_2 n_3).$$

It is identically one on P . The remaining 2-cocycle ψ is removed by its inflation along $G \rightarrow P$. Thus the category is $(\text{Vec}_{C_7}^\alpha)^P$ with strict permutation action. A trivial action is pointed. Otherwise its kernel C_2 acts strictly trivially and its characters grade the category, with neutral $(\text{Vec}_{C_7}^\alpha)^{C_2}$. This has the $\text{Rep}(D_{14})$ fusion ring; the same argument with a reflection complement supplies the previous order-two case.

Finally this neutral ring is adjoint to itself. For its three dimension-two objects, indexed by nonzero elements modulo inversion, $X_i^2 = 1 + \varepsilon + X_{2i}$, and doubling cycles the three indices. The adjoint of the full category is contained in the displayed neutral component and contains its adjoint, so it equals that component. Consequently the universal grading is exactly C_2 . $\square$

Proposition 10.11. *There is no fusion category with dimension fifty-six, type $(1^4, 2, 4^3)$ and trivial universal grading. In particular $R8_2_694$ is generally excluded.*

Proof. Solvability and trivial universal grading force a last equivariantization. A C_7 equivariantization would supply seven invertibles, so the last group is C_2 . Its descent $\mathcal{D}$ has dimension twenty-eight and is integral by the orbit/stabilizer dimension formula. It cannot be pointed, since a C_2 equivariantization of a pointed category has all dimensions at most two. Lemma 10.10 therefore applies. The outer involution preserves $\mathcal{D}_{\text{ad}}$ and permutes its three dimension-two simples, fixing at least one. Its cyclic stabilizer has trivial projective multiplier, so its two characters produce two distinct dimension-two equivariant simples. The proposed category has only one. This is a contradiction. $\square$

10.9. A two-object orbit obstruction in dimension ninety-six.

Proposition 10.12. *A fusion category of type $(1^4, 2, 4, 6^2)$ has a nontrivial universal grading. In particular $R8_2_404$, $R8_2_418$, $R8_2_604$, $R8_2_692$ have no complex fusion categorification.*

Proof. Suppose such a category has trivial universal grading. Its dimension is ninety-six, and its invertible group has order four. Solvability forces a last C_2 equivariantization $\mathcal{C} = \mathcal{D}^{C_2}$. Whichever nontrivial character occurs, it fixes the unique dimension-two and dimension-four simples. Thus the outer involution exchanges the two dimension-two simples of $\mathcal{D}$. Depending on whether it fixes or exchanges the dimension-six pair upstairs, $\mathcal{D}$ has type $(1^4, 2^2, 3^4)$ or $(1^4, 2^2, 6)$ and dimension forty-eight.

Its canonical adjoint is preserved by the outer involution. A proper adjoint's dimension therefore has the form $a + 4b + 9c + 36e$, where $a \in \{1, 2, 4\}$, $b \in \{0, 2\}$, c is even, and e is zero or one according to the displayed type. The proper divisors of forty-eight among these possibilities are only 1, 2, 4, 12. A pointed adjoint of dimension at most four cannot contain the square of a dimension-three or dimension-six simple, since each invertible occurs there with coefficient at most one. An adjoint of dimension twelve consists of all four invertibles and both dimension-two simples. Every other grading component would then have dimension twelve using only dimension-three or dimension-six simples; a sum of their squared dimensions is divisible by nine, so this is impossible. Hence $\mathcal{D}_{\text{ad}} = \mathcal{D}$.

A second prime descent gives $\mathcal{D} = \mathcal{E}^{C_2}$, $\text{FPdim}(\mathcal{E}) = 24$. The complete cyclic-orbit dimension partitions are $(1^6, 3^2)$ and $(1^2, 2, 3^2)$. In the first case the dimension-two pair of $\mathcal{D}$ comes from two free orbits in the invertible group A of order six. The inner involution fixes exactly two elements of

A. For $A = C_6$ it is inversion, and for $A = S_3$ it is conjugation by a transposition. In either group the coefficients of the two induced objects in their own squares are respectively zero and one, by ordinary orbit addition. The output stabilizers are trivial, so associator or tensorator phases cannot change these coefficients. The outer involution cannot exchange this pair as required.

In the second case the unique dimension-two simple x of $\mathcal{E}$ is fixed by the inner involution, with $x^2 = 1 + g + x$. Its two equivariant lifts x_+, x_- are exchanged by the central character ε . Therefore their squares are equal, because $\varepsilon^2 = 1$ and ε is central. Each square has precisely one constituent from the dimension-two pair, by forgetting to the single x channel. The outer involution exchanges x_+, x_- but preserves their common square. Its unique dimension-two constituent would have to be fixed, which is impossible. Both dimension partitions and every initial character choice have therefore been excluded. The supplementary check binds the full four source tensors and exhausts these small dimension and orbit cases. $\square$

10.10. An order-four action on rank-one components.

Proposition 10.13. *The ring $R8_2_68$ has no complex fusion categorification.*

Proof. Its exact type is $(1^4, 4^2, 6^2)$, its invertibles form C_4 , its adjoint is the whole ring, and its order-two invertible fixes all four noninvertibles. Solvability in dimension 108 therefore forces $\mathcal{C} = \mathcal{D}^{C_2}$ with $\mathcal{D}$ integral of dimension 54 and type $(1^2, 2^4, 3^4)$. The outer involution exchanges both noninvertible dimension classes in pairs and fixes the invertibles. The possible proper dimensions of its invariant adjoint are 1, 2, 9, 18, 27. Pointed dimensions one and two cannot contain the square of a dimension-three object; dimensions nine and twenty-seven would contain dimension-two objects, contradicting strong Frobenius divisibility [3]. Thus either $\mathcal{D}_{\text{ad}} = \mathcal{D}$ or its adjoint $\mathcal{E}$ has type $(1^2, 2^4)$ and dimension eighteen.

In the first case prime descent gives $\mathcal{D} = \mathcal{F}^{C_2}$, with $\mathcal{F}$ of type $(1^9, 3^2)$. In the second case $U(\mathcal{E}) = 1$: a grading of order two would have a dimension-nine neutral containing a dimension-two simple, again contradicting strong Frobenius; order-three components of dimension six would each require both invertibles; higher grading orders cannot accommodate a dimension-two simple. Hence $\mathcal{E} = (\text{Vec}_A^\omega)^{C_2}$, $|A| = 9$, with inversion on A . Its unique nontrivial central invertible g is the canonical Tannakian lift. The relative-center C_3 action of the grading of $\mathcal{D}$ fixes this lift. Since $H^2(C_3, \mathbb{C}^\times) = 0$, its central equivariant lifts, together with the unit lifts, form an abelian group of order six. The unique order-two lift restricts to g and inherits its self-braiding $+1$; it therefore gives a central $\text{Rep}(C_2)$ in $\mathcal{D}$. This explicitly verifies bosonicity, rather than inferring it from centrality alone. De-equivariantization again gives $\mathcal{F}$ of type $(1^9, 3^2)$: a dimension-three object fixed by g would descend to dimension $3/2$, which is not an algebraic integer.

In both cases the order-two central lift in $\mathcal{D}$ is unique: its kernel is trivial when $U(\mathcal{D}) = 1$, and has order three when $U(\mathcal{D}) = C_3$. The outer involution preserves this Tannakian subcategory. Combining the two equivariantizations gives $\mathcal{C} = \mathcal{F}^{C_4}$, because the intermediate symmetric subcategory has dimension four and its four characters form the census C_4 . Let β generate this coherent action. Its square acts by inversion on $A = \text{Inv}(\mathcal{F})$ and fixes the two dimension-three objects.

The adjoint of $\mathcal{F}$ is Vec_A^ω , with a C_3 grading of ranks 9, 1, 1. Rank-one components force $[\omega] = 0$ by their module pentagon. The action of β exchanges these two components; otherwise the original $\mathcal{C}$ would inherit a nontrivial C_3 grading. For $A = C_9$ no automorphism squares to inversion. For $A = C_3^2$, its matrix B satisfies $B^2 = -I$, so its minimal polynomial is the irreducible $t^2 + 1$ over $\mathbb{F}_3$ and $\det(B) = 1$. Its tensorator class $\eta \in H^2(A, \mathbb{C}^\times) = C_3$ is fixed by B ; order-four coherence gives $4\eta = 0$, hence $\eta = 0$. Such a tensor autoequivalence fixes every rank-one left-module class, since those classes are $H^2(A, \mathbb{C}^\times)$. But the two nonneutral components have different left-module classes: an equivalence, tensored with the inverse of one invertible bimodule, would identify the rank-nine regular module with a rank-one component. Thus β cannot exchange them, a contradiction. The supplement checks every source coefficient, adjoint dimension partition and matrix calculation; the central lift and action tower use [47, 4, 3]. $\square$

10.11. Integral extensions from pointed Morita data.

Lemma 10.14. *Suppose $\mathcal{B}$ is Morita equivalent to Vec_G^ω and every normal abelian subgroup of G has square order. Then every faithful finite-group graded extension of $\mathcal{B}$ is integral.*

Proof. An invertible Vec_G^ω bimodule is, as a left module, a transitive pointed module associated to a subgroup $N \leq G$ and a cochain trivializing $\omega|_N$. Its dual is pointed [7, Proposition 4.2]. In the twisted double-coset description [46, Theorems 4.1 and 5.1], a simple over NaN has FP dimension

$$[N : N \cap aNa^{-1}] \dim(\pi),$$

where π is an irreducible projective representation of the intersection. Pointedness forces N to be normal. The identity coset subcategory is ordinary $\text{Rep}(N)$, even with the general associator and subgroup cochain [46, Theorem 4.1]; thus N is abelian as well. The left module has $[G : N]$ simples with equal normalized FP dimension. Since a graded component has total squared dimension $|G|$, each dimension is $\sqrt{|N|}$, an integer. Every component of an extension of Vec_G^ω is an invertible bimodule, so that extension is integral.

Morita equivalence transports the complete graded-extension data, including multiplication and associativity, and the resulting complete extensions are Morita equivalent [7, Corollary 3.13 and Theorem 1.3]. Integrality is preserved under this equivalence in the present setting. Indeed, the center of an integral category is integral because forgetting preserves FP dimensions. A Morita equivalent category $\mathcal{C}$ has the same center and integer total dimension. For a simple $W \in \mathcal{C}$, induction has dimension $\text{FPdim}(\mathcal{C}) \text{FPdim}(W)$, an integer. Hence $\text{FPdim}(W)$ is rational and an algebraic integer, and therefore is an integer. $\square$

Proposition 10.15. *The rings $R8_2_53$ and $R8_2_93$ have no categorification over $\mathbb{C}$.*

Proof. Each source tensor has FP type $1^4, 4^2, (3\sqrt{2})^2$ and a faithful C_2 grading with integral neutral subring D of dimension 36. Its invertibles form $C_4 = \langle g \rangle$. They fix both noninvertibles X, Y , which are self-dual, and

$$X^2 = 1 + g + g^2 + g^3 + X + 2Y, \quad Y^2 = 1 + g + g^2 + g^3 + 2X + Y.$$

In particular $D_{\text{ad}} = D$. Any categorification $\mathcal{D}$ is solvable by the two-prime theorem [3, Theorem 1.6]. Its last prime step must be an equivariantization since its universal grading is trivial. The prime cannot be three because $\text{Inv}(\mathcal{D}) = C_4$ contains no C_3 . Thus $\mathcal{D} = \mathcal{E}^{C_2}$, with $\text{FPdim}(\mathcal{E}) = 18$. The sign object is g^2 and fixes X, Y . The equivariantization orbit formula [6] makes $\mathcal{E}$ of type $1^2, 2^4$.

Let η be its nontrivial invertible. In every ZZ^* with $\text{FPdim}(Z) = 2$, omitting η would leave dimension three after the unit, which cannot be a sum of two-dimensional simples. Therefore $ZZ^* = 1 + \eta + T$ with $\text{FPdim}(T) = 2$. The adjoint subcategory contains both units and a two-dimensional simple. If proper, its integer dimension divides 18; the only possibilities for a subcategory of type $1^2, 2^n$ are 2 and 6. Dimension two omits T . At dimension six, the other universal-grading components contain only two-dimensional simples and cannot have total squared dimension six. Hence $\mathcal{E}_{\text{ad}} = \mathcal{E}$. Solvability again forces $\mathcal{E} = \mathcal{P}^{C_2}$, where $\text{FPdim}(\mathcal{P}) = 9$ and $\mathcal{P}$ is pointed [21, 22].

The canonical central $\text{Rep}(C_2) \subset Z(\mathcal{E})$ has underlying object η . Its half-braiding is unique: two half-braidings on an invertible differ by a tensor character, and $\text{Aut}_\otimes(\text{Id}_\mathcal{E}) = \text{Hom}(U(\mathcal{E}), \mathbb{C}^\times) = 1$. The outer C_2 action therefore preserves this central subcategory. Its equivariantization embeds as a symmetric subcategory of $Z(\mathcal{E}^{C_2})$: equivariant half-braidings act on every equivariant object, and the morphism conditions give full faithfulness. Forgetting equivariance followed by the fiber functor on $\text{Rep}(C_2)$ gives a fiber functor, so this subcategory is $\text{Rep}(H)$ for a group of order four. Its four underlying invertibles exhaust the C_4 in $\mathcal{D}$, forcing $H = C_4$. Central de-equivariantization [4, 22] now gives

$$\mathcal{D} \simeq (\text{Vec}_A^\alpha)^{C_4}, \quad |A| = 9.$$

The dimension-four noninvertibles force C_4 to act freely on all nonidentity elements of A . Thus $A = C_3^2$, since $\text{Aut}(C_9)$ has order six. Its generator acts by an irreducible matrix J over $\mathbb{F}_3$ with $J^2 = -I$.

The pointed crossed product is Morita dual to this equivariantization. Consequently $\mathcal{D}$ is Morita equivalent to Vec_G^ω , where $G = C_3^2 \rtimes_J C_4$; the group extension splits by coprimality. The only normal abelian subgroups of G are 1 and C_3^2 . Indeed, an element with nonzero C_4 exponent has commutators with C_3^2 filling $(J^k - I)C_3^2 = C_3^2$. A normal subgroup containing such an element contains C_3^2 and cannot be abelian. Within C_3^2 , irreducibility of J leaves only 1 and C_3^2 . Their orders are squares, so Lemma 10.14 makes every extension of $\mathcal{D}$ integral. The alleged census category would be a C_2 extension with two simples of dimension $3\sqrt{2}$, a contradiction. The exact verifier checks both full source tensors, the grading and every finite-group premise. $\square$

Proposition 10.16. *The rings $R8_2_191$ and $R8_2_299$ have no categorification over $\mathbb{C}$.*

Proof. Their source tensors have FP type $1^4, 2^2, (\sqrt{6})^2$ and a faithful C_2 grading with integral neutral subring D of dimension 12. Its invertibles form V_4 and fix two simples X, Y , with

$$X^* = Y, \quad X^2 = 2Y, \quad Y^2 = 2X, \quad XY = YX = \sum_{g \in V_4} g.$$

Thus a hypothetical neutral category $\mathcal{D}$ is faithfully C_3 graded over its pointed subcategory $\mathcal{B}$ on V_4 , with rank-one nonneutral components. A rank-one module trivializes the pointed associator, so $\mathcal{B} \simeq \text{Vec}_{V_4}$.

The ambient odd component $\mathcal{M}$ has two simples permuted transitively by V_4 , with a stabilizer $H = C_2$; its stabilizer in source labels is $(0, 5)$ for $R8_2_191$ and $(0, 7)$ for $R8_2_299$. Its restriction $\mathcal{N}$ to $\mathcal{B}$ is therefore an indecomposable rank-two module. Since $H^2(C_2, \mathbb{C}^\times) = 0$, its cochain is trivial. A decomposition $V_4 = H \times K$ identifies its dual with

$$\mathcal{B}_{\mathcal{N}}^* \simeq \text{Rep}(H) \boxtimes \text{Vec}_K \simeq \text{Vec}_{V_4},$$

including the trivial associator. The actual multiplication action of $\mathcal{D}$ on $\mathcal{M}$ is a coherent extension of this $\mathcal{B}$ module.

Apply [58, Theorem 1.3] to the *induced* module $\mathcal{L} = \mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{N}$. It makes the dual $\mathcal{D}_{\mathcal{L}}^*$ a semidirect C_3 product with neutral component $\mathcal{B}_{\mathcal{N}}^*$, and hence pointed. The theorem concerns this induced-module dual; it does not assert that the dual of the original two-object $\mathcal{M}$ is pointed.

The original C_3 Brauer–Picard action on $\mathcal{B}$ is nontrivial: a rank-one component is not the regular rank-four module. Its class therefore has order three. Morita transport conjugates the action and preserves this order. The tensor-automorphism classes of Vec_{V_4} have kernel $H^2(V_4, \mathbb{C}^\times) = C_2$ over $\text{Aut}(V_4) = S_3$ [22]. An order-three class must therefore act nontrivially on V_4 . The pointed Morita group is $V_4 \rtimes C_3 = A_4$, so $\mathcal{D}$ is Morita equivalent to $\text{Vec}_{A_4}^\omega$ for some ω . The only normal abelian subgroups of A_4 are 1 and V_4 , of square orders 1 and 4. Lemma 10.14 makes every extension of this compatible neutral category integral, contradicting the odd FP dimensions $\sqrt{6}$. The source checker verifies both gradings, the complete neutral tensors, the odd module action and all ten subgroups of A_4 . The neutral ring by itself is not claimed to have only pointed Morita models; the coherent odd module is essential. $\square$

Proposition 10.17. *The rings $R8_2_158$, $R8_2_186$, $R8_2_579$ and $R8_2_682$ have no categorification over $\mathbb{C}$.*

Proof. All four source tensors have FP type $1^4, 2, 4, (2\sqrt{3})^2$ and a faithful C_2 grading. Their common integral neutral subring D , of dimension 24, has V_4 invertibles fixing Y, Z and

$$Y^* = Y, \quad Z^* = Z, \quad Y^2 = \sum_{g \in V_4} g, \quad YZ = ZY = 2Z, \quad Z^2 = \sum_{g \in V_4} g + 2Y + 2Z.$$

Thus $D_{\text{ad}} = D$. Solvability [3, Theorem 1.6] forces a neutral categorification to be $\mathcal{D} = \mathcal{E}^{C_2}$ with $\text{FPdim}(\mathcal{E}) = 12$; its invertibles exclude a final C_3 equivariantization. The sign ϵ fixes Y, Z . The orbit formula makes $\mathcal{E}$ of type $1^4, 2^2$, with an outer involution swapping the two noninvertibles and swapping a pair of invertibles.

The finite rank-six check exhausts both groups $A = C_4, V_4$, their trivial and transitive actions on the two nonunits, both dualities, and every oriented nonunit product allowed by FP dimension and reciprocity. There are 100 candidate tables and six associative ones, of which exactly two admit the required outer involution. Their rings are $R_{3,A}$:

$$X^* = X', \quad X^2 = 2X', \quad (X')^2 = 2X, \quad XX' = X'X = \sum_{g \in A} g,$$

with all units fixing X, X' . The transitive C_4 action in this exhaustion is the homomorphism $(0, 1, 0, 1)$ in cyclic labels. Jordan–Larson [45, Theorem 1.4] excludes $A = C_4$, since its cyclic prime-power factor occurs with odd multiplicity. Thus $\mathcal{E}$ is C_3 graded over $\mathcal{B} \simeq \text{Vec}_{V_4}$; its rank-one components trivialize the pointed associator.

Let $\mathcal{M}$ be the two-object ambient odd module over $\mathcal{D}$. The central regular algebra $1 + \epsilon$ in $\mathcal{D}$ gives an actual de-equivariantized module $\mathcal{M}_{1+\epsilon}$ over $\mathcal{E}$. This construction uses the central algebra inside $\mathcal{D}$; it requires no half-braiding on ϵ in the ambient category. Its rank is one if ϵ swaps the source simples, and four if it fixes them: each fixed simple has two algebra-module structures, using $H^2(C_2, \mathbb{C}^\times) = 0$.

The six indecomposable Vec_{V_4} modules have ranks 4, 1, 1, 2, 2, 2, by their stabilizers and cochains. The C_3 action sends the regular rank-four class to the X component, then to the X' component. These are distinct rank-one classes, so they form a three-cycle of total rank six. The three rank-two classes are either fixed or form another orbit of total rank six. Galindo’s Clifford theorem [58, Proposition 4.1 and Theorem 1.1] now excludes rank one for $\mathcal{M}_{1+\epsilon}$, and makes its rank-four alternative two fixed rank-two submodules. The group C_3 cannot permute two actual summands; each is a coherent $\mathcal{E}$ -module extension of a rank-two $\mathcal{B}$ module.

For one such module, the induced-module dual of [58, Theorem 1.3] is a semidirect C_3 product with neutral Vec_{V_4} . Its action class has order three, as in Proposition 10.16. Since the tensor-automorphism classes are $C_2 \times S_3$, this class is a pure order-three group action. The remaining action coherence lies in $H^2(C_3, \widehat{V_4}) = 0$ by coprimality [57, Proposition 3.2]. Hence the action is equivalent to the strict group action and the dual is the untwisted Vec_{A_4} . Its graded tensor section is part of Galindo’s semidirect-product conclusion; no uncontrolled C_3 associator twist is discarded.

The equivariantization $\mathcal{D}$ is Morita equivalent to $\mathcal{E} \rtimes C_2$. Morita transport gives a C_2 extension $\mathcal{T}$ of Vec_{A_4} with the same center as $\mathcal{D}$. The center of Vec_{A_4} has exactly three Lagrangian subcategories, all in the orbit of the canonical $\text{Rep}(A_4)$ [57, Proposition 7.6 and Section 8.3]. Consequently each has untwisted Vec_{A_4} as pointed Morita dual. The involution fixes at least one of these three. For its invariant pointed module $\mathcal{N}$, the dual of $\mathcal{T} \boxtimes_{\text{Vec}_{A_4}} \mathcal{N}$ is a C_2 crossed product with pointed neutral Vec_{A_4} [58, Proposition 5.3]; every component is pointed. This step requires module-class invariance, rather than a coherent extension of $\mathcal{N}$. It gives a pointed Morita group

$$1 \longrightarrow A_4 \longrightarrow G \longrightarrow C_2 \longrightarrow 1.$$

Because A_4 is centerless with $\text{Out}(A_4) = C_2$, G is S_4 or $A_4 \times C_2$. The latter has six identity-supported central characters for every associator. But $\text{Inv}(Z(\mathcal{D})) \rightarrow \text{Inv}(\mathcal{D})$ is injective: its kernel is $\text{Hom}(U(\mathcal{D}), \mathbb{C}^\times) = 1$. The four source invertibles therefore exclude $A_4 \times C_2$. Thus $\mathcal{D}$ is Morita equivalent to $\text{Vec}_{S_4}^\omega$.

The only normal abelian subgroups of S_4 are 1 and its V_4 . Lemma 10.14 makes every extension of $\mathcal{D}$ integral, contradicting the two odd dimensions $2\sqrt{3}$. The exact checker validates all four source tensors, the small-ring exhaustion, module-rank alternatives and all thirty subgroups of S_4 . $\square$

11. FURTHER GENERAL OBSTRUCTIONS

The following arguments exclude nine further source tensors without a pivotality assumption. The proofs use a conductor bound, a Frobenius–Perron index, cyclic-module divisibility, and changes of Morita representative.

Proposition 11.1. *The rings $R8_2_35$, $R8_2_38$, $R8_2_346$ and $R8_2_406$ have no categorification over $\mathbb{C}$.*

Proof. Each tensor has four invertibles and four simples of FP dimension $d = 3 + \sqrt{10}$. By the published [25, Lemma 3.1], existence of any categorification implies existence of a pseudo-unitary Galois conjugate, whose canonical spherical structure has global dimension

$$D = 4(1 + d^2) = 80 + 24\sqrt{10}, \quad DD' = 640.$$

Its simple-dimension and global-dimension fields both equal $\mathbb{Q}(\sqrt{10})$. The nontrivial quadratic character has conductor 40, so [63, Theorem 5.1] requires

$$\left| \frac{1}{D} - \frac{1}{D'} \right|^2 \leq \frac{1}{40}.$$

The left side is $9/160 > 1/40$, a contradiction. $\square$

Proposition 11.2. *The ring $R8_2_209$ has no categorification over $\mathbb{C}$.*

Proof. Put $d = (5 + \sqrt{29})/2$. The source dimension vector is $(1, 1, 1, d, d-1, d+1, d, d)$, and its first three basis elements form a pointed subcategory of dimension 3. Hence

$$\text{FPdim}(\mathcal{C}) = 5(1 + d^2) = \frac{145 + 25\sqrt{29}}{2}.$$

By [21, Proposition 8.15], its quotient by 3 must be an algebraic integer. Its field trace is $145/3$, which is not an integer. $\square$

Lemma 11.3 (Cyclic-module divisibility). *Let $H \cong C_p$, with p prime, be a group of invertibles in a fusion category over $\mathbb{C}$. Suppose that simple objects X, Y, Z satisfy $hX \cong X$, $Yh \cong Y$, and $hZ \cong Z \cong Zh$ for every $h \in H$. If $\text{FPdim}(Z)/p$ is not an algebraic integer, then $p \mid N_{XY}^Z$.*

Proof. The rank-one module category generated by X trivializes the pointed associator on H . Thus $A = \bigoplus_{h \in H} h$ admits its regular connected separable algebra structure. Since $H^2(C_p, \mathbb{C}^\times) = 0$, the stabilizer cocycles can be trivialized to give X a left A -module structure and Y a right A -module structure. Hence $X \otimes Y$ is an A -bimodule.

The A -bimodules supported on the fixed simple Z correspond to projective representations of C_p^2 : the projective-representation dimension equals the underlying multiplicity of Z . A scalar two-cocycle on C_p^2 has either zero or nondegenerate commutator pairing. Its twisted group algebra is accordingly $\mathbb{C}^{p^2}$ or $M_p(\mathbb{C})$. The first possibility would give a simple bimodule of FP dimension $\text{FPdim}(Z)/\text{FPdim}(A) = \text{FPdim}(Z)/p$, contrary to algebraic integrality. In the second possibility every simple bimodule supported on Z has underlying object pZ . Decomposing the Z -isotypic bimodule summand of $X \otimes Y$ proves the assertion. This is the regular cyclic-algebra method of [64, Lemmas 6.1–6.2], with the stabilizer cocycle retained rather than assumed trivial. $\square$

For later use, a free left C_p orbit whose right orbit agrees with it has cyclic left-right stabilizer of order p . Its block in $A\text{-bimod}(\mathcal{C})$ therefore consists of p simples of the original FP dimension. A label fixed on both sides instead contributes either p^2 simples of dimension $\text{FPdim}(X)/p$ or one simple of dimension $\text{FPdim}(X)$, by the preceding proof.

Proposition 11.4. *The ring $R8_2_260$ has no categorification over $\mathbb{C}$.*

Proof. Put $d = 3 + 2\sqrt{3}$. Its dimension vector is $(1, 1, 1, d, d, d, d, 3)$, and the first three basis elements form C_3 . They fix label 3 on both sides. Since $\text{Norm}(d/3) = -1/3$, Lemma 11.3 requires $3 \mid N_{33}^3$. The source tensor has $N_{33}^3 = 2$. $\square$

Proposition 11.5. *The rings $R8_2_380$ and $R8_2_673$ have no categorification over $\mathbb{C}$.*

Proof. Write $x = 4 + \sqrt{6}$ and $y = 3 + 2\sqrt{6}$. Each ring has a subcategory $\mathcal{E}$ with the representation ring of the dihedral group D_{10} of order ten. It occupies labels $(0, 3, 4, 5)$ in $R8_2_380$ and $(0, 1, 6, 7)$ in $R8_2_673$. Its nontrivial invertible g fixes the two dimension-two labels, so the regular algebra $A = 1 + g$ exists. By [64, Lemma 6.2 and the proof of Theorem 6.3], $A\text{-bimod}(\mathcal{E})$ is pointed with group D_{10} .

The outside labels form two free g -orbits, of dimensions x and y , with the same left and right actions. They are $(1, 2), (6, 7)$ in the first tensor and $(2, 5), (3, 4)$ in the second. The cyclic stabilizer calculation gives two x -dimensional and two y -dimensional simples in the dual category $\mathcal{B} = A\text{-bimod}(\mathcal{C})$; all its other simples are the ten invertibles of D_{10} .

The rotation subgroup $H = C_5$ fixes every noninvertible on both sides, since each dimension fiber has only two elements. Moreover

$$\text{Norm}(x/5) = \frac{2}{5}, \quad \text{Norm}(y/5) = -\frac{3}{5}.$$

For an x -dimensional simple X , Lemma 11.3 therefore makes every noninvertible multiplicity in XX^* divisible by 5. Write $x^2 = a + bx + cy$ for the resulting dimension equation, where a, b, c are nonnegative integers. The $\sqrt{6}$ coefficient gives $b + 2c = 8$, contradicting $5 \mid b, c$. $\square$

Proposition 11.6. *The ring $R8_2_745$ has no categorification over $\mathbb{C}$.*

Proof. Put $d = 1 + \sqrt{2}$ and let g be label 1. The dimension vector is $(1, 1, d, d, 2, 2, 2d, 2d)$. The subsets $\mathcal{D} = (0, 1, 2, 3)$ and $\mathcal{E} = (0, 1, 4, 5)$ are fusion subcategories, the latter with the representation ring of the dihedral group D_{10} of order 10. The element g exchanges labels 2, 3 and fixes labels 4, 5, 6, 7 on both sides. Thus $A = 1 + g$ exists.

In $\mathcal{B} = A\text{-bimod}(\mathcal{C})$, the subcategory $\mathcal{E}' = A\text{-bimod}(\mathcal{E})$ is pointed with group D_{10} , by [64, Lemma 6.2 and the proof of Theorem 6.3]. The subcategory $\mathcal{D}' = A\text{-bimod}(\mathcal{D})$ has two invertibles and two simples of dimension d , by the cyclic-stabilizer calculation above. Each of labels 6, 7 contributes either four simples of dimension d or one of dimension $2d$. These are all the simples of $\mathcal{B}$, so it has precisely ten invertibles and 2, 6 or 10 simples of dimension d .

For a d -dimensional simple X , the dimension of XX^* is $a + bd$ with $a, b \in \mathbb{Z}_{\geq 0}$, where a is its invertible stabilizer order. The identity $d^2 = 1 + 2d$ forces $a = 1$. Thus the D_{10} action on these simples is free, and their number must be 10. It follows that $\mathcal{B}$ is generalized near-group, with no $2d$ -dimensional simples remaining.

By [65, Proposition 3.6(1)], the adjoint subcategory $\mathcal{B}_{\text{ad}}$ is nonpointed and is contained in every nonpointed fusion subcategory, hence in $\mathcal{D}'$. There is no proper nonpointed fusion subcategory of $\mathcal{D}'$: one noninvertible would give the forbidden rank-two rule $X^2 = 1 + 2X$ [23], while two noninvertibles force the other invertible by reciprocity. Therefore $\mathcal{B}_{\text{ad}} = \mathcal{D}'$. Its order-two invertible group must be normal in D_{10} , since conjugation preserves the adjoint subcategory. No such normal subgroup exists. $\square$

The source-bound verifiers in `evidence/theoretical_20261016/rank8` check all dimension characters and subcategory tensors used above, including the indicated left and right actions. The categorical implications are the preceding proofs and their explicitly cited dependencies.

12. AN ANTI-INVARIANT MODULE OBSTRUCTION

We give eight further exclusions that require no pivotal structure.

Lemma 12.1. *Let g be an invertible simple of order two in a fusion category, and let $x_1, \dots, x_f$ be all simples fixed by right tensoring by g , where $f > 0$. There is an integral f -dimensional representation T of the fusion ring such that, with $B_X = (N_{Xx_a}^{x_b})_{a,b}$,*

$$|(T_X)_{ab}| \leq (B_X)_{ab}, \quad (T_X)_{ab} \equiv (B_X)_{ab} \pmod{2}, \quad T_{X^*} = T_X^t.$$

In particular T_X is normal if $XX^* = X^*X$.

Proof. A fixed simple supplies a rank-one module category over the pointed category generated by g , trivializing its associator. Hence the regular connected separable algebra $A = \mathbf{1} + g$ exists; compare [64, Lemmas 6.1–6.2]. Each fixed simple has two simple right A -module structures m_a^+, m_a^- , whereas each free right orbit gives one simple module. Twisting the right action by the algebra automorphism that acts as -1 on g exchanges m_a^+ with m_a^- and fixes the free-orbit modules. It commutes with left tensoring by every object. Thus the anti-invariant subgroup has basis $v_a = [m_a^+] - [m_a^-]$. The two lift multiplicities have sum $(B_X)_{ab}$ and difference $(T_X)_{ab}$, proving the bound and parity. Rigidity gives the transpose relation, since the orthogonal difference vectors all have squared norm two. All fusion relations are preserved. $\square$

Proposition 12.2. *The rings R8_2_318, R8_2_319, R8_2_320, R8_2_368, R8_2_370, R8_2_735, R8_2_758 and R8_2_760 have no categorification over $\mathbb{C}$.*

Proof. Set $q = z^2 - 6z + 1$, $r = z^2 + 2z - 1$, and $h = z^2 - 6z - 1$, and put

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

The following source identities use zero-based labels. Each indicated X commutes with X^* ; the last column is a polynomial vanishing on X . The stated fixed labels are all the right-fixed simples of g .

suffix	g	fixed labels	X	B_X	polynomial
318	4	(1, 2, 3, 7)	5	A	$(z^2 - 1)qr$
319	4	(1, 2, 3, 7)	5	A	$(z^2 - 1)q$
320	6	(1, 2, 5, 7)	3	A	$(z^2 - 1)(z^2 + 1)q$
368	3	(4, 5, 6, 7)	1	C	$(z^2 - 1)(z^2 + 1)h$
370	3	(4, 5, 6, 7)	1	B	$(z^2 - 1)(z^2 + 1)q$
735	1	(4, 5, 6, 7)	2	B	$(z^2 - 1)qr$
758	1	(4, 5, 6, 7)	2	C	$(z^2 - 1)h$
760	1	(4, 5, 6, 7)	2	B	$(z^2 - 1)q$

Let $T = T_X$ from Lemma 12.1. Normality gives $\|T\|_F^2 = \sum_{j=1}^4 |\lambda_j|^2$. Integrality forces conjugate quadratic eigenvalues to occur with equal multiplicity. For bounds A, B , parity gives $\|T\|_F^2 \in \{12, 16\}$. The roots $3 \pm 2\sqrt{2}$ of q contribute 34 and so are absent. For suffixes 319, 320, 370, 760, all remaining roots have modulus one, forcing $\|T\|_F^2 = 4$, a contradiction.

For 318, 735, the additional roots $-1 \pm \sqrt{2}$ contribute 6. The possible squared norms are therefore 4, 8, 12, according to the number of copies of this pair. Only two copies are possible, giving norm squared 12 and trace -4 . But norm squared 12 forces the last diagonal entry to vanish, while the other diagonal entries are two signs and a zero. Their trace lies in $[-2, 2]$, a contradiction.

For bound C , the squared norm is in $\{12, 16, 20\}$. The roots $3 \pm \sqrt{10}$ of h contribute 38 and are absent. All remaining roots for 368, 758 have modulus one, again giving the impossible squared norm 4. These arguments give unrestricted exclusions for 318, 735, in addition to their pivotal obstructions. $\square$

The portable verifier `evidence/theoretical_20261016/rank8/verify_anti_module_eight.py` checks the displayed source identities and spectral arithmetic directly; the proof does not rely on an enumeration of candidate matrices.

12.1. Ordered exclusion ledger. An excluded tensor is removed after the first verified certificate in the following order. For each row, “left” still counts rings subsequently recognized as positive, so this is an exclusion table rather than a table of outstanding questions.

Criterion	New $m = 1$	New $m = 2$	Left $m = 1$	Left $m = 2$
Near-group stabilizer	1	1	95	876
Rank-two closed subring	0	10	95	866
Zero-spectrum tensor condition	7	80	88	786
Proved rank 2–6 subring	3	121	85	665
Four proved rank-seven subrings	0	12	85	653
Commutative codegree d -number	23	462	62	191
Rank-eight noncommutative codegree	1	17	61	174
Integral formal-codegree divisibility	1	6	60	168
Published no-induction-matrix lemma	2	0	58	168
Involution-module obstruction	0	2	58	166
Field-automorphism obstruction	0	0	58	166
Fusion-subcategory dimension quotient	0	1	58	165
Primitive formal-codegree test	0	0	58	165
Cyclotomic eigenvalue test	0	0	58	165
Complete oriented signed quotient	3	71	55	94
Graded Brauer–Picard obstruction	1	2	54	92
Rank-one fiber-functor obstruction	0	2	54	90
Solvability and equivariantization	3	0	51	90
Prime descent and Hopf obstructions	0	9	51	81
Dimension-96 two-object orbit	0	4	51	77
Dimension-108 order-four action	0	1	51	76
Galois-invariant pointed associator	0	3	51	73
Normal abelian square-order kernels	0	8	51	65
Dimension-block conductor bound	0	4	51	61
Quadratic subcategory index	0	1	51	60
Cyclic-module divisibility (direct or Morita)	0	3	51	57
Morita adjoint normality	0	1	51	56
Conductor-32 signed Jacobi obstruction	0	1	51	55
Anti-invariant involution module	0	8	51	47
Algebraic $2D2$ graph-pair rigidity	0	4	51	43
Odd-involution Morita	0	1	51	42
Twisted half-braiding support	0	2	51	40
Quantum-group inverse Morita	0	1	51	39
Pivotal parity and inverse Morita	1	0	50	39
Fibonacci character parity	0	2	50	37
Conductor-seventeen indicator balance	0	5	50	32
Weighted Galois third indicator	0	1	50	31
Conductor-forty center trace	0	1	50	30
Cyclic induction phase multiplicity	1	0	49	30
Full center STS and fourth moment	1	0	48	30
Inner square and tensor-functor rigidity	0	1	48	29

The 77 tensors left by the exclusion ledger are exactly the positive models. Every negative stage is a theorem applied to tensor data or an exact based-tensor transfer of a published and proved no-go. Proposition 10.1 and the integral-divisibility stage both have certified first applications already at

multiplicity one; the latter also resolves two multiplicity-two cases not excluded by the preceding criteria. These *first appearances* refer only to the two multiplicities of the complete rank-eight census. Nothing in this calculation establishes a multiplicity from which no new criterion is needed.

13. EXACT POSITIVE MODELS

Each standard family in this section admits a unitary complex realization. A pointed group ring of order eight is $K_0(\text{Vec}_G)$ for the group reconstructed from its tensor. The five groups are C_8 , $C_4 \times C_2$, C_2^3 , D_8 and Q_8 . The near-group ring $C_7 + 0$ is realized by a Tambara–Yamagami category [41]. If A is a rank-four unitary model from the complete catalogue of [14] and B is either Vec_{C_2} or Fibonacci, the Deligne product $A \boxtimes B$ is a unitary rank-eight model with multiplication tensor $N_A \otimes N_B$. Its multiplicity is at most two precisely for the displayed matches.

The three group-theoretical models $\mathcal{C}(F_5, 1, D_5, 1)$, $\mathcal{C}(S_4, 1, A_4, 1)$, and $\mathcal{C}(\text{PSL}(2, 7), 1, C_7 \rtimes C_3, 1)$ are untwisted group-theoretical fusion categories, hence admit unitary realizations. Their exact appendix tensors are matched against the census in all 512 positions, with a stored basis permutation for every match [20, Appendix C.3.2(3), (6), (11)]. Some admit a second subgroup construction, also recorded in the model catalogue.

Proposition 13.1 (Representation and quantum-group templates). *All three character-ring templates $\text{Rep}(D_{20})$, $\text{Rep}(D_{26})$, and $\text{Rep}(\text{Dic}_5)$ have rank eight and multiplicity one. The standard unitary $SU(2)_7$ category, and the even subcategories of $SU(2)_{14}$ and $SU(2)_{15}$, also have rank eight and multiplicity one. Their generated multiplication tensors occur in the supplied census and are matched by full based-ring isomorphisms.*

Proof. For $SU(2)_k$, the simples are the labels $0, \dots, k$ and

$$N_{ab}^c = 1 \iff |a - b| \leq c \leq \min(a + b, 2k - a - b), \quad a + b - c \equiv 0 \pmod{2};$$

all other coefficients are zero. The even labels close under multiplication and yield eight objects for $k = 14, 15$, while $k = 7$ already has eight labels [29]. The explicit tensor generator implements this formula directly.

For the dihedral group of order $2n$, take $n = 10$ or 13 . Its two-dimensional characters ρ_j obey $\rho_i \rho_j = \rho_{i+j} + \rho_{|i-j|}$, with $\rho_0 = 1 + \beta$ and $\rho_{n-j} = \rho_j$. For $n = 10$, $\rho_5 = \alpha + \alpha\beta$, the four linear characters form C_2^2 , $\beta\rho_j = \rho_j$ and $\alpha\rho_j = \rho_{5-j}$. For $n = 13$ there are two linear characters, with $\beta\rho_j = \rho_j$. Thus their ranks are respectively $4 + 4$ and $2 + 6$. For Dic_5 of order twenty the four linear characters χ_i form C_4 , with

$$\rho_0 = \chi_0 + \chi_2, \quad \rho_5 = \chi_1 + \chi_3, \quad \rho_{10-j} = \rho_j, \quad \chi_i \rho_j = \begin{cases} \rho_j & i \text{ even,} \\ \rho_{5-j} & i \text{ odd;} \end{cases}$$

the same product $\rho_i \rho_j = \rho_{i+j} + \rho_{|i-j|}$ completes the ring. These are ordinary finite-group character identities [42]. The supplied generator expands each product and compares all 8^3 ordered coefficients after unit-fixing relabelling. A match is a ring isomorphism and the corresponding representation category supplies the construction. $\square$

Multiplicity	Ring ID	One exact unitary model
1	R8_1_1	$\text{Rep}(\text{Dic}_5)$
1	R8_1_3	$\mathcal{C}(C_5 \rtimes C_4, \omega_2, C_2, 1)$
1	R8_1_4	$\text{Vec}(G), G =8$
1	R8_1_5	$\text{Vec}(C_4) \times \text{Fib}$
1	R8_1_6	Bosonic quotient of $PSU(2)_6 \boxtimes \mathcal{C}(C_4, q)$
1	R8_1_7	Central square-root extension of pseudo- $PSU(2)_6$
1	R8_1_10	$\text{Vec}(G), G =8$
1	R8_1_11	Central-twisted flip extension of pseudo- $PSU(2)_6$

Multiplicity	Ring ID	One exact unitary model
1	R8_1_12	Isotropic orbifold of $NG(C_{16}, 16)$
1	R8_1_13	$\text{Vec}(\mathbf{G}), \mathbf{G} =8$
1	R8_1_14	Central-twisted flip extension of $PSU(2)_6$
1	R8_1_15	$\text{HI}(C_4)$
1	R8_1_16	$\text{Vec}(\mathbf{G}), \mathbf{G} =8$
1	R8_1_19	$(\text{Fib} \boxtimes \text{Fib}) \rtimes C_2$
1	R8_1_20	$\text{TY}(C_7)$
1	R8_1_22	Diagonal parity in $\text{TY}(C_3) \boxtimes \text{Vec}_{C_4}$
1	R8_1_23	$\text{Vec}_{Q_8}^{C_3}$
1	R8_1_24	$\mathcal{C}(\mathbf{F5}, 1, \mathbf{D5}, 1)$
1	R8_1_25	$PSU(2)_6 \rtimes C_2$ with the flip action
1	R8_1_27	$\text{TY}(C_3) \times \text{Vec}(C_2)$
1	R8_1_30	$\text{TY}(C_6)^{C_2}$ with odd coherence -1
1	R8_1_32	$\text{pseudo-PSU}(2)_6 \times \text{Vec}(C_2)$
1	R8_1_34	$\text{TY}(C_9)^{C_2}$ with odd coherence -1
1	R8_1_35	$\text{TY}(C_3^2)^{C_2}$ with odd coherence -1
1	R8_1_36	$\text{TY}(C_3)$ crossed C_2
1	R8_1_39	$\text{TY}(C_3) \times \text{Fib}$
1	R8_1_42	Pseudo- $PSU(2)_6 \rtimes C_2$ with the flip action
1	R8_1_43	$\text{pseudo-PSU}(2)_6 \times \text{Fib}$
1	R8_1_53	Isotropic orbifold of $NG(C_4^2, 16)$
1	R8_1_59	$\text{Rep}(\mathbf{D10}) \times \text{Vec}(C_2)$
1	R8_1_61	One-half $S_0(12)_2$
1	R8_1_65	$\text{Vec}(\mathbf{G}), \mathbf{G} =8$
1	R8_1_66	$\text{Vec}(C_2 \times C_2) \times \text{Fib}$
1	R8_1_67	$PSU(2)_6 \times \text{Vec}(C_2)$
1	R8_1_69	$\text{HI}(C_2^2)$
1	R8_1_70	$\text{Fib} \times \text{Vec}(C_2) \times \text{Fib}$
1	R8_1_71	$PSU(2)_6 \times \text{Fib}$
1	R8_1_72	$PSU(2)_7 \times \text{Vec}(C_2)$
1	R8_1_73	$PSU(2)_{14}$
1	R8_1_78	$S_0(9)_2$
1	R8_1_79	$Z(\text{Vec}(S_3))$
1	R8_1_83	Integral part of the $m = 0$ triality gauging
1	R8_1_85	Neutral part of S_3 -gauged $S_0(8)_1$
1	R8_1_91	$\text{Rep}(\mathbf{D}_{\{26\}})$
1	R8_1_92	$PSU(2)_{15}$
1	R8_1_93	$\text{Rep}(\mathbf{D10}) \times \text{Fib}$
1	R8_1_94	$PSU(2)_7 \times \text{Fib}$
1	R8_1_95	$\text{Fib} \times \text{Fib} \times \text{Fib}$
2	R8_2_37	Nonsplit C_2 extension of the unitary $2D2$ category
2	R8_2_39	Asaeda–Haagerup AH_4
2	R8_2_143	$\text{Vec}_{C_{17}}^{C_4}$
2	R8_2_154	$\text{TY}(C_2^2)^{S_3}$ with parity coherence
2	R8_2_174	$\text{Rep}(\mathbf{A4}) \times \text{Vec}(C_2)$
2	R8_2_179	$\text{Vec}_{C_2^2}^{C_5}$
2	R8_2_202	$\mathcal{C}(S_4, 1, \mathbf{A4}, 1)$
2	R8_2_212	$\text{Rep}(\mathbf{A4}) \times \text{Fib}$
2	R8_2_237	$\text{Rep}(\mathbf{AFL}(1, 8))$
2	R8_2_239	C_3 -algebra Morita dual of an S_3 extension of Vec_{C_7}
2	R8_2_241	$\mathcal{C}(\text{PSL}(2, 7), 1, C_7 \rtimes C_3, 1)$
2	R8_2_244	$\text{Vec}_{C_4^2}^{C_3}$

Multiplicity	Ring ID	One exact unitary model
2	R8_2_289	$\text{Vec}_{C_2}^{C_3}$
2	R8_2_290	Charge-conjugation defects of $PSU(3)_4$
2	R8_2_313	Diagonal bosonic quotient of the two pseudo- $PSU(2)_6$ categories
2	R8_2_477	<code>even 2D2 x Vec(C2)</code>
2	R8_2_484	Asaeda–Haagerup AH_5
2	R8_2_491	<code>even 2D2 x Fib</code>
2	R8_2_493	Haagerup H1 x $\text{Vec}(C_2)$
2	R8_2_535	Dual of the $1 + \rho$ Q-system in $HI(C_6)$
2	R8_2_557	Diagonal bosonic quotient of $PSU(2)_6 \boxtimes PSU(2)_6$
2	R8_2_575	Davydov–Runkel extension of $\text{Rep}(H_{C:\sigma_1})$
2	R8_2_611	$TY(C_2^2)^{S_3}$ with strict action
2	R8_2_666	Pointed centralizer in $Z(NG(C_4, 4))$
2	R8_2_820	$NG(C_5, 5)$ center factor, auxiliary C_3^2
2	R8_2_839	Haagerup H1 x <code>Fib</code>
2	R8_2_855	Even D_{14} category
2	R8_2_868	Dual Haagerup–Izumi C_7 category
2	R8_2_869	$NG(C_5, 5)$ center factor, auxiliary C_9

Five further multiplicity-one models are imported from exact AnyonWiki tensor matches: $R8_1_36$ is the crossed product of $TY(C_3)$ by the order-two inversion action; $R8_1_61$ is the rank-eight fusion subcategory called one-half $SO(12)_2$; $R8_1_78$ is $SO(9)_2$; $R8_1_79$ is $Z(\text{Vec}_{S_3}) \simeq \text{Rep}(D(S_3))$; and $R8_1_85$ is the neutral component of the universal $\mathbb{Z}_2$ grading after gauging the full S_3 symmetry of $SO(8)_1$ [1, 2]. For the crossed product, the standard bicharacter $\chi(a, b) = \exp(2\pi i ab/3)$ and positive $\tau = 1/\sqrt{3}$ are invariant under simultaneous inversion. The induced tensor action squares to the identity, and the crossed product is unitary. Its independently constructed tensor is matched in all 512 positions. The other four are standard unitary categories or fusion subcategories thereof; their complete sourced tensors and unit-preserving permutations are included in the positive catalogue. Existence of these models proves general complex categorifiability. No negative conclusion is imported merely from an AnyonWiki flag; $R8_1_26$ is excluded by the theoretical proof in Proposition 10.3.

13.1. Twenty-two further constructions from extensions and subfactors. The following constructions are verified by generating their multiplication tensors directly and comparing all 512 coefficients. No associator search is needed.

For $G = C_4$ and $G = C_2^2$, write $X_g = g\rho$ and use

$$gh = g + h, \quad gX_h = X_{g+h}, \quad X_g h = X_{g-h}, \quad X_g X_h = (g - h) + m \sum_{a \in G} X_a.$$

At $m = 1$, Izumi’s unitary generalized Haagerup constructions [8, Section 9] give $R8_1_15$ and $R8_1_69$, respectively. At $m = 2$, these are the Asaeda–Haagerup rings AH_4 and AH_5 , giving $R8_2_39$ and $R8_2_484$. The C_4 construction is [9, Theorem 4.9]; existence of the C_2^2 category follows from its Lemma 3.15, Theorem 3.17 and Section 4. These are unitary subfactor categories.

The ring $R8_1_19$ is $(\text{Fib} \boxtimes \text{Fib}) \rtimes C_2$, using the canonical action exchanging the two factors. The ring $R8_1_22$ is the fusion subcategory of $TY(C_3) \boxtimes \text{Vec}_{C_4}$ whose simple objects have matching parity: (a, z^{2j}) , (m, z) and (m, z^3) . Both operations preserve unitarity.

For $R8_1_30$, take the standard unitary $TY(C_6)$ with bicharacter $\chi(a, b) = \exp(2\pi i ab/6)$ and positive parameter $\tau = 1/\sqrt{6}$. Inversion fixes m , preserves every standard associator coefficient, and gives a tensor functor T with identity tensorator. Equip the C_2 action with coherence $T^2 \Rightarrow \text{Id}$ equal to $+1$ on invertibles and -1 on m . This is a T -invariant monoidal natural automorphism, since the

TY grading is $\mathbb{Z}/2$. In the equivariantization, the two structures on m have eigenvalues $\pm i$. The tensor decomposition is obtained by splitting the permutation action on each fusion space into its two characters; in particular two odd eigenvalues contribute the additional minus sign. The result is the stated full tensor, with four invertibles, two objects of dimension two, and two of dimension $\sqrt{6}$.

Let $F_{20} = C_5 \rtimes C_4$ with

$$(a, b)(c, d) = (a + 2^b c, b + d), \quad \omega(x, y, z) = (-1)^{b(x) \lfloor (b(y) + b(z))/4 \rfloor},$$

where the two coordinates are reduced modulo five and four, respectively, and representatives $0 \leq b < 4$ are used in the cocycle. For $H = \langle (0, 2) \rangle$, the restriction $\omega|_H$ is trivial. The unitary group-theoretical category $\mathcal{C}(F_{20}, \omega, H, 1)$ realizes $R8_1_3$. The supplied code constructs the eight simple homogeneous $\mathbb{C}[H]$ -bimodules and their balanced tensor products. All phases are fourth roots of unity and are stored as integer exponents modulo four; the cocycle identity, bimodule coherence, balanced quotient, decomposition multiplicities, and full census tensor match are checked exactly.

The ring $R8_1_6$ has a bosonic quotient construction. Let $\mathcal{D} = PSU(2)_6$, with simples $1, f, \rho, f\rho$, where f is a fermion and $\rho^2 = 1 + \rho + f\rho$. Let $\mathcal{P} = \mathcal{C}(C_4, q)$ be the unitary pointed modular category with $q(z^a) = \exp(\pi i a^2/4)$. Then $b = f \boxtimes z^2$ in $\mathcal{B} = \mathcal{D} \boxtimes \mathcal{P}$ is an order-two boson: its self-braiding is $+1$. It generates a Tannakian $\text{Rep}(C_2)$, whose regular algebra is $A = 1 \oplus b$. The central inclusion supplied by the braiding gives a fusion category $\mathcal{B}_A$ of all right A -modules, with tensor free-module functor F [22, Theorems 8.4.9 and 8.23.3]. The unitary crossed-product construction preserves its positive $*$ -structure [40]. The boson is not transparent in $\mathcal{B}$; thus this uses all modules, rather than only local modules, and asserts no descended braiding.

Tensoring by b acts freely on the sixteen simples by $(V_j, z^a) \mapsto (V_{6-j}, z^{a+2})$, with $j \in \{0, 2, 4, 6\}$. By adjunction every free module $F(V_j \boxtimes z^a)$ is simple, and two are isomorphic exactly when their labels belong to the same orbit. Dominance of F gives precisely eight simples. With $g = F(1 \boxtimes z)$ and $r = F(\rho \boxtimes 1)$, their rules are

$$g^4 = 1, \quad gr = rg, \quad r^2 = 1 + r + g^2 r.$$

The ordered model basis $(1, g, g^2, g^3, r, gr, g^2 r, g^3 r)$ maps to census labels $(0, 1, 7, 2, 5, 3, 6, 4)$. The verifier reconstructs all coefficients from the truncated $SU(2)_6$ rules and the eight free orbits.

For $R8_1_7$, use either of the two unitary pseudo- $PSU(2)_6$ categories $\mathcal{D}$ of [30, Theorem 4.1]. Their basis $1, g, x, y$ obeys $g^2 = 1$, $gx = y$, $x^2 = y^2 = g + x + y$ and $xy = 1 + x + y$. The two source-bound AnyonWiki associator tables, labelled $(4, 1, 2, 4, 1, 0, 1)$ and $(4, 1, 2, 4, 2, 0, 1)$, are included in the supplement. Over $\mathbb{Q}(\zeta)$, $\zeta^4 + 1 = 0$, the verifier checks all 224 coefficients, 136 invertible associator blocks, unit and rigidity conditions and 2688 pentagons for each table.

Write $a(i) = gi$ and let $F_\ell^{ijk}[e, f]$ take left intermediate e to right intermediate f . In the two tables respectively, the scalars

$$(\lambda_1, \lambda_g, \lambda_x, \lambda_y) = (1, \zeta^2, \zeta, \zeta^3), \quad (1, -\zeta^2, -\zeta^3, -\zeta)$$

define a half-braiding of g . The exact check is the following identity for each of the 24 fusion channels $N_{ij}^k = 1$:

$$\lambda_i \lambda_j F_{a(k)}^{igj}[a(i), a(j)] = \lambda_k F_{a(k)}^{gij}[a(i), k] F_{a(k)}^{ijg}[k, a(j)].$$

Thus $\tilde{g} = (g, c)$ is an invertible object of $Z(\mathcal{D})$. Its self-braiding satisfies $\lambda_{\tilde{g}}^2 = -1$, so a fermionic condensation argument would be inappropriate. The equivalence between the unitary and ordinary centers permits a unitary representative of this lift [32, corrected Remark 3.1].

Instead take the trivial homomorphism $C_2 \rightarrow \text{BrPic}(\mathcal{D})$ and modify its product equivalences by the normalized central 2-cocycle $t(1, 1) = \tilde{g}$, with other normalized values the unit. The only nontrivial cocycle identity compares $\tilde{g}$ with itself, regardless of its order in $\text{Inv } Z(\mathcal{D})$. The remaining extension obstruction belongs to $H^4(C_2, \mathbb{C}^\times) = 0$ [7, Theorem 1.3]; this is also the trivial-extension zesting of [31, Section 6.2].

This extension is unitary. Choose regular unitary $\mathcal{D}$ -bimodule components and unitary multiplication equivalences from the unitary lift. Preliminary associativity intertwiners can be chosen unitary by polar decomposition, so their normalized pentagon defect is $U(1)$ -valued. A normalized 3-cochain on C_2 is determined by u at $(1, 1, 1)$, and its coboundary at $(1, 1, 1, 1)$ is u^2 . A unit-modulus square root therefore removes the defect while keeping every coherence map unitary. The resulting regular-component extension is a unitary fusion category; this argument uses no general assertion that arbitrary extensions of unitary categories are unitarizable.

Its simples are (X, a) , $a \in C_2$, with

$$(X, a)(Y, b) = (XYg^{ab}, a + b).$$

For $z = (1, 1)$, this gives $z^2 = g$, $zx = xz$ and $x^2 = z^2 + x + z^2x$. The model basis $(1, g, x, y, z, zg, zx, zy)$ maps to census labels $(0, 7, 3, 4, 1, 2, 6, 5)$, and all 512 coefficients agree. The exact associator and central-lift replay is invoked by the finite-update verifier.

The same two rank-four bases have nontrivial flip extensions. For $\mathcal{D} = PSU(2)_6 = \text{Ad}(A_7)$, a tensor autoequivalence τ exchanges ρ and $f\rho$ [33, Lemma 4.10]. Its square is tensor-isomorphic to the identity. The fusion rules imply $\text{Aut}_{\otimes}(\text{Id}_{\mathcal{D}}) = 1$, so this square isomorphism is unique after normalization and obeys the C_2 action coherence. Reutter's unitarization and automatic-unitarity theorems give a unitary representative of the entire action [34, Theorems 1–2].

For the second pseudo- $PSU(2)_6$ source table, the permutation $\tau = (x \ y)$ also gives a coherent tensor action, now checked explicitly. Its tensorator J_{ij}^k is 1 when an input is the unit. On the seventeen other admissible channels, ordered lexicographically, write $J = (\sqrt{2})^a \zeta^b$; the exponent vectors are

$$\begin{aligned} a &= (-1, -1, 0, -1, -1, -1, -1, -1, 0, -1, 0, -1, 0, 0, 0), \\ b &= (29, 15, 10, 17, 1, 5, 5, 11, 0, 1, 16, 15, 4, 5, 0, 0, 0). \end{aligned}$$

All 224 tensor-functor identities hold over $\mathbb{Q}(\zeta)$. The natural isomorphism $\eta : \tau^2 \Rightarrow \text{Id}$ has scalars

$$\eta = \left(1, \frac{1 + \zeta^2}{2}, -\frac{1 + \zeta^2}{2}, -\frac{1 + \zeta^2}{2} \right)$$

in the order $(1, g, x, y)$. The verifier checks all 24 equations $\eta_i \eta_j = \eta_k J_{ij}^k J_{\tau(i), \tau(j)}^{\tau(k)}$ and $\eta_{\tau(i)} = \eta_i$. This is therefore a coherent action, not just a symmetry of the ring. Transport to unitary functors preserves its coherence; the transported monoidal natural isomorphism is automatically unitary by [34, Theorems 1–2].

For either rank-four base, its nontrivial invertible has a unique central lift: the kernel of $\text{Inv } Z(\mathcal{D}) \rightarrow \text{Inv } \mathcal{D}$ is $\text{Aut}_{\otimes}(\text{Id}_{\mathcal{D}}) = 1$, and a lift was already constructed above. The flip fixes this lift. Thus one may use either the crossed product or its central 2-cocycle modification. The remaining C_2 obstruction is removed by the unitary phase correction described above. With v the nontrivial invertible, their products are

$$(X, a)(Y, b) = (X\tau^a(Y)v^{\epsilon ab}, a + b), \quad \epsilon \in \{0, 1\}.$$

For $PSU(2)_6$, these give *R8_1_25* at $\epsilon = 0$ and *R8_1_14* at $\epsilon = 1$; for pseudo- $PSU(2)_6$, they give *R8_1_42* and *R8_1_11*, respectively. The exact tensorator and coherent-square replay is `software/verify_pseudo_psu_swap.py`.

The ring *R8_1_23* is $\text{Vec}_{Q_8}^{C_3}$ for the strict automorphism $i \mapsto j \mapsto k \mapsto i$ of the quaternion group. The fixed objects $1, -1$ each have three equivariant structures, giving six invertibles. The two free orbits $\{i, j, k\}$ and $\{-i, -j, -k\}$ give two simples of dimension three. For a free output orbit, the fusion coefficient is the number of pairs of input group elements with product equal to its chosen representative. For a fixed output, the product fibre is a sum of free C_3 -orbits, each contributing its regular character, together with any fixed pairs carrying the product character. These rules

independently generate every coefficient. In the order of the three characters over 1, the three over -1 , and the two free orbits, the census permutation is $(0, 5, 6, 7, 1, 2, 3, 4)$.

The inversion construction for $R8_1_30$ also applies to $A = C_9$ and $A = C_3^2$, producing $R8_1_34$ and $R8_1_35$. Use respectively $\chi(a, b) = \exp(2\pi i ab/9)$ and $\chi(a, b) = \exp(2\pi i(a_1 b_1 + a_2 b_2)/3)$, with $\tau = 1/3$. Inversion preserves these nondegenerate bicharacters, hence the standard unitary TY associators. The parity coherence is again -1 on m and $+1$ on group elements. There are two equivariant simples above 0, four free pairs of nonzero group elements, and two structures with eigenvalues $\pm i$ above m . Their dimensions are 1, 1, 2, 2, 2, 2, 3, 3; the same exact character projection generates the recorded tensors.

Finally take $A = \mathbb{F}_2^2$, $\chi(a, b) = (-1)^{a_1 b_2 + a_2 b_1}$ and $\tau = 1/2$. The whole group $S_3 = \text{GL}_2(\mathbb{F}_2)$ preserves χ , so label permutations give a strict unitary action on the standard TY presentation. For $t = 0, 1$, modify its action coherence on m by

$$\alpha_t(g, h) = (-1)^{tp(g)p(h)},$$

where $p : S_3 \rightarrow \mathbb{F}_2$ is parity, leaving the group simples unchanged. This is monoidal by the TY grading, and the action pentagon is the elementary normalized 2-cocycle identity for $p(g)p(h)$. Both equivariantizations are therefore unitary. Their orbits $\{0\}$, $A \setminus \{0\}$ and $\{m\}$ have stabilizers S_3, C_2, S_3 . Denote their simples by A_ρ , X_ϵ and M_ρ , where ρ ranges over 1, sign, std and $\epsilon = \pm$. Above m , the equivariance is $i^{tp(g)}\rho(g)$, so the product of two such factors contributes sign^t .

The fusion coefficients are ordinary character inner products. Multiplication by A_ρ is tensoring with ρ , restricted to C_2 on an X -object. The other rules are

$$X_\epsilon X_\delta = A_{\text{Ind}_{C_2}^{S_3}(\epsilon\delta)} + X_+ + X_-,$$

$$X_\epsilon M_\rho = M_{(\text{Ind}_{C_2}^{S_3} \epsilon)\rho},$$

$$M_\rho M_\sigma = A_{\rho\sigma \text{sign}^t} + X_{\text{Res}_{C_2}^{S_3}(\rho\sigma \text{sign}^t)},$$

where subscripts extend additively to reducible representations. For XX , the zero fibre is the induced character and the nonzero fibre is a regular C_2 permutation representation. These formulas give $R8_2_611$ at $t = 0$ and $R8_2_154$ at $t = 1$, both of global dimension 48. The model dimensions in the order $(A_1, A_{\text{sign}}, A_{\text{std}}, X_+, X_-, M_1, M_{\text{sign}}, M_{\text{std}})$ are $(1, 1, 2, 3, 3, 2, 2, 4)$. All character arithmetic and tensor comparisons are exact.

One more quadratic model is $R8_1_53$. Yu-Zhang construct a unitary near-group category $NG(G, 16)$ for $G = C_4^2$ [35, Proposition 3.1]. Its bicharacter and quadratic refinement may be taken as

$$\chi(g, h) = i^{g_1 h_1 - g_2 h_2}, \quad a(g) = \zeta_8^{3(g_1^2 - g_2^2)}.$$

For $H = 2G$, exact finite-group arithmetic gives $H = H^\perp$ and $a(h) = \chi(h, g_a)$ with $g_a = (1, 1) \notin H$. Izumi's general orbifold theorem applies to $H \subseteq H^\perp$ even when $a|_H$ is nontrivial [36, Theorem 12.5]; its crossed-product-factor construction remains unitary. With $A = G/H = C_2^2$, the resulting rules are

$$\alpha_g \alpha_h = \alpha_{g+h}, \quad \alpha_g \sigma = \sigma \alpha_g, \quad \sigma^2 = \alpha_{g_a} + \sum_{g \in A} \alpha_g \sigma.$$

There are four invertibles and four noninvertibles of dimension $(8 + 4\sqrt{5})/4 = 2 + \sqrt{5}$, and every noninvertible has a distinct dual. This uses the non-self-dual theorem, rather than its self-dual Lagrangian corollary. In the model order $(00, 01, 10, 11, \sigma, 01\sigma, 10\sigma, 11\sigma)$, the census permutation is $(0, 5, 6, 7, 1, 3, 4, 2)$. The portable verifier checks the subgroup hypotheses, the duality shift and all fusion coefficients; the cited theorem supplies categorical existence.

A cyclic version gives $R8_1_12$. Budinski constructs $NG(C_{16}, 16)$ [37, Theorem 3.1.1 and Table 3.1], with data

$$\chi(x, y) = \exp(2\pi i 3xy/16), \quad a(x) = \exp(-\pi i 3x^2/16).$$

Although the table prints decimal arguments, the thesis’s Section 3.3.2 gives an exact algebraic certification, with the defining-polynomial scope explicitly including $n = 16$. We use that exact existence statement as a citation dependency; no numerical reconstruction is counted here as a category. For $H = 4C_{16}$, one has $H = H^\perp$ and $a(h) = \chi(h, 2)$. The same unitary orbifold theorem [36, Theorem 12.5] therefore gives

$$g^4 = 1, \quad \sigma g = g^{-1}\sigma, \quad \sigma^2 = g^2 + (1 + g + g^2 + g^3)\sigma.$$

In the order $(1, g, g^2, g^3, \sigma, g\sigma, g^2\sigma, g^3\sigma)$, the census permutation is $(0, 1, 7, 2, 3, 6, 4, 5)$. The four noninvertibles are paired by duality, distinguishing this ring from the standard C_4 Haagerup–Izumi ring. The finite verifier checks all subgroup hypotheses and the full tensor exactly.

The ring `R8_1_83` has a triality-gauging model. The S_3 gauging of the three-fermion category exists by [2, Section 6.2]; the unobstructed gauging data form the usual $H^3(S_3, U(1))$ torsor, including the twist denoted $m = 0$ in [38, Section VI.D and Table IX]. This supplies an actual category. Starting from a pointed category, gauging produces a weakly group-theoretical fusion category, hence one admitting a unitary realization [39, Theorem 5.20]. Its eight integral simples, with dimensions $(1, 1, 2, 3, 3, 4, 4, 4)$, form a fusion subcategory of dimension 72.

The integer fusion formulas for this subcategory are independently checked against the full twelve-object S -matrix of [38, Eq. (6.53), $m = 0$]. Its entries lie in $\mathbb{Q}(\sqrt{2})$; using the unnormalized matrix, the verifier checks $S^2 = 144I$, all 1728 nonnegative integral Verlinde coefficients, and fusion closure of the eight integral simples. It then compares the entire tensor with the pinned census under the stored permutation. Category existence comes from the gauging construction, not merely from the feasible modular datum.

The table provides one construction per positive tensor. Some have two isomorphic presentations; `data/realized_models.json` records every matched constructor, exact model tensor, precise model source and source-to-model isomorphism. Labels such as “pseudo- $PSU(2)_6$ ” denote the conventional fusion-ring name, not a weaker existence assertion: the cited rank-four table supplies a unitary category [14]. The table exhausts both positive lists: 48 tensors at exact multiplicity one and 29 at exact multiplicity two.

13.2. Pointed equivariantizations and an affine group. Four further models are strict unitary equivariantizations of untwisted pointed categories. Their group automorphisms are

$$\begin{aligned} R8_2_244 : \quad & A = C_4^2, \quad (x, y) \mapsto (-y, x - y), \quad G = C_3; \\ R8_2_289 : \quad & A = C_2^4, \quad (x, y, z, w) \mapsto (y, x + y, w, z + w), \quad G = C_3; \\ R8_2_143 : \quad & A = C_{17}, \quad x \mapsto 4x, \quad G = C_4; \\ R8_2_179 : \quad & A = C_2^4, \quad (x, y, z, w) \mapsto (w, x + w, y + w, z + w), \quad G = C_5. \end{aligned}$$

In each case the displayed generator has the stated order and acts freely off zero. The zero orbit supplies the ordinary characters of G , and each nonzero orbit has trivial stabilizer. Tensor products are calculated by orbit addition and projection to stabilizer characters, giving the exact full source tensors. Strict permutation functors with identity tensorators supply coherent unitary actions.

The ring `R8_2_237` is $K_0(\text{Rep}(\text{Aut}(1, 8)))$. This group of order 168 has elements (v, a, t) acting on $\mathbb{F}_8$ as $x \mapsto v + ax^{2^t}$. Its simple dimensions are $(1^3, 3^2, 7^3)$. The three dimension-seven characters have the exact expression

$$\chi_s(v, a, t) = \zeta_3^{st} \sum_{\substack{r \in \mathbb{F}_8^\times \\ ar^{2^t-1}=1}} (-1)^{\text{Tr}_{\mathbb{F}_8/\mathbb{F}_2}(v/r)}.$$

Together with the characters inflated from $\mathbb{F}_8^\times \rtimes C_3$, they give an exact character table in $\mathbb{Q}(\zeta_{21})$. Orthogonality and all tensor multiplicities are checked independently by finite group sums.

13.3. Bosonic quotients and the D_{14} model. The ring $R8_2_557$ is the central bosonic quotient of $PSU(2)_6 \boxtimes PSU(2)_6$ by its diagonal invertible. Each factor invertible is a fermion, so their product has self-braiding $+1$ and trivial pointed associator. It generates a genuine central $\text{Rep}(C_2)$. All sixteen simples have free diagonal orbits; the eight free-module orbits give the stated ring.

For $R8_2_313$, use instead the two unitary pseudo- $PSU(2)_6$ categories of [30], with the exact sourced associators discussed above. Their central invertibles have self-braidings $+i$ and $-i$, and each has associator $F_{ggg} = -1$. The diagonal product therefore has both trivial self-braiding and trivial associator; its central $\text{Rep}(C_2)$ gives the same eight-orbit quotient construction. For each rank-four factor the universal grading is trivial, so a central lift of an invertible is unique; its central square is the unit. This verifies the order-two condition as well as the associator and self-braiding conditions required for the categorical quotient.

The ring $R8_2_855$ is the even D_{14} category, or the local module category of the $SU(2)_{24}$ algebra $1 \oplus V_{24}$. It is unitary modular [48, 49]. Its basis is $f_0, f_2, f_4, f_6, f_8, f_{10}, P, Q$, with P, Q self-dual. Fold the bulk $SU(2)_{24}$ products by $f_{2t} = f_{2(12-t)}$ and $f_{12} = P + Q$. The remaining rules are

$$Pf_{2j} = \sum_{k=6-j}^5 f_{2k} + \begin{cases} P & j \text{ even,} \\ Q & j \text{ odd,} \end{cases} \quad Qf_{2j} \text{ interchanges } P, Q,$$

$$P^2 = P + f_0 + f_4 + f_8, \quad Q^2 = Q + f_0 + f_4 + f_8, \quad PQ = f_2 + f_6 + f_{10}.$$

All these constructions preserve the unitary structure.

13.4. A Morita construction over C_7 . The ring $R8_2_239$ is a Morita dual of an S_3 extension of Vec_{C_7} . In $\text{BrPic}(\text{Vec}_{C_7}) = O(C_7 \oplus \widehat{C_7})$ take $r = \text{diag}(2, 4)$ and let s exchange the two coordinates. They generate S_3 . Its degree-three obstruction vanishes by coprime cohomology with seven-primary coefficients; the next obstruction lies in $H^4(S_3, \mathbb{C}^\times) = 0$. Thus the actual extension exists [7, Theorem 1.3]. Its C_3 restriction is pointed, on $F_{21} = C_7 \rtimes C_3$. The associator class there is inflated from C_3 : the order-three action kills the $H^3(C_7, \mathbb{C}^\times)$ invariant part. Surjectivity of $H^3(S_3, \mathbb{C}^\times) \rightarrow H^3(C_3, \mathbb{C}^\times)$ permits an outer twist that makes this restriction untwisted.

Take the separable algebra $B = \bigoplus_{h \in C_3} h$ in this restriction. The category of B -bimodules is an actual Morita dual, whose neutral category is $\text{Rep}(F_{21})$. Write its simples as α_i ($i \in \mathbb{Z}/3$), X, Y , and odd m_j ($j \in \mathbb{Z}/3$). The nontrivial rules, with all m_j self-dual, are

$$X^2 = X + 2Y, \quad Y^2 = 2X + Y, \quad XY = \sum_i \alpha_i + X + Y,$$

$$\alpha_i m_j = m_{i+j}, \quad m_j \alpha_i = m_{j-i}, \quad X m_j = Y m_j = m_j X = m_j Y = \sum_k m_k,$$

$$m_j m_k = \alpha_{j-k} + X + Y.$$

They follow from the S_3 bimodule action and reciprocity, and match all 512 source coefficients. The extension and its Morita dual are weakly group-theoretical, hence unitarizable [39, Theorem 5.20].

13.5. Charge-conjugation defects of $PSU(3)_4$. The ring $R8_2_290$ is a unitary C_2 crossed extension of the rank-five modular category $\mathcal{B} = PSU(3)_4$. Charge conjugation is an actual braided symmetry [50, Corollary 2.7]. The invertible group of $\mathcal{B}$ is trivial, so the degree-three gauging obstruction vanishes, and $H^4(C_2, U(1)) = 0$ removes the remaining obstruction. The unitary gauging theorem [2, Sections 3.2–3.4] therefore supplies an actual unitary extension. Its odd rank is three, the number of charge-conjugation-fixed neutral simples [51, Proposition 1.1].

For completeness, its tensor is forced by a small exact module calculation. In the neutral basis $(0, 0), (3, 0), (1, 1), (0, 3), (2, 2)$ the nonselfdual generator X has regular characteristic polynomial $(t^2 + t + 2)(t^3 - 2t^2 - t + 1)$, computed by quantum Weyl alternants in $\mathbb{Q}(\zeta_{21})$. Its odd action A has the cubic factor as characteristic polynomial, since its Perron root has degree three. The

commuting neutral action and the dual-transpose relation make A normal. Its three roots are real, so A is symmetric. Nonnegative integer entries, $\text{Tr}(A) = 2$ and $\text{Tr}(A^2) = 6$ give, up to relabelling,

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The five neutral actions are $I, A, A^2 - I, A, A^2 - A$. Distinct odd dimensions force each odd object to be self-dual; reciprocity then determines every odd product. The independent constructor matches all 512 census coefficients.

13.6. A self-dual Hopf extension in dimension thirty-two. For $R8_2_575$ the neutral Hopf algebra is $H = H_{C:\sigma_1}$ of dimension sixteen from [44]. A symmetric nondegenerate Hopf pairing supplies the extra data of the Davydov–Runkel construction [52, Theorem 2.5]. Here is enough explicit information to fix the construction and its normalization. Write $h(p, q, j)$ for its basis, with p modulo four and q, j modulo two. Put $\zeta = \exp(2\pi i/16)$ and $i = \zeta^4$. The pairing is

$$P(h(p, q, j), h(r, s, k)) = \delta_{s,j} \delta_{k,q} i^{-3rp} \zeta^{jq} / 4.$$

The supplementary Hopf presentation and checker verify nondegeneracy, symmetry and every pairing identity exactly in $\mathbb{Q}(\zeta)$. Davydov–Runkel use $H^\vee = (H^*)^{\text{op}, \text{cop}}$; consequently their self-duality map is $\Gamma = S \circ P^{-1}$, rather than the unmodified inverse pairing. The right cointegral $\lambda(h(p, q, j)) = \delta_{j,0}/2$ satisfies $\Gamma(\lambda) = 4\Lambda$ and $\lambda(\Gamma(\lambda)) = 1$, where Λ is the normalized integral. The distinguished grouplike is one, $S^2 = 1$, and Γ is symmetric, so all conditions of the cited theorem hold. Its rigidity criterion [52, Proposition 3.25] gives an actual fusion category $\text{Rep}(H) \oplus \text{Vec}$. Its neutral ring is $\text{Rep}(D_{16})$; for the odd object M of dimension four,

$$XM = MX = \text{FPdim}(X)M, \quad M^2 = \sum_{X \in \text{Irr}(H)} \text{FPdim}(X)X.$$

The dimension-32 solvability theorem and [39, Theorem 5.20] make this fusion category unitarizable. The complete model tensor, Hopf identities, and source comparison are separate exact checks.

13.7. The dual Haagerup–Izumi C_7 category. The ring $R8_2_868$ is the dual even part of the unitary Haagerup–Izumi C_7 subfactor. Its existence is supplied by [53, Theorem 5(d)]. The exact center identification follows from [87, Theorem A(2), proved as Theorem 5.1]: the metric-group discriminant is $7^2 + 4 = 53$, which is prime, so the positive spherical branch determines its quadratic form up to isometry. This supplies the actual-center premise below; the generator matrix of that source alone would not determine the full fusion tensor. Put $d = (7 + \sqrt{53})/2$, so $d^2 = 1 + 7d$. Izumi’s dual Q -system analysis [8, Proposition 7.4 and Theorem 7.5] gives rank eight, with dimensions $1, d, (d-1)^3, (d+1)^3$.

We verify the canonical-algebra hypothesis needed to use the full fusion rules in [53, Theorem 7]; matching modular data alone would not suffice. In its published double, the twist-one simples are $1, b$ of dimension d^2 , and six A_i of dimension $1 + d^2$. The nonzero C_{53} sector has no twist one, and the other twist-one labels are precisely those with zero second coordinate. For the actual dual category $\mathcal{D}$, its canonical Lagrangian algebra $L = I_{\mathcal{D}}(1)$ is connected, has trivial twist and dimension $7(1 + d^2)$. Therefore

$$L = 1 + m_b b + \sum_{i=1}^6 m_i A_i.$$

Writing $T = \sum_i m_i$, rational and irrational coefficients in the dimension identity give $m_b + T = 7$ and $m_b + 2T = 13$, hence $m_b = 1, T = 6$. Adjunction gives $\dim \text{End}(L) = \text{rank}(\mathcal{D}) = 8$, so $\sum_i m_i^2 = 6$. Thus every $m_i = 1$. The actual canonical algebra has exactly the underlying object required by the cited theorem; no assumption about a guessed algebra multiplication is made.

For a direct exact constructor, use basis $1, a, b_1, b_2, b_3, c_1, c_2, c_3$. Put $B = \sum_i b_i$, $C = \sum_i c_i$, $b_0 = c_0 = 0$, and $b_{-i} = b_i, c_{-i} = c_i$ for indices modulo seven. The rules are

$$\begin{aligned} a^2 &= 1 + a + B + C, & ab_i &= a + B - b_i + C, \\ ac_i &= a + B + c_i + C, & b_i c_j &= a + B + C, \\ b_i b_j &= \delta_{ij} 1 + (1 - \delta_{ij})a + B - b_{i+j} - b_{i-j} + C, \\ c_i c_j &= \delta_{ij} 1 + (1 + \delta_{ij})a + B + c_{i+j} + c_{i-j} + C. \end{aligned}$$

They match every source coefficient; all 4096 associativity equations and all dimension products in $\mathbb{Q}(d)$ are checked. Both even parts of the cited finite-depth C^* subfactor are unitary.

Model	Model-to-census permutation
<i>R8_2_143</i>	(0, 1, 7, 2, 3, 5, 6, 4)
<i>R8_2_179</i>	(0, 1, 4, 3, 2, 5, 6, 7)
<i>R8_2_237</i>	(0, 1, 2, 5, 6, 7, 4, 3)
<i>R8_2_239</i>	(0, 1, 2, 3, 4, 5, 7, 6)
<i>R8_2_244</i>	(0, 1, 2, 3, 7, 4, 5, 6)
<i>R8_2_289</i>	(0, 1, 2, 3, 4, 5, 6, 7)
<i>R8_2_290</i>	(0, 1, 6, 2, 7, 3, 4, 5)
<i>R8_2_313</i>	(0, 7, 1, 2, 3, 4, 6, 5)
<i>R8_2_557</i>	(0, 1, 2, 7, 3, 6, 5, 4)
<i>R8_2_575</i>	(0, 5, 1, 2, 3, 7, 4, 6)
<i>R8_2_855</i>	(0, 1, 7, 4, 5, 6, 2, 3)
<i>R8_2_868</i>	(0, 3, 1, 2, 4, 5, 7, 6)

The model bases are fixed by the independent constructors in the supplement. Each permutation is checked on the entire ordered tensor; it is not inferred from a character table or dimension vector.

14. A NONSPLIT EXTENSION OF THE 2D2 CATEGORY

Proposition 14.1. *The ring $R8_2_37$ admits a unitary categorification.*

Proof. We use the factor realization of a graded extension of the known Haagerup–Izumi category for C_4 in [66, Section 3 and Theorem 4.1]. It supplies automorphisms α_j, β of a factor M , an endomorphism ρ , and unitaries $u, v \in M$ such that

$$\beta \alpha_j = \alpha_j \beta, \quad \beta \rho = \text{Ad}(u) \alpha_1 \rho \beta, \quad \beta^2 = \text{Ad}(v) \alpha_{1+z}, \quad z \in \{0, 2\}.$$

The α_j form an honest C_4 action. The character identities in [66, Lemma 3.2, (3.10) and (3.18)] give $\alpha_2(u) = -u$ and $\alpha_2(v) = -v$.

Form $P = M \rtimes_{\alpha_2} C_2$, with implementing involution λ . Grossman’s construction [55, Section 4.1] extends the neutral generators by

$$\tilde{\alpha}_j(\lambda) = (-1)^j \lambda, \quad \tilde{\rho}(\lambda) = \lambda.$$

They generate the unitary 2D2 category with simple sectors $(1, g, r, gr)$, where $g = [\tilde{\alpha}_1]$, $r = [\tilde{\rho}]$, and

$$g^2 = 1, \quad gr = rg, \quad r^2 = 1 + 2r + 2gr.$$

This construction depends only on the algebra $1 + \alpha_2$ in the neutral category, so it applies to the neutral realization just chosen.

Extend β to $b \in \text{Aut}(P)$ by $b(\lambda) = \lambda$. The factor identities extend to

$$b\tilde{\rho} = \text{Ad}(u)\tilde{\alpha}_1\tilde{\rho}b, \quad b^2 = \text{Ad}(v)\tilde{\alpha}_{1+z}.$$

Indeed, they hold on M by construction. On λ the two right-hand sides are respectively $u(-\lambda)u^* = \lambda$ and $v(-\lambda)v^* = \lambda$. The anticommutation of u, v with λ is essential here.

Put $a = [b]$. These identities imply

$$a^2 = g, \quad a^4 = 1, \quad ara^{-1} = gr = a^2r, \quad ra = a^{-1}r.$$

The full tensor category generated by b and $\tilde{\rho}$ has exactly the eight simple sectors

$$(1, a, a^2, a^3, r, ar, a^2r, a^3r).$$

The four invertibles are distinct because $a^2 = g \neq 1$. If $a^i r \cong a^j r$, Frobenius reciprocity puts a^{j-i} in $rr^* = 1 + 2r + 2gr$, so $i = j$. The noninvertibles have dimension $2 + \sqrt{5}$. The displayed relations show that these eight sectors are closed under products and duals, and their associativity and unitarity are inherited from $\text{End}_0(P)$.

For $i, j \in \mathbb{Z}/4$, their multiplication is

$$\begin{aligned} a^i a^j &= a^{i+j}, & a^i(a^j r) &= a^{i+j} r, \\ (a^i r)a^j &= a^{i-j} r, & (a^i r)(a^j r) &= a^{i-j} + 2a^{i-j} r + 2a^{i-j+2} r. \end{aligned}$$

The displayed basis maps to zero-based census labels $(0, 1, 7, 2, 5, 3, 6, 4)$. The exact verifier compares all 512 structure constants under this bijection. Thus it identifies an already constructed category with $R8_2_37$; no coherent order-two autoequivalence is assumed. $\square$

14.1. Odd isotypic multiplicities and involution algebras.

Lemma 14.2. *Let X be a self-dual simple in a pivotal fusion category over $\mathbb{C}$. If a self-dual simple S occurs with odd multiplicity in $X^{\otimes n}$, then*

$$\nu_2(S) = \nu_2(X)^n.$$

More generally, in a palindromic tensor word of self-dual simples, every self-dual constituent of odd multiplicity has indicator equal to the product of the indicators of the factors.

Proof. Use pivotal coherence to suppress associators and double-dual identifications. Choose a self-duality $f : X \rightarrow X^*$ whose pivotal transpose is $t(f) = \epsilon f$, where $\epsilon = \nu_2(X) \in \{1, -1\}$. The reversed dual word has the same identical factors, so

$$F = f^{\otimes n} : X^{\otimes n} \longrightarrow (X^{\otimes n})^*, \quad t(F) = \epsilon^n F.$$

This construction needs no braiding: reversal changes neither the object word nor the sequence of component morphisms. On the S -isotypic block the nondegenerate form is $f_S \otimes B$, with $t(f_S) = \nu_2(S)f_S$ and B an invertible bilinear-form matrix on the complex multiplicity space. Hence

$$B^\top = \frac{\epsilon^n}{\nu_2(S)} B.$$

An invertible alternating matrix has even size, which proves the assertion for an odd multiplicity. Semisimplicity ensures that the form on a self-dual isotypic block is nondegenerate. The same calculation works for a palindrome, using the same chosen self-duality at each pair of matching positions. $\square$

Proposition 14.3. *The ring $R8_1_49$ has no categorification over $\mathbb{C}$.*

Proof. Its invertibles are $G = \{0, 3, 4, 5\} \cong C_2^2$, and its four noninvertibles have dimension $d = 2 + \sqrt{5}$, with $d^2 = 1 + 4d$. The published two-dimension theorem [25, Lemma 3.1] therefore replaces any hypothetical categorification by a pseudo-unitary Galois conjugate with the same fusion rules. Equip this conjugate with its canonical positive spherical structure. This supplies the pivotal structure without assuming one on the initial category.

Set $u = 3$, $g = 5$ and $\rho = 6$. The source tensor gives

$$\rho^* = \rho, \quad \rho^2 = 1 + \sum_{h \in G} h\rho, \quad \rho^3 = (1, 4, 4, 1, 1, 1, 5, 4)$$

in source-label order. The unit and every invertible occur once in the cube. Lemma 14.2 first gives $\nu_2(\rho) = 1$ and then $\nu_2(h) = 1$ for every $h \in G$. In the positive spherical structure of a pointed C_2 category, the indicator of its nontrivial object is the sign of its associator class. Thus the restriction to $H = \{1, u\}$ is trivial, and its regular algebra $A = 1 \oplus u$ actually exists and is separable. No central lift of u is required.

Consider $\mathcal{B} = {}_A\mathcal{C}_A$. The source relation $u\rho u = g\rho$ gives two pointed H -by- H orbits of size two, $\{0, 3\}$ and $\{4, 5\}$, and the free noninvertible orbit $\{1, 2, 6, 7\}$. Each pointed orbit has stabilizer C_2 and contributes two simple bimodules of dimension one, since every complex projective representation of C_2 can be linearized. The free orbit contributes the single simple $A\rho A$ of bimodule dimension $4d/2 = 2d$. Thus $\mathcal{B}$ has four invertibles, forming $G' \cong C_4$ or C_2^2 , and one noninvertible Z . Reciprocity and dimensions force

$$Z^2 = \sum_{h \in G'} h + 8Z.$$

For $G' = C_2^2$ this contradicts [25, Theorem 1.1]; for $G' = C_4$ it contradicts the general cyclic-four criterion [16, Theorem 6.3] at $k = 2$. The source-bound verifier `verify_r49_fs_morita.py` checks the cube, all double orbits, and the same six-branch terminal certificate used below. Hence the original ring cannot be categorified. $\square$

14.2. An inverse-Morita obstruction for the split involution.

Proposition 14.4. *The ring `R8_2_170` admits no fusion-category categorification over $\mathbb{C}$.*

Proof. Its source tensor has a faithful C_2 grading with neutral labels $(0, 5, 6, 7)$. Put $g = 5$, $r = 6$, and $u = 3$. Its four invertibles form C_2^2 , with u odd, and $uru = gr$. The subgroup $H = \{1, u\}$ maps isomorphically to the grading group. If a category with this tensor exists, multiply its associator by the inflation of the inverse of its pointed associator class on H . This is a normalized grading three-cocycle, so it preserves the pentagon and every fusion coefficient. After this twist $A = 1 \oplus u$ is a connected separable algebra. This step does not assume that A exists before twisting.

Consider the fusion category $\mathcal{B} = {}_A\mathcal{C}_A$. For a free bimodule $F(Y) = A \otimes Y \otimes A$, adjunction gives

$$\dim \text{End}_{\mathcal{B}}(F(Y)) = \dim \text{Hom}_{\mathcal{C}}(Y, A \otimes Y \otimes A) = |\{(h, k) \in H^2 : hYk \cong Y\}|.$$

The two pointed double orbits are $\{0, 3\}$ and $\{4, 5\}$, each of stabilizer size two. Each free bimodule therefore splits into two distinct simple bimodules, once each: its semisimple endomorphism algebra has dimension two. The underlying support of either summand is stable under H on both sides, so each contains both orbit objects once. Dividing its underlying FP dimension by $\text{FPdim}(A) = 2$ shows that it is invertible. Thus these orbits give four invertibles in $\mathcal{B}$.

The four noninvertibles form one double orbit $\{1, 2, 6, 7\}$ with trivial stabilizer. Hence $F(r)$ is a single simple Z of dimension $2(2 + \sqrt{5}) = 4 + 2\sqrt{5}$. Every simple bimodule occurs in a free one, so this is the complete list. The four invertibles form a group G equal to C_4 or C_2^2 . The unique noninvertible is self-dual and fixed by G ; reciprocity and dimensions give

$$Z^2 = \sum_{h \in G} h + 8Z.$$

For $G = C_2^2$, this contradicts [25, Theorem 1.1]. For $G = C_4$, it contradicts the general cyclic-four central-support criterion of [16, Theorem 6.3], at parameter $k = 2$ and moment height six. The archived optimizer-free replay checks all six branches: two force a fusion coefficient $1/2$, and four have exact separating vectors with nonnegative $A^\top y$ and respective values $b^\top y = -800, -220, -320, -80$. This terminal criterion applies to arbitrary complex categorifications through its proved Galois

pseudo-unitarity reduction. Both possible groups are excluded. No classification of the neutral $2D2$ category is used. $\square$

15. THE CYCLIC ORDER-SIX HAAGERUP–IZUMI ALGEBRA DUAL

Proposition 15.1. *The dual of the algebra $1 + \rho$ in a known unitary Haagerup–Izumi category for C_6 has Grothendieck ring $R8_2_535$.*

Proof. Set $d = 3 + \sqrt{10}$ and $D = 6(1 + d^2) = 120 + 36\sqrt{10}$. The category and its algebra $1 + \rho$ exist by [67, Example 4.5 and condition (2.24)]. Denote the unitary dual by $\mathcal{E}$. Izumi’s description [8, Proposition 7.4 and Theorem 7.5] gives its simple objects and dimensions as

	1	b	r	rb	e_1	e_2	s_1	s_2
FPdim	1	1	d	d	$d + 1$	$d + 1$	$d - 1$	$d - 1$

Here $b = \beta_3$, $e_j = \eta_j$, and s_1, s_2 are the two σ objects. Its canonical module has seven simples $(i_0, \dots, i_5, \tau)$ with relative dimensions $(1, \dots, 1, d - 1)$.

Let P be the cyclic permutation matrix on the six i -labels. The upper-left six-by-six blocks of the eight action matrices are

$$I, \quad P^3, \quad I, \quad P^3, \quad P + P^{-1}, \quad P^2 + P^{-2}, \quad 0, \quad 0.$$

The matrices of $1, b$ have zero off-diagonal τ entries and τ diagonal 1. Each of the other six has every off-diagonal τ entry equal to 1, with respective τ diagonals $(5, 5, 6, 6, 4, 4)$. These formulas follow from the decompositions in [8, Proposition 7.4]; the last entries also follow from the relative FP dimension vector.

These matrices generate a commutative algebra of dimension five. Their three independent dependencies are

$$M_1 - M_b - M_r + M_{rb} = 0, \quad M_1 - M_r + M_{s_1} = 0, \quad M_1 - M_r + M_{s_2} = 0.$$

The complex fusion algebra is semisimple. A noncommutative simple block would have dimension at least four and would have to lie in the three-dimensional kernel of this representation. Hence the entire fusion algebra of $\mathcal{E}$ is commutative.

Write $d' = 3 - \sqrt{10}$ and $D' = 120 - 36\sqrt{10}$. Fourier diagonalization of the module yields five characters:

	1	b	r	rb	e_1	e_2	s_1	s_2	f
χ_{FP}	1	1	d	d	$d + 1$	$d + 1$	$d - 1$	$d - 1$	D
χ'_{FP}	1	1	d'	d'	$d' + 1$	$d' + 1$	$d' - 1$	$d' - 1$	D'
u	1	-1	1	-1	1	-1	0	0	6
v	1	1	1	1	-1	-1	0	0	6
w	1	-1	1	-1	-2	2	0	0	12.

There are three further one-dimensional characters. Orthogonality to the displayed rows forces each to have the form

$$(1, \epsilon, -1, -\epsilon, 0, 0, s, t), \quad \epsilon \in \{1, -1\}, \quad s + t = 1 + \epsilon. \quad (22)$$

We determine their codegrees using the center of the known category, which is Morita invariant. The complete published inventory [67, Lemma 3.5 and Theorem 3.9] is

simple dimension	1	d^2	$1 + 3d$	$2 + 6d$	$3d$	$6d$
multiplicity	2	2	4	16	4	18.

The dimensions squared sum to D^2 . Formal-codegree induction [24] associates to each character a simple constituent of $I(1)$ of dimension D/f . Thus the possible codegrees are

$$D, \quad D', \quad 12, \quad 6, \quad 4\sqrt{10}, \quad 2\sqrt{10}.$$

Formal codegrees are totally positive: in this commutative case each is the squared norm of a character and every Galois conjugate is another formal codegree. This eliminates the last two values. For a missing character, the Perron bound gives

$$f = 4 + |s|^2 + |t|^2 \leq 4 + 2(d-1)^2 = 8 + 8d < D.$$

So no missing codegree is D . The five known rows form a Galois-stable set; a missing codegree D' would have a missing Galois conjugate of codegree D , also impossible. Finally $\sum_{\chi} f_{\chi}^{-1} = 1$ and $D^{-1} + (D')^{-1} = 1/6$ force the three missing codegrees to be 6, 6, 12.

The object b permutes s_1, s_2 . If it fixed both, a missing character with $\epsilon = -1$ would have $s = t = 0$ and codegree 4. All three missing rows would then have $\epsilon = 1$, but the vectors in (22) with $\epsilon = 1$ span a space of dimension at most two. Therefore b exchanges s_1 and s_2 , and $t = \epsilon s$. There is exactly one missing row with $\epsilon = 1$, namely $(1, 1, -1, -1, 0, 0, 1, 1)$ of codegree 6. The two remaining rows have $\epsilon = -1$, one of each codegree 6, 12. Complex conjugation preserves the missing set, ϵ , and the codegree, so each of these two rows is real. Their norms and orthogonality determine, up to exchanging s_1, s_2 , the full list

$$\begin{array}{c|cccccccc} x & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 6 \\ y & 1 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & 6 \\ z & 1 & -1 & -1 & 1 & 0 & 0 & -2 & 2 & 12. \end{array}$$

All characters are real, hence all simple objects are self-dual. Character inversion now determines every structure constant:

$$N_{ij}^k = \sum_{\chi} \frac{\chi(i)\chi(j)\chi(k)}{f_{\chi}}.$$

The resulting tensor is identified with $R8_2_535$ by the basis map $(0, 1, 3, 6, 4, 5, 2, 7)$. The verifier reconstructs all 512 coefficients exactly and checks the canonical module equations. No multiplicity bound or completeness claim about the census is used to determine the unknown dual ring. $\square$

16. ALGEBRAIC RIGIDITY OF THE 2D2 MODULE GRAPH

The next argument gives four general complex obstructions. It does not assert that every categorification of the rank-four $2D2$ ring is unitary: the specific module graph is part of the hypothesis.

Lemma 16.1. *Let $\mathcal{D}$ be a pseudo-unitary fusion category and $\mathcal{M}$ an indecomposable module category with a simple object X . Suppose its rooted principal graph is $2D2$, with modulus $\delta = \sqrt{3 + \sqrt{5}}$. Then the associated algebraic shaded planar algebra is isomorphic to the known unitary $2D2$ planar algebra. In particular its dual even category has rank six.*

Proof. We first justify the algebraic planar-algebra construction. Schaumann's module-trace theorem [68, Proposition 5.8] supplies positive FP trace weights on $\mathcal{M}$. His Theorem 4.5 constructs natural double-adjoint isomorphisms for module functors between any two traced module categories, compatible with composition. Applied to $\mathcal{D}$ and $\mathcal{M}$, it equips the full Morita linking category with a pivotal structure. Corollary 4.6 and Proposition 5.10(iv) give the compatible spherical dual. Normalize the module trace so that the two loop values of X are δ .

For completeness, the graph embedding in this setting requires no star operation. Act on a column of the linking category, and decompose every object into simples. A morphism acts on the resulting path spaces as a direct sum of matrices, faithfully. Choose dual bases for the nondegenerate trace pairings. Composition is matrix multiplication, while evaluation and coevaluation are the weighted graph cups and caps. Thus all planar operations are preserved. This is the algebraic pivotal construction of [43, Theorems 3.45 and 3.70], with the cup and cap calculation of its Theorem 3.75 performed with dual bases. Its dagger-specific Theorem 3.77 is not needed. Denote the resulting planar algebra by $\mathcal{P}$ and its embedding into the graph planar algebra by $\mathcal{P} \subseteq \mathcal{G}$.

The graph is two-supertransitive. Its depth-three Jones–Wenzl corner is $f_3\mathcal{P}_{3,+}f_3 = \mathbb{C}p \oplus \mathbb{C}q$, where p, q are orthogonal primitive idempotents and $p + q = f_3$. The two depth-three PF weights coincide, so $T = p - q$ has trace zero and $T^2 = f_3$. The bilinear trace pairing is nondegenerate, and

$$\mathcal{P}_{3,+} = \text{TL}_{3,+} \oplus \mathbb{C}T$$

is a trace-orthogonal decomposition. Pairing a cap of T with every element of $\mathcal{P}_{2,+}$ shows that it vanishes, since the corresponding cups lie in $\text{TL}_{3,+}$. Rotation preserves the one-dimensional orthogonal complement, so T is an uncappable rotational eigenvector, whose eigenvalue is a cube root of unity. We verify the exclusion of the two primitive roots directly below; this also independently replays [69, Lemma 3.2] without a self-adjointness hypothesis.

We describe the exact complex calculation needed at this point. Label the graph vertices $0, \dots, 7$, with edges

$$01, 12, 23, 24, 35, 45, 56, 67,$$

rooted at 0 and with even vertices $0, 2, 5, 7$. Put $s = \sqrt{5}$, $\varphi = (1 + s)/2$, and assign weights

$$(w_0, \dots, w_7) = (1, 1, 2 + s, \varphi, \varphi, 2 + s, 1, 1).$$

These are the PF weights with odd entries divided by δ . For a path $p = (p_0, p_1, p_2, p_3)$, set $A_p = \sqrt{w_{p_1}w_{p_2}}$. The rectangular lopsided matrix is $T_{\text{rect}} = AT_{\text{sph}}A^{-1}$. Thus the coefficient at a loop $(v_0, \dots, v_5)$ is multiplied by $\sqrt{w_{v_1}w_{v_2}/(w_{v_5}w_{v_4})}$. In these coordinates a Temperley–Lieb projection has entries $w_{p_i}/\sum_{a \sim p_{i-1}} w_a$ on paths differing only at the backtracking vertex p_i , and zero otherwise. Caps therefore give unweighted sum equations. Two-click rotation replaces $(v_0, \dots, v_5)$ by $(v_4, v_5, v_0, v_1, v_2, v_3)$, with coefficient $w_{v_0}w_{v_5}/(w_{v_2}w_{v_1})$.

For a primitive root ω , work in $\mathbb{Q}(s, \omega)$, $\omega^2 + \omega + 1 = 0$. The cap and ω -rotation equations on the 152 loops have rank 144. After their elimination, let E_j be the 90 distinct normalized nonzero entries of $T^2 - f_3$, in the deterministic path-matrix order of the verifier. Direct polynomial arithmetic gives

$$\begin{aligned} 1 &= (-11 - 5s)E_{26} + \left(-25 - 11s + \frac{5 + 3s}{2}\omega\right)E_0 \\ &\quad + \left(-12 - 6s + \frac{9 + 3s}{2}\omega\right)E_1 + (44 + 20s)E_4 \\ &\quad + (4 + 2s - (7 + 3s)\omega)E_{13}. \end{aligned}$$

The portable checker reconstructs the linear elimination and every E_j from the graph before checking this identity. It excludes both primitive cubic roots simultaneously, without complex conjugation equations. Hence the eigenvalue is one.

For eigenvalue one the cap and rotation system again has rank 144. Its eight free loop coordinates $x_0, \dots, x_7$ may be chosen, in order, at

$$012321, 212323, 212354, 212453, 232323, 232354, 232453, 235453.$$

Put $k = (3 - s)/2$ and $c = (5s - 11)/2 > 0$. On this eight-dimensional linear space, the ideal of all path-matrix entries of $T^2 - f_3$ is generated by

$$\begin{aligned} x_0 + \frac{3 + s}{2}x_7, \quad x_1 + \frac{3 - s}{2}x_7, \quad x_2 + 2x_5, \quad x_3 + 2x_6, \\ x_4 + \frac{s - 1}{2}x_7, \quad x_5x_6 - c, \quad x_7^2 - k^2. \end{aligned} \tag{23}$$

This is an equality of polynomial ideals over $\mathbb{Q}(\sqrt{5})$, not just equality of their real or self-adjoint solution sets. The portable certificate generates all square equations from the graph and verifies both inclusions: reduction proves that (23) implies every square equation, and explicit polynomial identities of degree at most three express its seven generators in the original equations. The linear rank and all coefficients are checked exactly.

Thus every complex solution is $\pm T(\lambda)$ for $\lambda \in \mathbb{C}^\times$, with free coordinates

$$\left(-1, -k^2, -2\sqrt{c}\lambda, -2\sqrt{c}\lambda^{-1}, -\frac{s-1}{2}k, \sqrt{c}\lambda, \sqrt{c}\lambda^{-1}, k\right).$$

The base solution $T(1)$ is self-adjoint after the diagonal change to spherical coordinates. Assign an oriented-edge character λ to the edge $3 \rightarrow 5$, its inverse to the reverse edge, and 1 to all others. Multiplying each loop coefficient by its edge-character product is a graph planar algebra automorphism: the factors on internal edges cancel in every state sum. The eight coordinates have winding numbers $(0, 0, 1, -1, 0, 1, -1, 0)$, so this gauge carries $T(1)$ to $T(\lambda)$. The two signs generate the same subalgebra.

After applying this gauge, $\mathcal{P}$ contains the planar subalgebra generated by $T(1)$. By [69, Lemma 3.4 and Theorems 7.8–7.9], this is the known unitary $2D2$ planar algebra. Its positive box dimensions are the rooted-loop dimensions of $2D2$, exactly those of $\mathcal{P}$. The containment is therefore equality in every positive box space; one-click rotation gives equality in the negative spaces of positive degree, and both zero-box spaces are $\mathbb{C}$. This rules out additional higher-depth data without using positivity on $\mathcal{P}$. The known dual graph has level sizes $(1, 1, 1, 2, 4, 1)$, hence six even vertices. $\square$

Proposition 16.2. *The rings $R8_2_152$, $R8_2_153$, $R8_2_166$ and $R8_2_495$ admit no fusion-category categorification over $\mathbb{C}$.*

Proof. Each tensor has a faithful C_2 grading with rank-four neutral ring $g^2 = 1$, $gr = rg$, $r^2 = 1 + 2r + 2gr$. Its odd component contains X with $XX^* = X^*X = 1 + r$. The source witnesses are

ring	$(1, g, r, gr)$	X	graph vertices $0, \dots, 7$
$R8_2_152$	$(0, 6, 5, 7)$	1	$(0, 1, 5, 3, 4, 7, 2, 6)$
$R8_2_153$	$(0, 6, 5, 7)$	1	$(0, 1, 5, 3, 4, 7, 2, 6)$
$R8_2_166$	$(0, 5, 7, 6)$	3	$(0, 3, 7, 1, 2, 6, 4, 5)$
$R8_2_495$	$(0, 1, 6, 3)$	2	$(0, 2, 6, 4, 5, 3, 7, 1)$

Both left and right incidence matrices for tensoring with X are the rooted graph of Lemma 16.1. The object r generates the full neutral category, so both even parts have rank four.

Suppose a complex categorification exists. Its neutral category has exactly two FP dimensions. By [25, Lemma 3.1], it is Galois conjugate to a pseudo-unitary category. Apply the same field automorphism to the entire category and its odd module; their based graphs are unchanged. Lemma 16.1 then forces the dual even part to have rank six, contradicting the rank-four graph computed from the tensor. Equivalently, the odd component is an invertible neutral bimodule and its dual category is the rank-four neutral category. No pivotal structure on the original rank-eight category is assumed. $\square$

17. POINTED SUPPORT IN A TWISTED HALF-BRAIDING

Proposition 17.1. *The rings $R8_2_314$ and $R8_2_432$ admit no fusion-category categorification over $\mathbb{C}$.*

Proof. Both tensors have a fusion subring $\mathcal{D}$ with basis $(1, g, r, gr)$ and rules

$$g^2 = 1, \quad gr = rg, \quad r^2 = g^\epsilon + r + gr,$$

where $\epsilon = 1$ for $R8_2_314$ and $\epsilon = 0$ for $R8_2_432$. The complement $\mathcal{M}$ is stable under both $\mathcal{D}$ actions. Its four simples are aX for $a \in \text{Irr } \mathcal{D}$, with

$$\begin{aligned} Xa &\cong F(a)X, & X^2 &= g^\epsilon + X + gX, \\ F(1) &= 1, & F(g) &= g, & F(r) &= gr, & F(gr) &= r. \end{aligned}$$

The complete source witnesses are

ring	$(1, g, r, gr)$	X	(X, gX, rX, grX)
$R8_2_314$	$(0, 7, 1, 2)$	3	$(3, 4, 6, 5)$
$R8_2_432$	$(0, 3, 4, 5)$	6	$(6, 7, 1, 2)$

Here F is initially only notation for the displayed label permutation. We explain why a hypothetical categorification forces the coherence needed below.

The functor $L : \mathcal{D} \rightarrow \mathcal{M}$, $a \mapsto aX$, is an equivalence of left module categories: it maps simples bijectively to simples and is fully faithful on their Hom spaces. Transport the right $\mathcal{D}$ action through L . Left module endofunctors of the regular category are right tensor functors, so this action supplies a strong monoidal functor $F : \mathcal{D} \rightarrow \mathcal{D}$ with coherent natural isomorphisms $Xa \cong F(a)X$. Its displayed permutation of simples makes it a tensor autoequivalence.

Applying this relation twice gives coherent isomorphisms $X^2a \cong F^2(a)X^2$. The decomposition $\mathcal{C} = \mathcal{D} \oplus \mathcal{M}$ is a decomposition of $\mathcal{D}$ -bimodule categories, so its projection to $\mathcal{M}$ preserves these constraints. Write $z = 1 \oplus g$. The $\mathcal{M}$ part of X^2 is zX . Using $Xa \cong F(a)X$ and cancelling the fully faithful functor L gives

$$\eta_a : zF(a) \longrightarrow F^2(a)z.$$

If J^F, J^{F^2} denote the tensorators, its compatibility is

$$\eta_{ab}(\text{id}_z \otimes J_{a,b}^F) = (J_{a,b}^{F^2} \otimes \text{id}_z)(\text{id}_{F^2(a)} \otimes \eta_b)(\eta_a \otimes \text{id}_{F(b)}),$$

with associators understood. Equivalently this is the bimodule constraint of the $\mathcal{M}$ projection of multiplication $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M} \rightarrow \mathcal{C}$. Applying a monoidal quasi-inverse of F , and using $F^{-1}(z) \cong z$, gives an F -twisted half-braiding

$$\theta_a : za \longrightarrow F(a)z, \quad \theta_{ab} = (J_{a,b}^F \otimes \text{id}_z)(\text{id}_{F(a)} \otimes \theta_b)(\theta_a \otimes \text{id}_b). \quad (24)$$

No identification of F^2 with the identity is required.

Fix the two summands $z_0 = 1, z_1 = g$. Since $r \neq gr$ and $F(r) = gr$, the map θ_r must send $z_i r$ to $F(r)z_{i+1}$, with indices modulo two. Thus it exchanges the two support labels. Equation (24) for $a = b = r$ applies two exchanges, and hence $\theta_{r \otimes r}$ preserves these labels. Its associators and tensorator preserve the relevant direct-sum idempotents by naturality.

Choose a nonzero inclusion $v : r \rightarrow r \otimes r$, which exists in both rings. Naturality requires

$$\theta_{r \otimes r}(\text{id}_z \otimes v) = (F(v) \otimes \text{id}_z)\theta_r.$$

On $z_i r$ the left side lands in $F(r \otimes r)z_i$ and the right side in $F(r \otimes r)z_{i+1}$. Both maps are nonzero: $v, F(v)$ are split monomorphisms, tensoring preserves them, and the θ maps are isomorphisms. The two target direct summands have zero intersection, a contradiction. The argument uses only semisimplicity, rigidity and tensor coherence, with no pivotality assumption. $\square$

18. A FORCED QUANTUM-GROUP MORITA DUAL

Proposition 18.1. *The ring $R8_2_342$ has no categorification over $\mathbb{C}$.*

Proof. Suppose $\mathcal{C}$ categorifies this ring. The invertible $g = 5$ fixes simples 3, 7. Its pointed associator is therefore trivializable, since either fixed simple gives a rank-one module category over $\langle g \rangle$. Thus $A = 1 \oplus g$ is a connected separable algebra. Put $\mathcal{M} = \mathcal{C}_A$ and let $\mathcal{E}$ be its Morita dual.

The simple module labels are

$$(0, 0), (1, 0), (3, +), (3, -), (4, 0), (7, +), (7, -).$$

The exact certificate `verify_r342_inverse_morita.py` exhausts the two-dimensional signed representations on the two split labels. Certified annihilating polynomials and the three source generators 1, 4, 6 leave eight generator tuples and exactly one full representation up to sign gauge. Consequently all seven-by-seven left action matrices L_i are forced. In particular $L_g = I_7$.

The distinguished induction Gram matrix is $2I_7$. Its nonnegative integral columns must be exactly two copies of each standard basis vector. Hence $\mathcal{E}$ has fourteen simples, two above each module label. Their right action matrices R_y commute with every L_i and satisfy

$$Q_a := \sum_i (L_i)_{aa} L_i = \sum_y R_y[:, a] R_y[:, a]^t, \quad 0 \leq (R_y)_{ba} \leq \lfloor \sqrt{(Q_a)_{bb}} \rfloor.$$

These are linking-category associativity identities. The common commutant has dimension eleven; a prescribed standard first column leaves four free matrix entries. Exhausting the displayed bounds gives candidate counts $(2, 4, 2, 2, 2, 2, 2)$. The first domain contains the identity and a unique non-trivial involutive permutation P , representing the second invertible u . Right multiplication by P pairs every domain freely. Of the resulting two joint candidate tuples, exactly one satisfies all seven Gram identities.

The certificate reconstructs the full dual fusion tensor, rather than stopping at these necessary identities. For each product its coefficient vector n obeys

$$\sum_w n_w R_w = R_y R_z, \quad n_1 = \delta_{z,y^*}, \quad n_u = \delta_{zu,y^*}.$$

The first column bounds each coefficient by the corresponding entry of $R_y R_z$. Complete enumeration forces the ten fusion matrices with labels $0, 1, 4, 5, 6, 7, 10, 11, 12, 13$. The fourteen words

$$1, u, 4, 5, 6, 7, 10, 11, 12, 13, 44, 46, 411, 413$$

have independent class vectors. Their associativity therefore forces all remaining matrices. The resulting tensor passes all original product domains and all 14^4 associativity equations.

The labels $(0, 2, 5, 6, 9, 10, 13)$ form a fusion subcategory $\mathcal{B}$, with complement $\mathcal{B}u$ and faithful C_2 grading. The restriction $\mathcal{M}|_{\mathcal{B}}$ is regular. Its based ring is $\text{PSU}(3)_5$: the generator matrix in [72, Section 2.4] matches under the local map $(0, 4, 5, 6, 1, 3, 2)$, and its seven powers on the unit span the whole ring. Conjugation by u induces precisely its duality permutation $(0, 1, 3, 2, 4, 6, 5)$. The checker also verifies $\mathcal{B}_{\text{ad}} = \mathcal{B}$ and that its based automorphism group is C_2 .

The Kazhdan–Wenzl theorem, in the general monoidal form [73, Theorem 5.1], applies to $\mathcal{B} \boxtimes \text{Vec}_{C_3}$, whose based ring is $\text{SU}(3)_5$. Restrict its resulting equivalence to adjoint subcategories. The grading three-cocycle is trivial there, so $\mathcal{B}$ is a Galois conjugate of the standard quantum $\text{PSU}(3)_5$ category. This step assumes neither a braiding nor pivotality.

The gauge-autoequivalence kernel of $\text{SU}(3)_5$ is trivial by [50, Theorem 1.1 and Section 3]. Since the standard category factors as $\text{PSU}(3)_5 \boxtimes \mathcal{P}$, with $\mathcal{P}$ pointed of rank three, any based-trivial tensor autoequivalence of $\text{PSU}(3)_5$ extends by the identity on $\mathcal{P}$. It is therefore trivial. Galois transport proves the same statement for $\mathcal{B}$. Thus its nontrivial based duality permutation determines a unique tensor autoequivalence class. Moreover $U(\mathcal{B}) = 1$, so $\text{Aut}_{\otimes}(\text{Id}_{\mathcal{B}}) = 1$; the coherent C_2 action realizing this class is unique up to equivalence when it exists.

Finally, the first-column multiplicities give the actual internal-end algebra $\text{End}_{\mathcal{E}}(m_0) = 1 \oplus u$. Its associativity trivializes the pointed associator of u . The grading is consequently a crossed product $\mathcal{E} = \mathcal{B} \rtimes C_2$, and Morita reflection through this algebra identifies $\mathcal{C}$ with $\mathcal{B}^{C_2}$. The existing category in Proposition 19.1 has a central Tannakian boson, and its de-equivariantization is a Galois conjugate of $\text{PSU}(3)_5$ by [72, Section 2.4]. Its action swaps exactly the two dual pairs and is the same nontrivial based action. The uniqueness above forces $K_0(\mathcal{C}) = R8_2_666$. Every simple in that ring is self-dual, whereas labels $1, 2$ in $R8_2_342$ form a non-self-dual pair. This is a contradiction. $\square$

19. NEAR-GROUP CENTRALIZER CONSTRUCTIONS AND A COMMON SPHERICAL REDUCTION

Proposition 19.1. *The ring $R8_2_666$ admits a unitary braided categorification: the centralizer of the full pointed subcategory in the center of a near-group $C_4 + 4$ category.*

Proof. The smallest hyperbolic case in [71, Section 3.2.3.1] is explicitly identified with the actual centers of the two complex-conjugate near-group $C_4 + 4$ categories. This finite case precedes the conjecture there for larger groups. A convenient description is given in [72, Section 2.4]:

$$G = C_4^2, \quad q_G(x, y) = i^{-xy}, \quad \theta_G(x, y) = (y, x),$$

$$H = C_{16} \times C_2, \quad q_H(x, y) = \zeta_{32}^{3x^2} i^{y^2}, \quad \theta_H(x, y) = (3x + 8y, x + y).$$

The fixed groups identify with $K = C_4$ through $k \mapsto (k, k)$ and $k \mapsto (4k, k \bmod 2)$. The center has 28 simple objects, labelled by K , $K \times \{\pi\}$ and representatives of the nonfixed involution orbits in G, H . Its pointed subcategory $\mathcal{P}$ has dimension four and twist $q_K(k) = i^{-k^2}$. Trivial monodromy with every pointed object selects exactly the eight labels

$$(0, 0), (2, 2), (0, 0, \pi), (2, 2, \pi), (3, 1)_G, (2, 1)_H, (6, 1)_H, (4, 0)_H.$$

Thus $\mathcal{P}'$ is an existing unitary braided fusion category, of dimension $96 + 64\sqrt{2}$. Its Müger center is the Tannakian order-two radical of $\mathcal{P}$.

The source-bound certificate `verify_c4_centralizer.py` reconstructs the full 28-label matrix from [72, Equation (1)]. It works in $\mathbb{Z}[\zeta_{32}]$, using $\zeta_{32}^{16} = -1$ and $\sqrt{2} = \zeta_{32}^4 + \zeta_{32}^{-4}$. It checks unitarity and all 1792 Verlinde coefficients whose first two labels belong to this centralizer, including vanishing of every coefficient outside it. The resulting 512 coefficients identify its based ring with $R8_2_666$ under the unital map

$$(0, 1, 3, 4, 7, 2, 5, 6).$$

The proof note records the entire matrix formula, transversals, normalization and source hash, and the normal and optimized replays also verify associativity and self-duality. $\square$

Proposition 19.2. *The rings $R8_2_820$ and $R8_2_869$ admit unitary modular categorifications. They are the nonpointed Müger factors of the Drinfeld centers of the first and second near-group $C_5 + 5$ categories in [70, Proposition 6 and Table 2].*

Proof. Use the actual unitary near-group categories of [70, Proposition 6 and Table 2], with parameters $(m, \kappa) = (1, -1)$ and $(2, \omega)$, where $\omega = e^{2\pi i/3}$ and $\zeta = e^{2\pi i/5}$. Only the directly computed tube inventory and its known blocks in [70, Section 4.3] are used below. In particular, we do not identify the unknown tube block by numerical proximity to candidate data. Put

$$a = \frac{5 + 3\sqrt{5}}{2}, \quad b = a + 1, \quad c = a + 2, \quad a^2 = 5a + 5.$$

The pointed center objects form the nondegenerate metric category on C_5 with pairing ζ^{2mgh} . Its Müger centralizer is therefore an actual unitary modular category. The tube inventory gives its eight simples $(1, B, C_1, C_2, D_1, \dots, D_4)$, with dimensions $\mathbf{d} = (1, b, c, c, a, a, a, a)$ and global dimension $\mathcal{D} = 5c^2 = 9a^2$. None of the labels D_i presupposes a candidate metric group or duality.

Partial fusion from known columns. Let M be the unnormalized S matrix of the centralizer, so $MM^* = \mathcal{D}I$. The first four columns are real and known directly:

$$\begin{aligned} \mathbf{d}^\top, \quad \mathbf{s}_B &= (b, 1, c, c, -a, -a, -a, -a)^\top, \\ \mathbf{s}_k &= (c, c, c(\zeta^{2mk} + \zeta^{-2mk}), c(\zeta^{4mk} + \zeta^{-4mk}), 0, 0, 0, 0)^\top \quad (k = 1, 2). \end{aligned}$$

Every unknown D column has B -eigenvalue -1 . Consequently

$$N_B = -I + \frac{(b+1)\mathbf{d}^\top \mathbf{d} + (1+b^{-1})\mathbf{s}_B \mathbf{s}_B^\top + 2\mathbf{s}_1 \mathbf{s}_1^\top + 2\mathbf{s}_2 \mathbf{s}_2^\top}{\mathcal{D}}.$$

The C_k Verlinde sums use only the four known columns, because their entries on the other four columns vanish. Writing $E = \sum_i D_i$, exact evaluation gives, for $i \neq j$ in $\{1, 2\}$,

$$\begin{aligned} B^2 &= 1 + B + C_1 + C_2 + E, & BC_i &= B + 2C_i + C_j + E, \\ C_i^2 &= 1 + 2B + C_i + 2C_j + E, & C_1C_2 &= B + 2C_1 + 2C_2 + E, \\ BD_i &= B + C_1 + C_2 + E - D_i, & C_kD_i &= B + C_1 + C_2 + E. \end{aligned}$$

These formulas hold for both seeds and use no unknown D - D entry.

Every possible completion. Let $i \mapsto i^*$ be the unknown involution on the four D labels. Frobenius reciprocity and dimensions force

$$D_iD_j = \delta_{j,i^*}1 + (1 - \delta_{j,i^*})B + C_1 + C_2 + \sum_k n_{ij}^k D_k, \quad \sum_k n_{ij}^k = 2 + \delta_{j,i^*}.$$

The lowered coefficients $F_{ijk} = n_{ij}^{k^*}$ are symmetric, nonnegative integers, invariant under simultaneous duality. The three involution types on four labels exhaust all possibilities. Enumerating the unordered-triple orbits subject to these pair-sum equations gives

duality type	linear assignments	associative tensors	based classes
id	708	9	2
(12)	16	0	0
(12)(34)	0	0	0

Each variable is bounded by the smallest residual pair sum divided by its positive coefficient. The exhaustive recursion removes only a negative residual or one outside the exact remaining capacities; all 8^4 associativity identities are then checked. Thus this finite calculation does not assume census completeness or a guessed multiplicity bound. Both classes are self-dual. In the first, denoted R_E ,

$$D_i^2 = 1 + C_1 + C_2 + E - D_i, \quad D_iD_j = B + C_1 + C_2 + D_i + D_j \quad (i \neq j).$$

In the other, R_C , label the D objects by $x = 1, 2, 3, 4$, the nonzero classes in C_9 modulo sign. Its remaining lowered coefficients are

$$F_{xyz} = 1 - \sum_{s,t \in \{\pm 1\}} \mathbf{1}_{x+sy+tz=0 \pmod{9}}.$$

The analogous formula for C_3^2 , with representatives $(1, 0), (0, 1), (1, 1), (1, 2)$, gives R_E . These groups presently label integer tensors only, not asserted actual tube data.

An invariant of the actual seeds. In the orthonormal Cuntz basis T_h , the anti-linear rotations of [36, Remark 3.11 and Section 9] are

$$j_1(T_h) = \zeta^{2mh^2} T_{-h}, \quad j_2(T_h) = \frac{\bar{\kappa}}{\sqrt{5}} \sum_k \zeta^{-mhk} T_k.$$

Anti-linearity gives

$$\nu_3(\rho) = \text{Tr}(j_1 j_2) = \frac{\kappa}{\sqrt{5}} \sum_h \zeta^{mh^2} = \kappa \left(\frac{m}{5} \right),$$

namely -1 and $-\omega$. Let $t_1, \dots, t_4$ be the unknown actual D twists in the factor. The known other twists are $1, 1, \zeta^{-m}, \zeta^{-4m}$. Put $G_3(m) = \sum_g \zeta^{3mg^2}$ and $L_3(m) = \zeta^{-3m} + \zeta^{-12m}$. Every b, c, d center object contains ρ once under forgetting, and no a -type object contains it. The indicator induction formula [59, Theorem 4.1] therefore gives

$$\nu_3(\rho) = \frac{G_3(m)}{5c} \left(b + cL_3(m) + a \sum_i t_i^3 \right).$$

Here the ten unordered c labels are $(u + t, u - t)$ with $u \in C_5$ and $t = 1, 2$ modulo sign; their cubed-twist sum is $G_3(m)L_3(m)$. Thus the two actual seeds satisfy respectively

$$\sum_i t_i^3 = 4, \quad \sum_i t_i^3 = 1 + 3\omega^2 \neq 4.$$

Since the factors are unitary, the first equality forces every $t_i^3 = 1$.

Balancing distinguishes the two rings. Exact algebraic character tables of R_E, R_C have eight distinct normalized columns. Their four D columns are exactly the characters of formal codegree $\mathcal{D}/a^2 = 9$. Hence the actual D columns must be these columns up to permutation; symmetry constrains that permutation. This is character theory of the already enumerated rings, not a claim that a candidate modular matrix is the actual one. Set $T_D = \sum_i t_i$ and $T_C = \theta_{C_1} + \theta_{C_2}$. Balancing BD_i , with $\theta_B = 1$ and $M_{B,D_i} = -a$, gives $aT_D = -b - cT_C$. Each D_i^2 contains exactly three D labels once; let $f(i)$ denote the omitted label. Balancing that square now gives

$$r_i t_i^2 = -1 - t_{f(i)}, \quad r_i = M_{D_i, D_i}/a.$$

For R_E , $f(i) = i$ and each r_i is 1 or -2 . The respective equations are $t_i^2 + t_i + 1 = 0$ and $(t_i - 1)(2t_i + 1) = 0$; unit modulus forces $t_i^3 = 1$ in both cases. For R_C , the D character block divided by a is $Y_{xy} = -2\cos(4\pi xy/9)$. Row 3 has rational entries $1, -2$; each other row has its sole rational entry in column 3. A column permutation making Y symmetric must fix 3: otherwise compare entries $(3, \pi^{-1}(3))$ and $(\pi^{-1}(3), 3)$. The other three diagonal entries are roots of the irreducible polynomial $X^3 - 3X - 1$. If every $t_i^3 = 1$, the balancing equation would put those real cubic irrational numbers in $\mathbb{Q}(\omega) \cap \mathbb{R} = \mathbb{Q}$, a contradiction. The first actual seed therefore realizes R_E and the second realizes R_C .

Finally, exact comparison of every coefficient identifies these rings with $R8_2_820$ and $R8_2_869$ under

$$(0, 2, 6, 7, 1, 3, 4, 5), \quad (0, 3, 6, 7, 1, 2, 5, 4).$$

The four exact certificates and their normal/optimized replays are in `evidence/correctness_20261025/`. Their categorical premises are precisely the actual seed, tube inventory, modular factorization and indicator theorems used above. No numerical recognition of the unknown block enters this identification. $\square$

The following result, obtained before these constructions, also controls all possible categorifications of the two rings.

Proposition 19.3. *Every complex fusion category with based ring $R8_2_820$ or $R8_2_869$ admits a spherical structure. Every spherical dimension character belongs to the FP Galois orbit. Consequently, categorifiability of either ring is equivalent to pseudo-unitary categorifiability, and any categorification has a pseudo-unitary Galois conjugate.*

Proof. Both tensors are self-dual. In the canonical sphericalization choose the positive-second-indicator lift of each simple. The resulting signed quotient tensor T is fully symmetric, has $|T_{ij}^k| \leq N_{ij}^k$ and $T_{ij}^k \equiv N_{ij}^k \pmod{2}$, and satisfies all associativity equations. The odd-channel parity theorem [62, Theorem 3.9] gives the stronger restriction

$$N_{jj}^i - T_{jj}^i \equiv 0 \pmod{4}.$$

Every coefficient two in these tensors has a repeated label. Thus repeated coefficients one are fixed at $+1$, repeated coefficients two lie in $\{-2, 2\}$, and the remaining all-distinct coefficients one lie in $\{-1, 1\}$.

The source-bound checker reconstructs exactly 35 Boolean signs for $R8_2_820$ and 36 for $R8_2_869$. Expanding every ordered associativity identity and reducing sign squares to one leaves 215 distinct integer polynomial equations for either tensor. Exhaustive branching visits both signs at each unassigned variable and rejects a branch only when a fully assigned equation fails. It visits respectively

3071 and 1873 nodes and finds exactly one solution in each case: all signs are positive. Hence $T = N$. The chosen lifts form a full fusion subcategory of the sphericalization, and the forgetful functor restricts to a tensor equivalence with the original category. Restricting its spherical structure proves the first assertion.

The remaining reduction is common and theoretical. Write the basis as $(1, a_1, a_2, a_3, a_4, b, c_1, c_2)$, where b has source label 2 for $R8_2_820$ and label 3 for $R8_2_869$, the c -labels are 6, 7, and the a -labels are the four other nonunit labels. Both integral based rings satisfy

$$b^2 = 1 + \sum_{i=1}^4 a_i + b + c_1 + c_2, \quad (b+1)a_i = b^2 - 1, \quad (b-1)c_j = b^2 - 1.$$

Let $x = \dim(b)$ for any spherical structure. Nonzero simple dimensions exclude $x = 1$ and $x = -1$ by the last two identities. They therefore give $\dim(a_i) = x - 1$ and $\dim(c_j) = x + 1$. The first identity becomes

$$x^2 = 1 + 4(x-1) + x + 2(x+1) = 7x - 1, \quad x = \frac{7 \pm 3\sqrt{5}}{2}.$$

These are exactly the two Galois-conjugate FP characters. Conjugate the category together with its spherical structure to the positive character. A spherical category with positive simple dimensions is pseudo-unitary. This statement concerns every categorification; it does not assert that each categorification is unitarizable. $\square$

19.1. A unified character-parity exclusion for the Fibonacci pair.

Proposition 19.4. *Every complex categorification of $R8_2_190$ or $R8_2_489$ admits a spherical structure and has a pseudo-unitary Galois conjugate. For $R8_2_489$, the pointed C_2 subcategory has trivial associator, so $1 + g$ is a connected separable algebra. The following theorem uses this structural reduction to exclude both rings.*

Proof. Both rings have basis $g^i \tau^j x^k$, $i, j, k \in \{0, 1\}$, and commutative presentation

$$g^2 = 1, \quad \tau^2 = 1 + \tau, \quad x^2 = g^\epsilon + \tau(x + gx), \quad x^* = g^\epsilon x,$$

where $\epsilon = 1$ for $R8_2_190$ and $\epsilon = 0$ for $R8_2_489$. In the canonical sphericalization, choose the lift of g with dimension 1. It squares to the unit. A lift of τ is self-dual, and its signed square has the form $1 \pm \tau$; change the lift to make the sign positive. Choose a lift of x , and define the other lifts by the simple products $g\tau, gx, \tau x, g\tau x$. The relevant left and right products have the same underlying simple and equal nonzero spherical dimensions, so the chosen generators commute in the signed quotient. The same argument gives $x^* = g^\epsilon x$. The unit channel of xx^* therefore fixes the constant coefficient in

$$x^2 = g^\epsilon + a\tau x + b\tau gx, \quad a, b \in \{1, -1\}.$$

These relations determine the entire signed multiplication tensor. When $\epsilon = 1$, duality fixes x^2 and exchanges the last two terms, so $a = b$.

For $\epsilon = 0$, suppose instead $a = -b$. Writing $t = \dim(\tau)$ and $z = \dim(x)$ gives $t^2 = t + 1$, $z^2 = 1$, and global dimension $\delta = 4(t + 2)$. A source character with values $g \mapsto 1$, $\tau \mapsto t$, $x \mapsto d$, where $d^2 = 1 + 2td$, has formal codegree $f_E = 2(t + 2)(1 + d^2)$. Central induction in the sphericalization requires δ/f_E to be an algebraic integer [74, Theorem 2.13]: the sphericalization dimension and the source formal codegree both acquire a factor two. However,

$$d^4 - 2d^3 - 6d^2 + 2d + 1 = 0, \quad y = \frac{\delta}{f_E} = \frac{2}{1 + d^2}$$

has primitive minimal polynomial

$$5y^4 - 20y^3 + 25y^2 - 10y + 1.$$

Indeed the polynomial of d is irreducible modulo three, and $d = ((d^2)^2 - 6d^2 + 1)/(2(d^2 - 1))$ gives $\mathbb{Q}(y) = \mathbb{Q}(d)$; elimination gives the displayed polynomial. Its leading coefficient excludes integrality. Thus $a = b$ in both cases. Changing the lift of x and its three translates if necessary makes both signs positive. Now $T = N$, so the chosen lifts form a tensor-closed spherical section of the forgetful functor.

This section has $\dim(g) = 1$, $t^2 = t + 1$, and $z^2 = 1 + 2tz$. The irreducible quartic above shows that these four dimension characters form a single Galois orbit, containing the positive character

$$t = \varphi = \frac{1 + \sqrt{5}}{2}, \quad z = \varphi + \sqrt{\varphi^2 + 1}.$$

Conjugating to that character gives pseudo-unitarity.

For *R8_2_489*, work with this positive spherical structure. A homogeneous pairing on a self-dual object V induces one on V^n of sign $\nu_2(V)^n$. Any opposite-indicator self-dual summand consequently has even multiplicity. Since $\tau^2 = 1 + \tau$, we have $\nu_2(\tau) = 1$. The exact congruences

$$x^3 \equiv x + \tau + g\tau \pmod{2}, \quad (g\tau)^3 \equiv g \pmod{2}$$

then imply $\nu_2(x) = \nu_2(g\tau) = \nu_2(g) = 1$. For the positive spherical structure on $\text{Vec}_{C_2}^\omega$, the nontrivial associator class has indicator -1 on g . Hence ω is trivial, and $1 + g$ is its regular separable algebra. This algebra also exists before Galois conjugation. No half-braiding on g is inferred.

The source-bound replay `verify_fib_pair_theoretical.py` checks both complete multiplication tensors, the duality and cube identities, irreducibility modulo three, and the elimination polynomial. The proof does not require an enumeration of signed tensors. $\square$

Theorem 19.5. *Neither *R8_2_190* nor *R8_2_489* admits a complex fusion-category categorification.*

Proof. By Proposition 19.4, it suffices to exclude pseudo-unitary categorifications. Retain the common presentation above, put $y = \tau x$, and write

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad d = \varphi + \sqrt{\varphi + 2}, \quad D = 2(\varphi + 2)(1 + d^2).$$

The four dimension coordinates $1, \varphi, d, \varphi d$ are rationally independent: the norm of $\varphi + 2$ from $\mathbb{Q}(\varphi)$ to $\mathbb{Q}$ is the nonsquare 5, so d has degree two over $\mathbb{Q}(\varphi)$.

For a character χ , let Z_χ denote its central simple summand of $I(1)$ and let $\beta_\chi = D/f_\chi$. The character correspondence preceding [74, Theorem 2.13], together with its Example 2.9, gives $\dim Z_\chi = \beta_\chi$, $[FZ_\chi : 1] = 1$, and

$$S_{Z_\chi, W} = \beta_\chi \chi(FW)$$

for the unnormalized modular matrix of the center. Symmetry therefore yields

$$\beta_\chi \chi(FZ_\psi) = \beta_\psi \psi(FZ_\chi). \tag{25}$$

Take $\omega = 1$ for $\epsilon = 0$ and $\omega = i$ for $\epsilon = 1$. The two characters

$$\chi_\pm(g) = -1, \quad \chi_\pm(\tau) = \varphi, \quad \chi_\pm(x) = \pm\omega$$

have formal codegree $4(\varphi + 2)$ and central dimension $\beta_\pm = 1 + \varphi d$. Formal codegrees here use $\chi(b)\chi(b^*)$, which also gives this value when $x^* = gx$ and $\omega = i$. Their central restrictions consequently satisfy

$$FZ_\pm \in \{1 + y, 1 + gy\},$$

by the independence of the four dimension coordinates and unit multiplicity one.

Now consider the character

$$\psi(g) = 1, \quad \psi(\tau) = 1 - \varphi, \quad \psi(x) = z = (\varphi - 1)d - \varphi.$$

The relations give $z^2 = 1 + 2(1 - \varphi)z$, and direct exact reduction gives

$$f_\psi = 2(3 - \varphi)(1 + z^2), \quad \beta_\psi = (1 + \varphi) + (2 + 3\varphi)d.$$

Write $B = Z_\psi$. Because $\psi(g) = 1$, the values $\psi(FZ_+)$ and $\psi(FZ_-)$ coincide for every permitted choice of the two restrictions. Applying (25) twice and using $\beta_+ = \beta_-$ gives $\chi_+(FB) = \chi_-(FB)$.

The dimension of B requires total multiplicity two for x, gx in FB and total multiplicity three for y, gy . If Δ_x, Δ_y are the respective integer multiplicity differences, subtracting these two character evaluations gives

$$0 = 2\omega(\Delta_x + \varphi\Delta_y).$$

Irrationality of φ forces both differences to vanish. The two equal multiplicities of y and gy cannot sum to three, which is the required contradiction.

The source-bound replay `verify_fib_character_parity.py` checks all 512 tensor entries for each presentation, the three characters, the formal codegrees, and the exact dimension identities. The obstruction itself requires no induction-column enumeration. $\square$

19.2. Cyclic induction phase multiplicities exclude $R8_1_9$.

Lemma 19.6. *Let a be an invertible object of order n in a pseudo-unitary fusion category of dimension D . Every phase occurring in $I(a)$ is an n th root of $\nu_n(a)$, and the total dimension in each of these n phases, counted with its induction multiplicity, is D/n .*

Proof. The rotation spaces $\text{Hom}(1, a^m)$ vanish for $0 < m < n$, so $\nu_m(a) = 0$. For $m = n$ that space is one-dimensional, and $\eta = \nu_n(a)$ is a root of unity. The positive weighted trace $\text{Tr}(\theta^n I(a)) = D\eta$ has maximal absolute value $D = \dim I(a)$, so every constituent has $\theta^n = \eta$. Choose $\lambda^n = \eta$ and write W_j for the weight of phase $\lambda\zeta_n^j$. The zeroth and first $n-1$ trace identities are $\sum_j W_j = D$ and $\sum_j W_j \lambda^m \zeta_n^{jm} = 0$ for $1 \leq m < n$. Discrete Fourier inversion gives $W_j = D/n$. $\square$

Theorem 19.7. *The ring $R8_1_9$ has no complex fusion categorification.*

Proof. By [25, Lemma 3.1], take a pseudo-unitary Galois conjugate and put $d = 2 + \sqrt{5}$, $D = 8 + 16d$. The odd-cube pivotal identities force the second indicators, in source order, to be $(1, 0, 0, 0, 1, 1, 1)$. The pointed generator a has source label1, with a^{-1} of label2 and a^2 of label7. Its associator class $k \in \mathbb{Z}/4$ satisfies $\nu_2(a^2) = (-1)^k$, so $k = 0$ or 2 . The canonical pointed indicators give $\nu_4(a) = \nu_4(a^{-1}) = (-1)^{k/2}$.

The pointed induction Gram block is $8I_4$. Hence a central row v occurring over a has no other pointed coordinate, and its dimension has the form $v_a + Bd$, with $B \geq 0$. Lemma 19.6 gives weight $D/4 = 2 + 4d$ in each actual phase. Comparing constant coefficients therefore gives $\sum_{Z \text{ in one phase}} v_a(Z)^2 = 2$. Thus $v_1, v_2 \leq 1$ for every central row. This argument is uniform in the pointed associator class and does not average away the two imaginary phases.

The complete formal codegrees are $D, D', 8, 8, 8, 8, 8, 8$; the six other characters lie in $\mathbb{Q}(i)$. Actual character-labelled Galois symmetry transports a forgetful row by

$$T(v) = v + (0, 0, 0, 1, 1, 1, 1, 0) \left(v_0 + v_1 + v_2 + v_7 - \frac{v_3 + v_4 + v_5 + v_6}{2} \right),$$

up to the sign making it nonnegative. Partner rows have equal normalized relative moments, because their twists are related by σ^2 and σ^2 fixes $\mathbb{Q}(\sqrt{5})$. The two forced $I(1)$ rows and the six remaining character-labelled rows leave exactly eight possible low frames.

For arbitrary roots of unity, the first two relative traces lie in a fixed 17-vertex hull. It consists of the nine pairs from orders 1, 2, 4, 5, 10, 20, together with the needed negative-half endpoints: an extra odd factor coprime to 10 scales both moments by $\mu(m)/\varphi(m) \in [-1/2, 1]$. No root-order bound or prime-support assumption is used. The full elementary Gauss-sum proof is supplied in the evidence packet.

For $k = 0$, constituents over labels1 and2 have $\theta^4 = 1$; for $k = 2$ they have $\theta^4 = -1$ and their first two relative traces vanish. Those over label7 have $\theta^2 = 1$. Use these phase restrictions, the exact dimension-integrality condition $D/\dim(Z)$ integral, Galois pairing, and the residual Gram bound for the entire row orbit. With the Fourier bound, class zero leaves 64 row orbits and 405 moment

columns per frame. The original, weaker bounds suffice for class two and leave 92 orbits and 389 columns.

The Gram and two induction trace identities have 56 integer coordinates. The two covectors in `r9_center_trace_certificate.json` have nonnegative pairing with every column, but target pairings -64 in class zero and -6432 in class two, for all eight low frames. Their standalone standard-library verifier reconstructs every bounded domain and every positive semidefinite condition with exact rational arithmetic. Neither target is a nonnegative combination of its columns, excluding both possible pointed classes. The initial Galois reduction proves the general complex assertion. $\square$

19.3. The full modular identity and a fourth-moment projection.

Lemma 19.8. *Let $\mathcal{C}$ be a pseudo-unitary fusion category with commutative Grothendieck ring, and let b_χ be the forgetful vector of the actual summand $A_\chi \subset I(1)$ corresponding to the character χ . For a simple $Z \in Z(\mathcal{C})$ write v_Z for its forgetful vector. Then*

$$\sum_Z \theta_Z(v_Z)_i (v_Z)_j = \sum_\chi (b_\chi)_i \overline{\chi(X_j)}. \quad (26)$$

Proof. Let $C_{\chi i} = \chi(X_i)$, let f_χ be the formal codegrees, and put $X = \text{diag}(f_\chi^{-1})C$. The character identification in [74, Example 2.9 and Theorem 2.13] gives $S_{A_\chi, Z} = \chi(FZ)/f_\chi$ for every central simple Z , with $S_{1Z} = \dim(Z)/\dim(\mathcal{C})$. All A_χ have twist one, and the center has trivial anomaly. Thus the low block of $STS = T^{-1}ST^{-1}$ is

$$X \left(\sum_Z \theta_Z v_Z v_Z^\dagger \right) X^\dagger = XB^\dagger,$$

where B has rows b_χ . Character orthogonality gives $C\overline{C}^\dagger = \text{diag}(f_\chi)$, so $X^{-\dagger} = \overline{C}$. Cancellation proves the identity. $\square$

Theorem 19.9. *The ring $R8_1_18$ has no complex fusion categorification.*

Proof. By [25, Lemma 3.1], pass to a pseudo-unitary Galois conjugate. Put $d = 2 + \sqrt{5}$, $K = \mathbb{Q}(\sqrt{5})$ and $D = 8 + 16d$. In model order $(1, a, a^2, a^3, \rho, a\rho, a^2\rho, a^3\rho)$, mapped to source $(0, 1, 7, 2, 3, 6, 5, 4)$, the ring is commutative and satisfies

$$a^4 = 1, \quad \rho^2 = a + (1 + a + a^2 + a^3)\rho.$$

Its six non-FP characters have formal codegree eight and are

$$\chi_{k,\pm}(a) = \zeta_8^{2k}, \quad \chi_{k,\pm}(\rho) = \pm \zeta_8^k, \quad k = 1, 2, 3.$$

The other two codegrees are D, D' . The two forced low rows are 1 and $1 + X_3 + X_4 + X_5 + X_6$; the other six have form $1 +$ two nonpointed constituents. Actual character reciprocity and duality leave eight low assignments. Taking normalized relative trace t_K in (26) gives, for nonunit indices,

$$\sum_Z (v_Z)_i (v_Z)_j t_K(\theta_Z) = \begin{cases} 4 - d & i, j \in \{3, 4, 5, 6\}, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed every nonpointed value of the six $\mathbb{Q}(\zeta_8)$ characters has relative trace zero; the pointed entries cancel in each of the eight assignments.

The pointed induction Gram is $8I_4$. Lemma 19.6 therefore gives $(v_Z)_1, (v_Z)_2 \leq 1$ exactly as in the preceding theorem, uniformly over all four pointed associator classes. Besides the quadratic transport T displayed there, full Galois symmetry acts on rows by the source permutation

$$P = (0, 2, 1, 6, 5, 4, 3, 7).$$

It is induced by an automorphism fixing K and sending ζ_8 to ζ_8^3 : on the model basis it sends $a \mapsto a^{-1}$ and $a^j \rho \mapsto a^{3j+1} \rho$. Include also source duality $(0, 2, 1, 4, 3, 6, 5, 7)$. Every full row orbit

has a common relative moment histogram: Galois transport changes a twist by σ^2 , which fixes quadratic K , and duality preserves twists.

The Cauchy theorem for spherical fusion categories [61, Theorem 3.9] applies to the nonintegral dimension D . Its norm is $320 = 2^6 \cdot 5$, so the exponent has only primes two and five. For powers 1, 2, 4, every normalized relative twist triple is one of the triples from fortieth roots or the zero triple. This follows directly from the quadratic Gauss sums: fifth-adic exponent at least two, or 2-adic exponent at least four, gives zero in all three coordinates. There are exactly thirteen triples; this is not an assertion that the exponent divides forty.

There is no need to determine fourth indicators. Each $\nu_4(X_i)$ is a sum of fourth roots, so $t_K(\nu_4(X_i)) \in \mathbb{Z}$. The functional $\ell(a + bd) = 2a - b$ annihilates $D\mathbb{Q}$. Applying it to the fourth induction identity gives

$$\ell\left(\sum_Z (v_Z)_i \dim(Z) t_K(\theta_Z^4)\right) = 0.$$

Together with the first two induction traces, the full modular identity above, and the Gram identity, this gives 119 rational linear coordinates after low terms are subtracted.

The exact domain uses the diagonal Gram bounds, integrality of $D/\dim(Z)$, the Fourier multiplicity bound, full row-orbit closure, and positive semidefiniteness after subtracting that entire orbit. It has 25 row orbits for every low assignment. The pointed classes $k = 0$, $k = 1, 3$ and $k = 2$ impose their respective cyclic twist constraints and leave 167, 145 and 153 columns. The three integer covectors in `r18_sts_moment_certificate.json` pair nonnegatively with every column and pair to -256 with every corresponding target, for all eight low assignments. The standard-library verifier reconstructs the source, characters, thirteen twist triples and all bounded domains using exact rational arithmetic. Thus none of the four pointed classes is possible. The initial Galois reduction proves the general complex assertion. $\square$

20. AN INNER-SQUARE OBSTRUCTION FOR $R8_2_24$

The obstruction in this section combines a local pivotal calculation with rigidity of tensor functors. It applies to an arbitrary complex categorification after a proved Galois reduction to pseudo-unitarity.

Theorem 20.1. *The fusion ring $R8_2_24$ has no complex fusion categorification.*

20.1. The neutral indicators and the actual inner square.

Lemma 20.2. *Let $\mathcal{B}$ be pseudo-unitary with simple objects $1, g, r, s = gr$ and fusion rules*

$$g^2 = 1, \quad gr = rg = s, \quad r^2 = 1 + 2r + 2s. \quad (27)$$

For its canonical spherical structure, $\nu_2(g) = -1$ and $\nu_2(r) = \nu_2(s) = 1$.

Proof. Put $d = 2 + \sqrt{5}$ and $D = 2 + 2d^2 = 20 + 8\sqrt{5}$, and let a prime mean the nontrivial embedding of $\mathbb{Q}(\sqrt{5})$. The four formal codegrees are $D, D', 4, 4$. The corresponding constituents of $I(1)$ have dimensions $1, D/D', D/4, D/4$ and twist one. Use the normalized anomaly-free center modular matrices S, T , with $S_{1A} = \dim(A)/D$. In the span of these constituents let w be the unit vector proportional to the difference of the first two basis vectors. The modular-character formula for central induction gives

$$\text{proj}_{I(1)} Sw = \left(\frac{1}{D} - \frac{1}{D'}\right) w = -\frac{w}{\sqrt{5}}.$$

The diagonal torus acts on w by the quadratic character χ_5 . Indeed its Galois action interchanges the first two constituents precisely at the nonresidues modulo five; their signs are positive since $D, D' > 0$. The local compression bound [63, Lemma 4.1] is therefore attained at conductor five.

We recall why equality supplies the needed level statement. The congruence theorem [60] makes the modular representation a representation of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Decompose it into local irreducible

tensor factors. At the prime five, an irreducible factor with an upper-unipotent-fixed vector embeds in the permutation representation on primitive vectors modulo 5^a . In the quadratic torus eigenspace, the maximal singular subspace of compression is spanned by

$$F(p, q) = \chi_5(q)[q \text{ is a unit}], \quad G(p, q) = \chi_5(-p)[q = 0 \pmod{5}].$$

These are the equality vectors in the proof of the cited local bound; both are pullbacks from level five. Any irreducible constituent containing an equality vector consequently factors through level five. At every other prime the torus character is trivial. Equality in the norm bound one makes the vector fixed by both upper and lower unipotents, hence by the whole local group. The maximal singular subspace of a tensor product of contractions is the tensor product of their maximal singular subspaces, by diagonalizing A^*A in each factor. Thus the entire cyclic subrepresentation generated by w factors through level five. The known twist-one projection already attains the norm bound, so no additional twist-one coordinate occurs. This is the same equality argument as in Lemma 21.2.

For a central simple A with dimension z_A , its coordinate in Sw is proportional to

$$\frac{z_A}{D} - \left(\frac{z_A}{D}\right)'.$$

Consequently $\theta_A^5 = 1$ whenever this difference is nonzero. Let $\ell(z)$ be the rational coordinate of the nontrivial quadratic part of z/D , scaled so that its values on $(1, g, r, s)$ are $(2, 2, -1, -1)$. Choose a cyclotomic automorphism σ with unit exponent k modulo five, lifting that exponent by the Chinese remainder theorem to be coprime to the full conductor. Write π and ε_A for its modular permutation and signs. The canonical anomaly-free center normalization gives

$$z_{\pi A} = \varepsilon_A \frac{D}{\sigma D} \sigma(z_A), \quad \ell(z_{\pi A}) = \varepsilon_A \chi_5(k) \ell(z_A), \quad \theta_{\pi A} = \sigma^2(\theta_A).$$

The first induction trace, weighted by ℓ , is

$$\sum_A \ell(z_A) z_A \theta_A = D \ell(1).$$

The two Galois signs cancel in

$$\sigma(\ell(z_A) z_A) = \chi_5(k) \frac{\sigma D}{D} \ell(z_{\pi A}) z_{\pi A}.$$

Apply σ and reindex by π . Every nonzero term has twist dividing five; the constituents of $I(1)$ have twist one. If h is an inverse of k modulo five, then $\sigma(\theta_A) = \theta_{\pi A}^h$. Cancelling σD and using the induction indicator formula [59, Theorem 4.1] gives

$$\sum_X \ell(\dim X) \nu_h(X) = \chi_5(k) \ell(1).$$

Inversion permutes the unit residues and preserves χ_5 , so this holds with h renamed k . At $k = 2$ it becomes

$$2 + 2\nu_2(g) - \nu_2(r) - \nu_2(s) = -2.$$

The three unknowns are signs, and the asserted values are the unique solution. $\square$

Proposition 20.3 (Inner square supplied by the source tensor). *A complex categorification of $R8_2_24$ has a pseudo-unitary Galois conjugate containing a neutral subcategory $\mathcal{B}$ with (27) and the indicators of Lemma 20.2. It supplies a label-fixing tensor autoequivalence T of $\mathcal{B}$ satisfying*

$$T^2 \simeq_{\otimes} \text{Ad}(g). \quad (28)$$

Proof. The full source tensor has exactly two Frobenius–Perron dimensions, 1 and $2 + \sqrt{5}$. The published Galois-conjugacy statement [25, Lemma 3.1] applies to the entire category. Its pseudo-unitary conjugate has the same tensor and the faithful C_2 grading with neutral basis $(1, g, r, s) = (X_0, X_7, X_5, X_6)$. The odd invertible $a = X_1$ has inverse X_2 and square $g = X_7$. Commutativity

of the source tensor makes the actual inner tensor functor $T = \text{Ad}(a)|_{\mathcal{B}}$ fix all four neutral labels. Coherent composition of inner functors and the isomorphism $a^2 \simeq g$ give (28); no trivialization of the pointed associator is required. The neutral category inherits the canonical pseudo-unitary spherical structure, so the preceding lemma applies. $\square$

20.2. The local pivotal defect.

Theorem 20.4 (No label-fixing inner square root). *For a pseudo-unitary category $\mathcal{B}$ with (27) and $\nu_2(g) = -1$, $\nu_2(r) = \nu_2(s) = 1$, no label-fixing tensor autoequivalence satisfies (28).*

We prove the theorem in the next four steps. We use a strict monoidal rigid realization while retaining its pivotal map j ; we do not set j equal to the identity.

Let Z and gZ have indicator $+1$, and choose right dual objects $g^* = g$, $Z^* = Z$ with identity self-duality charts. Thus $j_g = -\text{id}_g$ and $j_Z = \text{id}_Z$. Write $e = \text{ev}_g : g^2 \rightarrow 1$. Pseudo-unitarity gives $\dim(g) = 1$, so the pivotal trace identity $\dim(g) = -e \text{coev}_g = 1$ fixes the coevaluation sign. The snake identities and dualization give

$$\text{coev}_g = -e^{-1}, \quad e \otimes g = -g \otimes e, \quad e^* = -e^{-1}. \quad (29)$$

An object written as a tensor factor in a morphism denotes its identity. Use $F(X) = gXg$ with tensorator and unit

$$F_2(X, Y) = -gX \otimes e \otimes Yg, \quad F_0 = e^{-1}.$$

This represents ordinary inner conjugation: its usual tensorator and unit in these dual conventions are $+gX \otimes e \otimes Yg$ and $-e^{-1}$, and the natural isomorphism with every component $-\text{id}$ changes both signs. The monoidal comparison of its square with the identity is

$$\eta_X = -e \otimes X \otimes e : ggXgg \longrightarrow X. \quad (30)$$

Indeed the adjacent contraction without the minus sign sends the unit of F^2 to -1 and has tensor-product defect -1 , by (29). Its negative satisfies both monoidal equations. The canonical dual comparison is

$$\delta_Z = -\text{id}_{gZg} : F(Z^*) \longrightarrow F(Z)^*. \quad (31)$$

To check the sign, the transported evaluation is

$$F_0^{-1}F(\text{ev}_Z)F_2(Z^*, Z) = -e(g \otimes \text{ev}_Z \otimes g)(gZ \otimes e \otimes Zg),$$

which is the negative of the ordinary nested evaluation of gZg .

Choose a nonzero $c : gZ \rightarrow Zg$. For arbitrary self-dualities ϕ_g, ϕ_Z , the self-duality $\psi = (\phi_Z \otimes \phi_g)c$ of gZ satisfies $\psi^* j_{gZ} = \nu_2(gZ)\psi$. Expanding this equation gives

$$(\phi_Z^{-1} \otimes \phi_g^{-1})c^*(\phi_g \otimes \phi_Z) = \frac{\nu_2(gZ)}{\nu_2(g)\nu_2(Z)}c = -c.$$

Thus $c^* = \epsilon c$, with $\epsilon = -1$, in our identity charts. Define

$$q = (Z \otimes e)(c \otimes g) : gZg \rightarrow Z$$

and the nonzero scalar κ by

$$(Z \otimes e)(c \otimes g)(g \otimes c) = \kappa(e \otimes Z) : ggZ \rightarrow Z. \quad (32)$$

Every map is invertible. Dualizing q yields

$$q^* = -\epsilon(g \otimes c)(e^{-1} \otimes Z), \quad qq^* = -\epsilon\kappa \text{id}_Z.$$

Define its form defect γ and composition defect λ by

$$q^*\phi_Zq = \gamma\delta_ZF(\phi_Z), \quad qF(q) = \lambda\eta_Z.$$

The first equation and (31) give $\gamma = \epsilon\kappa$. For the second, the following are typed maps $ggZgg \rightarrow Z$, with tensor symbols suppressed:

$$\begin{aligned} qF(q) &= (Ze)(cg)(gZeg)(gcgg) \\ &= (Ze)(Zgeg)(cggg)(gcgg) \\ &= -(Zee)(cggg)(gcgg) \\ &= -\kappa(eZe) = \kappa\eta_Z. \end{aligned}$$

The second line uses interchange, the third uses $e(geg) = -(e \otimes e)$, and the fourth uses (32). Consequently

$$\lambda = \kappa, \quad \frac{\lambda}{\gamma} = -1. \quad (33)$$

Rescaling q multiplies both defects by the same square.

Apply this calculation to $Z = r$. On $H = \text{Hom}(r^2, r)$, use full categorical transpose from the symmetric self-duality of r and set $\beta(f, h) = \text{tr}_r(fh^\dagger)$. The self-duality on r^2 is the tensor product of those on its two identical factors, with reversed dual order. Transpose is involutive; transposing the scalar composite proves that β is symmetric. Semisimplicity makes the composition pairing nondegenerate, and hence β is nondegenerate. These are bilinear forms over $\mathbb{C}$. The induced action

$$K(f) = qF(f)F_2(r, r)(q^{-1} \otimes q^{-1})$$

satisfies

$$K^2 = \lambda^{-1} \text{id}_H, \quad \beta(Kf, Kh) = \gamma^{-1} \beta(f, h) = -\lambda^{-1} \beta(f, h). \quad (34)$$

For the square, compose the monoidal comparison (30) with the output chart and two input charts. For the form, the output chart contributes γ and the inputs contribute γ^{-2} . A scalar K would have equal square and form multipliers, contrary to (34). Since $\dim H = 2$, it has two distinct opposite eigenvalues. Each eigenline is isotropic, and nondegeneracy pairs the two lines. Thus $[K]$ is a nontrivial projective involution.

20.3. One simultaneous chart for the square root. Suppose $T^2 \simeq_\otimes F$. A tensor autoequivalence preserves the canonical pseudo-unitary pivotal structure, by its uniqueness [21, Proposition 8.23]. Fix the unit chart by T_0^{-1} and choose charts $q_g : T(g) \rightarrow g$, $q_r : T(r) \rightarrow r$. Extend them to strict tensor words by

$$q_{uv} = (q_u \otimes q_v)T_2(u, v)^{-1}, \quad \hat{T}(f) = q_v T(f) q_u^{-1} \quad (f : u \rightarrow v).$$

Coherence makes this transported action strictly multiplicative on words. Rescale the two simple charts independently to make $\hat{T}(e) = e$ and $\hat{T}(\text{ev}_r) = \text{ev}_r$; each condition requires only a nonzero complex square root. Use the induced chart on $s = gr$. Distinct isomorphic words remain connected by the displayed morphisms, rather than by further identifications. We suppress the hat in the following calculations.

The one-dimensional crossing space has $T(c) = \alpha c$. Applying T to (32) gives $\alpha^2 \kappa = \kappa$, hence $\alpha = \pm 1$. Moreover $T(q) = \alpha q$, while T fixes the tensorator of F and commutes with its word operation. The explicit formula for K therefore implies the exact covariance

$$T_H K T_H^{-1} = \alpha^{-1} K. \quad (35)$$

The exponent is one output chart minus two input charts. A monoidal natural isomorphism $T^2 \Rightarrow F$ acts by a scalar on the simple r ; its induced change on H is therefore scalar. Thus $[T_H^2] = [K]$. If $\alpha = -1$, equation (35) makes T_H exchange the two K -eigenlines. Its square would be scalar, a contradiction. Hence $\alpha = 1$.

All composites of $e, c, c^{-1}, \text{ev}_r$ and their cups are now fixed in this same chart. In particular the nonzero evaluation

$$\text{ev}_s = (e \otimes \text{ev}_r)(g \otimes c^{-1} \otimes r) : grgr \rightarrow 1 \quad (36)$$

is fixed. Its associated self-duality is symmetric since $\nu_2(s) = 1$. For $X = r, s$ the full-transpose form

$$B_X(f, h) = \text{tr}_X(fh^\dagger), \quad f, h \in \text{Hom}(r^2, X),$$

is therefore nondegenerate, symmetric and preserved exactly by T . Putting $J = \text{Hom}(r^2, s)$, we obtain $T_H, T_J \in O(2, \mathbb{C})$ in their respective forms. A determinant-minus orthogonal matrix in dimension two squares to the identity. Thus $\det T_H = 1$, and in a hyperbolic basis

$$T_H = \text{diag}(u, u^{-1}), \quad u^4 = -1. \quad (37)$$

The latter equation expresses the eigenvalue ratio -1 of $[T_H^2] = [K]$. For J , either $\det T_J = 1$ and $T_J = \text{diag}(v, v^{-1})$ in a hyperbolic basis, or $\det T_J = -1$ and $T_J = \text{diag}(1, -1)$ in an eigenbasis.

20.4. All fusion spaces and all associator weights. For $a \in \mathbb{F}_2$ write $g_a = g^a$ and $r_a = g^a r$. The complete fusion rules are

$$g_a g_b = g_{a+b}, \quad g_a r_b = r_b g_a = r_{a+b}, \quad r_a r_b = g_{a+b} + 2r_{a+b} + 2r_{a+b+1}.$$

Every multiplicity-one vertex has a basis built from the fixed structural maps. For the two-dimensional spaces choose $A_{ab} : r_a r_b \rightarrow g_{a+b} r^2$ by

$$A_{00} = A_{10} = \text{id}, \quad A_{01} = c^{-1} \otimes r, \quad A_{11} = (e \otimes r^2)(g \otimes c^{-1} \otimes r).$$

For $m = a + b$ let $B_{abc} : r_c \rightarrow g_m r_{m+c}$ be the identity if $m = 0$ or $c = 1$, and $e^{-1} \otimes r$ if $m = 1, c = 0$. For $f : r^2 \rightarrow r_{a+b+c}$ transport the vertex by

$$f \longmapsto B_{abc}^{-1}(g_m \otimes f)A_{ab} \in \text{Hom}(r_a r_b, r_c). \quad (38)$$

Every structural transport map in this expression is fixed by T ; f itself carries the H or J action. These vertices therefore have exactly the H action for $a + b + c = 0$ and the J action for $a + b + c = 1$, without additional scalar characters. The thin vertex $r_a r_b \rightarrow g_m$ is $(g_m \otimes \text{ev}_r)A_{ab}$. The other thin vertices use e, c and unit maps. Composition-dual incoming bases have inverse eigenvalues. One chosen transport for each space suffices; equality of alternative transport paths is not required.

First suppose $\det T_J = 1$. Give the two H basis vectors weights $\pm(1, 0)$, the two J vectors weights $\pm(0, 1)$, and thin vertices weight $(0, 0)$. These define a diagonal gauge torus $U(z, w)$. Every associator coefficient has the signed sum of its four vertex weights, and every such weight belongs to

$$W = \{(p, q) \in (2\mathbb{Z})^2 : |p| + |q| \leq 4\}. \quad (39)$$

To prove this uniformly, use the six-edge, four-vertex incidence pattern of a full associator coefficient. A vertex has zero, two or three noninvertible incident edges, and it is thick precisely in the last case. Counting each marked edge twice makes the number of thick vertices even. Assign bit a to both g_a and r_a . The incident bit sum is zero at a thin or H vertex, and one at a J vertex. Counting each edge twice makes the number of J vertices even; the number of H vertices is then even as well. This proves evenness of both coordinates, and four vertices give the norm bound.

Let $L \subset \mathbb{Z}^2$ be generated by the weights of actual nonzero associator coefficients. Monoidality of T implies $u^p v^q = 1$ for every $(p, q) \in L$. If L has rank two, then

$$(4, 0), (0, 4) \in L. \quad (40)$$

Indeed choose two independent vectors from W and divide them by two. The possibilities are coordinate axes of length one or two and diagonals $(\pm 1, \pm 1)$. Two independent axes, an axis and a diagonal, or two independent diagonals each generate a lattice containing the two doubled coordinate axes. Multiplying back by two proves (40). It would force $u^4 = 1$, contradicting (37). Hence L has rank below two, and its annihilator in $(\mathbb{C}^\times)^2$ contains a positive-dimensional torus fixing every associator coefficient.

If $\det T_J = -1$, the J eigenvalues are $1, -1$. Track only the H exponent p , still even and with $|p| \leq 4$. At $p = \pm 2$, the H factor is $\pm i$, which no J sign can cancel. At $p = \pm 4$, all four vertices belong to H , so there is no J factor and the coefficient has multiplier -1 . All these coefficients

vanish. Every nonzero coefficient therefore has $p = 0$. The torus acting by $\text{diag}(t, t^{-1})$ on all transported H spaces and identically elsewhere fixes the full associator.

20.5. Tensor-functor rigidity and the exclusion. In either case a positive-dimensional diagonal gauge torus fixes all associator coefficients and all unit channels. In a skeletal semisimple realization, its matrices on the fusion spaces define tensorators for the identity underlying linear functor: extend them naturally from pairs of simple objects to direct sums. Equality of the two composites of these tensorators on each triple is exactly invariance of the full associator matrix. The fixed unit channels supply both unit equations. Thus every element of this torus is an actual tensor autoequivalence.

These functors have infinitely many monoidal natural-isomorphism classes. A natural isomorphism has scalar components on simple objects, so its change on each of H and J is scalar. It cannot remove a varying projective eigenvalue ratio. More explicitly, on the two-dimensional torus the parameters with scalar action on both spaces satisfy $z^2 = w^2 = 1$ and form a finite subgroup; the one-dimensional case has the same conclusion from $t^2 = 1$. A positive-dimensional torus remains infinite after this quotient. This contradicts the finiteness of tensor functors between fixed fusion categories [21, Theorem 2.31], and proves Theorem 20.4.

Proof of Theorem 20.1. Proposition 20.3 would supply precisely the pseudo-unitary neutral category and the tensor autoequivalence forbidden by Theorem 20.4. Therefore a complex categorification cannot exist. $\square$

The source tensor has SHA-256

2b4780441d9590e8e7269cd82b7675aa75cec11c39bd3e94bb372cca1ada6738. The source-bound evidence is in `evidence/rigidity_20261017/rank8/`. The verifier is `tools/verify_rigidity_20261017.py`. The finite checks verify the neutral tensor, the actual invertible and its conjugation permutation, and all 136 nonempty associator blocks with 1472 coefficient slots. They also check the 104 ordered independent weight pairs and 5888 reflection-case phases. These checks certify the specified finite identities; the categorical exclusion is the argument above.

21. A CONDUCTOR-SEVENTEEN INDICATOR BALANCE

Theorem 21.1. *The five rings*

$$R8_2_33, \quad R8_2_36, \quad R8_2_76, \quad R8_2_412, \quad R8_2_446$$

have no complex fusion categorification.

The same short indicator identity excludes all five rings, including the noncommutative cases. No enumeration of possible center rows or associator equations is needed.

Lemma 21.2. *Let S, T be the normalized unitary modular representation of a pseudo-unitary Drinfeld center, and let P project onto the T -fixed space. Suppose that a T -fixed vector w has diagonal-torus character the primitive quadratic character modulo 17. If a known subspace L of the T -fixed space contains w and*

$$\text{proj}_L Sw = -\frac{w}{\sqrt{17}},$$

then $PSw = -w/\sqrt{17}$ and the cyclic representation generated by w factors through level 17.

Proof. The compression bound in [63, Lemma 4.1] is already attained on L , so the full projection cannot have any additional coordinate. For completeness, the equality case also gives the asserted level bound. By the congruence theorem [60], decompose the finite representation into local irreducible tensor factors. At 17, an irreducible representation with an upper-unipotent-fixed vector embeds in the primitive-vector permutation representation over $\mathbb{Z}/17^a\mathbb{Z}$. In the quadratic torus eigenspace, the maximal singular subspace of the compression is spanned by the functions

$$F(p, q) = \chi_{17}(q)[q \text{ is a unit}], \quad G(p, q) = \chi_{17}(-p)[q = 0 \pmod{17}].$$

This is the equality subspace in the proof of the cited local bound. Both functions are pullbacks from level 17. An irreducible constituent containing an equality vector therefore factors through that level. At every other prime the character is trivial; equality in the bound one makes the vector fixed by both upper and lower unipotents, and hence by the entire local group. In a tensor product of contractions, the maximal singular subspace is the tensor product of the local maximal singular subspaces, as follows by diagonalizing the positive operators A^*A . Thus all constituents supporting w factor through level 17. The full equality argument is also recorded in the evidence note `compression/CONDUCTOR_EQUALITY_LEMMA.md`. $\square$

Lemma 21.3 (Saturated Galois indicator balance). *Let $\mathcal{C}$ be pseudo-unitary, with simple dimensions in a real quadratic field K of conductor f and irrational global dimension D . Suppose every center simple A outside $I(1)$ with $\dim(A)/D \notin \mathbb{Q}$ satisfies $\theta_A^f = 1$. Let $\ell : K \rightarrow \mathbb{Q}$ be a nonzero rational linear map with $\ell(D) = 0$. If k is coprime to f and $\zeta_f \mapsto \zeta_f^k$ fixes K , then*

$$\sum_{i \neq 0} \ell(\dim X_i) \nu_k(X_i) = 0. \quad (41)$$

Proof. Choose a common cyclotomic twist field $\mathbb{Q}(\zeta_N)$, with $f \mid N$. The Chinese remainder theorem gives an exponent congruent to k modulo f and coprime to N : at each prime dividing N but not f , impose the additional congruence 1. The corresponding Galois automorphism τ fixes K and raises f th roots to their k th powers.

For $z_A = \dim A$, induction reciprocity and the first trace identity [74, Theorem 2.5] give

$$\sum_A \ell(z_A) z_A \theta_A = D \ell(1).$$

All coefficients belong to K . The terms with $z_A/D \in \mathbb{Q}$ vanish since $\ell(z_A) = 0$. In every other term outside $I(1)$ the twist is an f th root by hypothesis; inside $I(1)$ it is one. Applying τ therefore replaces every remaining θ_A by θ_A^k , while fixing the coefficients and right side. The general induction indicator formula [59, Theorem 4.1] now gives

$$D \sum_i \ell(\dim X_i) \nu_k(X_i) = D \ell(1).$$

Cancel D and the unit term. The argument proves an exact complex identity, without taking real parts or relative traces, and does not assume that the source ring is commutative. $\square$

Proof of Theorem 21.1. Every source ring has four invertibles, forming a group G of order four, and four noninvertibles $\mathcal{O}$, all of dimension

$$d = 4 + \sqrt{17}, \quad d^2 = 1 + 8d, \quad D = 4 + 4d^2 = 8 + 32d = 136 + 32\sqrt{17}.$$

By [25, Lemma 3.1], any complex categorification is Galois conjugate to a pseudo-unitary one. This two-FP-dimension theorem does not assume commutativity. Work with the conjugate and its canonical positive spherical structure.

The source tensors give

$$\Omega = \sum_X X X^* = 8 \mathbf{1} + 8 \sum_{\rho \in \mathcal{O}} \rho, \quad \det(xI - L_\Omega) = (x - 8)^6 (x^2 - 272x + 1088).$$

Here L_Ω is the Casimir multiplication matrix; in the noncommutative cases we do not identify it with the full induction Gram matrix. If an irreducible ring representation E has dimension e_E , the Casimir eigenvalue is $e_E f_E$ with multiplicity e_E^2 [74, Remark 2.11]. Thus D and $D' = 136 - 32\sqrt{17}$ are the codegrees of one-dimensional representations, and every other ordinary codegree $f_E = 8/e_E$ is rational.

By [74, Theorem 2.13], the corresponding center constituent A_E of $I(1)$ has multiplicity e_E and dimension D/f_E . Let J correspond to D' . Then

$$[I(1) : J] = 1, \quad \dim J = D/D' = d^2.$$

All these constituents have twist one. Also $F(I(1)) = \Omega$ implies that none contains a nonunit invertible in its restriction.

Normalize $S_{00} = 1/D$. The multiplicity vector m of $I(1)$ satisfies $Sm = m$ [74, Proposition 2.8]. Therefore its support is Galois-stable with positive Galois signs. A Galois automorphism fixing $K = \mathbb{Q}(\sqrt{17})$ sends the central unit to the unique constituent of $I(1)$ of dimension one; one flipping K sends it to J , the unique constituent of dimension D/D' . Thus $w = e_1 - e_J$ has quadratic torus character χ_{17} .

The known twist-one block sends w to

$$\left(\frac{1}{D} - \frac{1}{D'}\right)w = -\frac{w}{\sqrt{17}}.$$

Indeed its $1, J$ block has diagonal $1/D$ and off-diagonal $1/D'$, and the other entries in these two columns agree because their codegrees are rational. This remains true when $e_E = 2$. Lemma 21.2 now gives $PSw = -w/\sqrt{17}$ and level 17 on the cyclic representation. For a center simple A of dimension z , the coordinate of Sw is

$$\frac{z}{D} - \sigma\left(\frac{z}{D}\right).$$

If z/D is nonrational and A is outside $I(1)$, its twist is therefore a primitive seventeenth root: it has order dividing 17, and twist one is excluded by the exact formula for PSw . Apply Lemma 21.3 with

$$\ell(a + b\sqrt{17}) = 4a - 17b, \quad k = 2.$$

Indeed $\ell(D) = 0$, and 2 is a quadratic residue modulo 17, so $\zeta_{17} \mapsto \zeta_{17}^2$ fixes K . The source weights are $\ell(1) = 4$ on the invertibles and $\ell(d) = -1$ on the noninvertibles. We obtain the common balance

$$\boxed{\sum_{\rho \in \mathcal{O}} \nu_2(\rho) = 4 \sum_{g \in G \setminus \{1\}} \nu_2(g).} \quad (42)$$

No restrictions on twists of constituents of $I(g)$, no relative moment hull, and no enumeration of center rows are needed.

A group of order four has an odd number of nonunit involutions: one for C_4 , three for C_2^2 . Its nontrivial pointed indicator sum is therefore a nonzero odd integer, so the right side of (42) has absolute value at least four. The left side has absolute value at most the number of self-dual noninvertibles. The exact source values are

ring	G	self-dual noninvertible labels	number
$R8_2_33$	C_4	5, 6	2
$R8_2_36$	C_4	$\emptyset$	0
$R8_2_76$	C_4	$\emptyset$	0
$R8_2_412$	C_2^2	6, 7	2
$R8_2_446$	C_2^2	$\emptyset$	0

Each row contradicts the balance, proving all five exclusions over $\mathbb{C}$. $\square$

The source-bound verifier `verify_q17_indicator_balance.py` checks the original tensors, hashes, associativity, FP dimensions, pointed groups, duality lists, Casimir polynomial, and the displayed arithmetic. It uses only exact integer operations and runs identically under standard and optimized Python. The proof and independent audits are stored in `evidence/unified_20261016/rank8/`, including `Q17_INDICATOR_BALANCE_PROOF.md`.

21.1. A spherical reduction for the two $\sqrt{10}$ cases.

Proposition 21.4. *Every complex categorification of $R8_2_630$ or $R8_2_671$ admits a spherical structure and has a pseudo-unitary Galois conjugate.*

Proof. Both tensors are self-dual. Choose the positive-second-indicator lifts in the canonical sphericalization. These lifts are self-dual, because their duals have the same nonzero spherical dimensions, whereas the other lifts have the opposite dimensions. Their signed quotient tensor T is fully symmetric and associative, with $|T_{ij}^k| \leq N_{ij}^k$ and $T_{ij}^k \equiv N_{ij}^k \pmod{2}$. The odd-channel parity theorem [62, Theorem 3.9] further gives $N_{jj}^i - T_{jj}^i \equiv 0 \pmod{4}$.

Thus repeated coefficients one are positive, repeated coefficients two have signs ± 2 , and all-distinct coefficients one have signs ± 1 . There are 27 Boolean signs for each tensor. Only $R8_2_671$ has an all-distinct coefficient two, with possible values $-2, 0, 2$. The source-bound exact checker regenerates every ordered associativity equation. For $R8_2_630$, exhaustive branching on the 157 distinct equations visits 195 nodes and leaves only $T = N$. For the three values of the extra coefficient of $R8_2_671$, it visits 57, 5, 103 nodes; the only solution is again $T = N$, in the last branch. Every rejection evaluates an integer associativity equation exactly. Consequently the chosen actual lifts are tensor closed, and restriction of the forgetful functor identifies their spherical fusion category with the original category.

The same checker verifies the complete tables of eight distinct characters over $\mathbb{Q}(\sqrt{3}, \sqrt{10})$. The formal codegrees are

$$120 \pm 36\sqrt{10}, \quad 4, 6, 6, 12, 12, 12.$$

All six characters outside the FP conjugate pair vanish on a simple basis object. Since spherical dimensions of simple objects are nonzero, the spherical dimension character belongs to that pair. A Galois automorphism therefore takes it to the positive FP character and produces a pseudo-unitary category. $\square$

The complete character tables, signed domains, source-bound replay, and independent audit are retained in `evidence/unified_20261016/rank8/HI6_PAIR_GALOIS_PU.md`. This proposition is a structural reduction; by itself it makes no existence decision and does not identify either center with a known Haagerup–Izumi center.

22. A THIRD-INDICATOR TRIANGLE OBSTRUCTION AT CONDUCTOR FORTY

Theorem 22.1. *The fusion ring $R8_2_630$ has no complex categorification.*

Proof. Proposition 21.4 reduces a hypothetical complex categorification to a pseudo-unitary Galois conjugate. Work with its canonical positive spherical structure. Put $s = \sqrt{10}$ and $K = \mathbb{Q}(s)$. In census order, the FP dimensions and global dimension are

$$(d_0, \dots, d_7) = (1, 1, 2 + s, 3 + s, 3 + s, 2 + s, 4 + s, 4 + s), \quad D = 120 + 36s.$$

The complete character table gives formal codegrees

$$D, \quad D' = 120 - 36s, \quad 4, 6, 6, 12, 12, 12.$$

The source ring is commutative, so each constituent A_E of $I(1)$ has multiplicity one and dimension D/f_E [74, Theorem 2.13]. Let J denote the constituent for D' ; then $\dim J = D/D' = 19 + 6s$.

The multiplicity vector of $I(1)$ satisfies $Sm = m$, so its support is Galois-stable with positive signs. The central unit has Galois orbit exactly $\{1, J\}$: these are the unique constituents of $I(1)$ of dimensions 1 and D/D' . Thus $w = e_1 - e_J$ has the primitive quadratic torus character χ_{40} . The known twist-one block satisfies

$$\text{proj}_{I(1)} Sw = \left(\frac{1}{D} - \frac{1}{D'} \right) w = -\frac{s}{20} w = -\frac{w}{\sqrt{40}}.$$

Indeed the $1, J$ submatrix has diagonal $1/D$ and off-diagonal $1/D'$, while all other entries in the two columns agree since their codegrees are rational.

The local compression bound [63, Lemma 4.1] is therefore attained. Its equality refinement, proved in the argument of Lemma 21.2, works at each prime power: for conductor p^c , the equality vectors are pullbacks from level p^c ; trivial local characters force the trivial representation. Applying this to the nontrivial local conductors 8 and 5 gives level 40 for the cyclic representation generated by w . The general equality proof is recorded in `compression/CONDUCTOR_EQUALITY_LEMMA.md`. Since the coordinate of Sw at a center simple of dimension z is

$$\frac{z}{D} - \sigma\left(\frac{z}{D}\right),$$

where σ flips K , every such object with z/D nonrational has $\theta^{40} = 1$.

Apply the same saturated Galois indicator lemma used for the five conductor-seventeen cases, Lemma 21.3, now with

$$\ell(a + bs) = 3a - 10b, \quad f = 40, \quad k = 3.$$

We have $\ell(D) = 0$, and $\chi_{40}(3) = 1$ means that $\zeta_{40} \mapsto \zeta_{40}^3$ fixes K . The source weights are

$$(\ell(d_i))_{i=0}^7 = (3, 3, -4, -1, -1, -4, 2, 2),$$

so the lemma gives

$$\sum_{i \neq 0} \ell(d_i) \nu_3(X_i) = 0. \quad (43)$$

This is an exact complex identity. The first trace is transformed into the third by a field automorphism fixing every coefficient; no reality of the third indicators is assumed.

Finally the source tensor gives

$$(\dim \operatorname{Hom}(1, X_i^3))_{i=0}^7 = (1, 0, 0, 1, 1, 1, 0, 0).$$

The only nonzero nonunit third indicators are consequently those at labels 3, 4, 5. Each is the trace of an order-three rotation on a one-dimensional space, hence a cube root of unity of modulus one. Equation (43) becomes

$$\nu_3(X_3) + \nu_3(X_4) + 4\nu_3(X_5) = 0.$$

The first two terms have sum of absolute value at most 2, whereas the last has absolute value 4. This strict triangle contradiction excludes the hypothetical category and proves the theorem. $\square$

The source-bound standard-library verifier `verify_r630_weighted_indicator.py` replays the structural sphericity gate, complete character table, dimensions, codegrees, conductor arithmetic, weights, and cubic invariant dimensions. The theoretical contradiction uses no enumeration of center rows. Its normal and optimized receipts and independent audit accompany `R630_WEIGHTED_GALOIS_INDICATOR_PROOF.md` and the common lemma `SATURATED_GALOIS_INDICATOR_LEMMA.md` in `evidence/unified_20261016/rank8/`.

23. A CONDUCTOR-FORTY CENTER TRACE OBSTRUCTION

Theorem 23.1. *The ring $R8_2_671$ has no complex fusion categorification.*

Proof. Proposition 21.4 reduces a hypothetical categorification to a pseudo-unitary one. Put $q = \sqrt{10}$. Its simple dimensions and global dimension are

$$d = (1, 1, 3 + q, 4 + q, 4 + q, 3 + q, 2 + q, 2 + q), \quad D = 120 + 36q.$$

The complete ordinary formal codegrees are $D, D', 4, 6, 6, 12, 12, 12$, where $D' = 120 - 36q$. The six remaining characters are defined over $\mathbb{Q}(\sqrt{3})$. All eight source objects are self-dual, and the

odd-isotypic pivotal identities force all their second indicators to be +1. The source induction row is

$$[FI(1)] = (8, 2, 4, 6, 8, 6, 6, 6).$$

Write a central forgetful row as v , with $\dim FZ = A + Bq$, where

$$A = v_0 + v_1 + 3v_2 + 4v_3 + 4v_4 + 3v_5 + 2v_6 + 2v_7, \quad B = \sum_{i=2}^7 v_i.$$

Choose a cyclotomic automorphism σ flipping q and fixing $\sqrt{3}$. The character-labelled correspondence [74, Example 2.9 and Theorem 2.13] and modular Galois symmetry give the exact transport

$$T(v) = v + (0, 0, 1, 1, 1, 1, 1, 1) \left(A - \frac{10B}{3} \right). \quad (44)$$

An actual row has partner $T(v)$ or $-T(v)$, with the sign making it nonnegative; the entries must be integral. In particular $3 \mid B$. The unit row is paired with $(1, 0, 1, 1, 1, 1, 1, 1)$.

Let J be the Galois image of the central unit and put $w = e_1 - e_J$. The multiplicity vector of $I(1)$ is fixed by S , so its Galois signs are positive. Its known twist-one block gives

$$\text{proj}_{I(1)} Sw = \left(\frac{1}{D} - \frac{1}{D'} \right) w = -\frac{w}{\sqrt{40}}.$$

The torus character of w is the primitive quadratic character χ_{40} . The equality refinement of the compression bound [63, Lemma 4.1] implies $PSw = -w/\sqrt{40}$ and level 40 for its cyclic representation. The quadratic Gauss-sum refinement gives negative χ_{40} on primitive 40th twist exponents. These refinements, with their full local proofs, are supplied in the common compression evidence packet. Since the Z -coordinate of Sw is $z/D - \sigma(z/D)$, every central simple outside $I(1)$ with nonrational z/D has twist order dividing 40 and cannot have twist one. The last assertion uses the exact full projection PSw , whose right side has no such coordinate.

Let t denote normalized relative trace to $\mathbb{Q}(q)$. For a root of unity of arbitrary order, the pair $(t(\theta), t(\theta^2))$ lies in the convex hull of

$$(1, 1), (-1, 1), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right), (0, -1), (\pm q/8, 0), (0, \pm q/8).$$

This is a root-order independent statement: rational traces are Ramanujan sums; an irrational first trace requires order $40m$, with m squarefree and coprime to 10, and has absolute value $q/(8\varphi(m))$ and second trace zero. An irrational second trace instead requires order $80m$ and has first trace zero. The displayed segments contain all these values. For a high row with nonrational dimension ratio, the allowed pair list reduces to

$$(-1, 1), (0, -1), \left(-\frac{1}{4}, -\frac{1}{4}\right), (0, 0), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(0, \frac{1}{4}\right), (-q/8, 0).$$

These are the orders 2, 4, 5, 8, 10, 20, 40, respectively. If $v_1 > 0$, positivity and $\text{Tr}(\theta^2 I(X_1)) = D = \dim I(X_1)$ force $\theta^2 = 1$. Thus only the two signs remain, and only -1 remains when the ratio is nonrational. Galois partners have equal relative moments. On the restricted rows one may also choose σ with exponent 59 modulo 120; its square is one modulo 40, so their actual twists agree.

Here is the complete finite arithmetic needed to finish the proof. The source tensor determines

$$C_{ij} = \sum_{k,l} N_{ki}^l N_{kj}^l = \sum_Z v_i(Z) v_j(Z).$$

Each constituent Z_χ of $I(1)$ has unit multiplicity and known dimension D/f_χ . Its actual character label imposes

$$\frac{\chi_i(v_j)}{f_i} = \frac{\chi_j(v_i)}{f_j}.$$

After the two forced rows, the six exact bounded domains have sizes 86, 74, 74, 22, 22, 22. Their prescribed sum and the entrywise and positive semidefinite residual Gram conditions leave exactly

six frames. For every remaining row, integral closure of the modular formal-codegree relation forces D/z integral; equivalently,

$$A^2 - 10B^2 \mid 120A - 360B, \quad A^2 - 10B^2 \mid 36A - 120B.$$

Together with (44) and the residual Gram bound for the whole partner orbit, the six frames have respectively 73, 85, 73, 28, 36, 28 allowable row orbits.

Combine their Gram columns with the first and second induction trace identities

$$\mathrm{Tr}(\theta I(X_i)) = 0, \quad \mathrm{Tr}(\theta^2 I(X_i)) = D \quad (i > 0).$$

After subtracting the known twist-one terms and multiplying the trace coordinates by eight, these are 56 integer equations. The moment domains above produce respectively 360, 398, 360, 138, 168, 138 columns. An actual center would express the target as a nonnegative real combination of them.

Two fixed integer covectors contradict all six systems. Their pairings with every allowable column are nonnegative; their target pairings are -225800939520 for the first three frames and -960 for the last three. The covectors are given explicitly in `r671_center_trace_certificate.json`. The standalone verifier reconstructs every domain, tests every rational positive semidefinite condition, and checks every integer pairing directly from the source tensor. It uses no optimizer or floating-point arithmetic. Thus every possible center is excluded, and the initial Galois reduction proves the general complex claim. $\square$

24. GALOIS-INVARIANT POINTED ASSOCIATORS

The next obstruction is intrinsic to a pointed fusion subcategory; it therefore does not require a pivotal structure on an ambient category.

Lemma 24.1. *Let D be a fusion ring with precisely two FP dimensions, 1 and $d > 1$, and let g be a specified order-two invertible. If every pseudo-unitary categorification of D has the same associator class on $\langle g \rangle$, then every complex categorification has that class. If the class is nontrivial, no fusion category containing such a subcategory can have a simple M with $gM \cong M$.*

Proof. By the published [25, Lemma 3.1], every categorification of D is Galois conjugate to a pseudo-unitary one. The two elements of $H^3(C_2, \mathbb{C}^\times)$ have normalized representatives $\omega(g, g, g) = +1$ and -1 , each fixed by every field automorphism. Thus their classes are preserved separately by Galois conjugation. If $gM \cong M$, the orbit $\{M\}$ is a rank-one module over $\mathrm{Vec}_{C_2}^\omega$. After choosing $gM \rightarrow M$, its scalar module pentagon says $d\psi = \omega$ for a two-cochain ψ . Hence $[\omega] = 0$, a contradiction. This is also the pointed module-category criterion of [22]. $\square$

Write the ordinary and pseudo- $PSU(2)_6$ rings in the basis $(1, g, x, y)$, with $g^2 = 1$, $gx = xg = y$ and $gy = yg = x$. Both have FP dimensions $(1, 1, 1 + \sqrt{2}, 1 + \sqrt{2})$. Their distinguishing products and dualities are

	$x^2 = y^2$	$xy = yx$	duality
ordinary	$1 + x + y$	$g + x + y$	$x^* = x, y^* = y$
pseudo	$g + x + y$	$1 + x + y$	$x^* = y.$

Proposition 24.2. *The rings $R8_2_312$, $R8_2_315$ and $R8_2_317$ have no fusion-category categorification over $\mathbb{C}$.*

Proof. In a pseudo-unitary pseudo- $PSU(2)_6$ category, Larson's [56, Proposition 5.5 and proof of Theorem 5.8, Case 1(c)(1)] forces $\nu_2(g) = -1$. In the $c = 1$ case at issue, the positive-sign branch would require at least eight roots of unity when at most seven are available. The canonical spherical structure has dimension one on pointed simples, so $\nu_2(g) = \omega(g, g, g)$ in $\mathrm{Vec}_{C_2}^\omega$ [59]. Thus the pseudo ring forces the nontrivial pointed class. Lemma 24.1 makes this a constraint on every complex categorification.

For completeness, the ordinary ring forces the trivial class. Equip a hypothetical pseudo-unitary categorification with its canonical spherical structure. The odd-multiplicity parity theorem [62, Theorem 3.9] gives $\nu_2(x) = \nu_2(y) = 1$. The remaining alternative $\nu_2(g) = -1$ is excluded by the following exact indicator certificate. Let A be the restriction matrix from the center and $C = AA^\top$ its induction Gram matrix. The complete low-column lists are precisely

$$\begin{aligned} L_1 &= [(1, 0, 0, 0), (1, 0, 0, 2), (1, 0, 1, 0), (1, 0, 1, 0)], \\ L_2 &= [(1, 0, 0, 0), (1, 0, 1, 1), (1, 0, 0, 1), (1, 0, 1, 0)], \\ L_3 &= [(1, 0, 0, 0), (1, 0, 2, 0), (1, 0, 0, 1), (1, 0, 0, 1)]. \end{aligned}$$

They follow from the complete formal-codegree list; these columns have trivial twist, and every other column has unit coordinate zero and belongs to the residual Gram matrix. For each L_i , the necessary center data yield an integer system $B_i n = b_i$, $n \geq 0$. The stored integer vectors v_i certify

$$v_i^\top B_i \geq 0, \quad (v_1^\top b_1, v_2^\top b_2, v_3^\top b_3) = (-48, -12, -48).$$

Each system has 38 equations and respectively 102, 152, 102 variables, including nonnegative slack variables. All three are therefore infeasible by direct integer arithmetic.

We specify the hypotheses of this finite certificate. Besides reciprocity, it imposes the center indicator identities

$$\sum_{Z \in \text{Irr } Z(\mathcal{D})} [F(Z) : X] \dim(Z) \theta_Z^j = \dim(\mathcal{D}) \nu_j(X), \quad 1 \leq j \leq 4,$$

and the trace bounds for an order- j pivotal rotation on $\text{Hom}(1, X^j)$ [59]. The Cauchy theorem [61], applied to $\dim(\mathcal{D}) = 8 + 4\sqrt{2}$ of norm 32, makes every twist order a power of two. The ten joint relative-trace vectors through height four to $\mathbb{Q}(\sqrt{2})$ cover all such orders: orders larger than the divisor box 32 have the zero vector, which is included. This is a finite trace classification, not a bound on the exponent. The remaining columns satisfy the d -number and dimension-quotient integrality conditions [24, 22]. Finally Galois symmetry of the normalized center S matrix [60] pairs simple dimensions by

$$z \mapsto \left| \frac{(8 + 4\sqrt{2})z'}{8 - 4\sqrt{2}} \right|,$$

with equal counts in each pair. These conditions give the displayed systems. The source-bound checker reproduces every column, trace vector, matrix and separating product; it uses no optimization solver. Hence $\nu_2(g) = 1$ in every pseudo-unitary ordinary categorification, and Lemma 24.1 makes its pointed class trivial in every complex categorification.

The exact census witnesses, in the displayed rank-four basis, are

ring	ordinary subring	pseudo subring	g
$R8_2_312$	none needed	$(0, 7, 1, 2)$	7
$R8_2_315$	$(0, 7, 3, 4)$	$(0, 7, 1, 2)$	7
$R8_2_317$	$(0, 7, 5, 6)$	$(0, 7, 1, 2)$	7.

For $R8_2_312$, each of the outside simples 3, 4, 5, 6 is fixed by g , contradicting the nontrivial pseudo-ring class by Lemma 24.1. For $R8_2_315$ and $R8_2_317$, the two subcategories restrict the same ambient associator to the same pointed subcategory, but force opposite classes. This is impossible. The argument conjugates the rank-four subcategories separately; it assumes no simultaneous pseudo-unitarization of the ambient category. $\square$

25. AUTOMORPHISMS OF THE UNITARY $2D2$ MODEL

The general complex exclusion is Theorem 20.1. The following model-specific argument describes the obstruction through the tensor automorphism group of the known unitary neutral category.

Proposition 25.1. *The ring $R8_2_24$ has no unitary categorification. More generally, it admits no categorification whose neutral $2D2$ category is Galois conjugate to the known unitary one.*

Proof. Let $\mathcal{Q}$ be the known unitary $2D2$ category. The construction in Proposition 14.1 gives a tensor autoequivalence $T = \text{Ad}(b)|_{\mathcal{Q}}$ swapping r, gr , with $T^2 = \text{Ad}(g)$. Grossman's [55, Lemma 4.1] shows that $\text{Ad}(g)$ exchanges two inequivalent algebra structures on $1 + r$, so it is nontrivial. Thus T has order four. The factor automorphism fixing M and sending λ to $-\lambda$ induces an outer, label-fixing involution S by [55, Lemma 4.2]. It commutes with b on both M and λ , hence S commutes with T and is independent of T^2 . Grossman's order-eight computation [55, Theorem 4.19 and its proof] therefore gives

$$\text{Aut}_{\otimes}(\mathcal{Q}) = \langle T, S \rangle \cong C_4 \times C_2.$$

Its label-fixing subgroup is $\langle T^2, S \rangle \cong C_2^2$; every label-swapping element has square $T^2 = \text{Ad}(g)$.

The neutral category of a unitary categorification is equivalent to $\mathcal{Q}$ by [30, Theorem 3.31]. In $R8_2_24$, an odd invertible u has $u^2 = g$ and fixes every neutral label by conjugation. Then $\text{Ad}(u)$ would be a label-fixing autoequivalence whose square is the nontrivial $\text{Ad}(g)$, impossible. Applying a Galois automorphism to the entire category proves the stated Galois-class assertion as well. $\square$

26. COMPRESSION OVER ALL CONSTITUENTS OF THE INDUCED UNIT

The modular-representation argument in [63, Lemma 4.1 and the proof of Theorem 5.1] can retain every simple constituent of $I(\mathbf{1})$, rather than only the Galois orbit of the dimension character. This gives two further exclusions in the pivotal and spherical settings.

Proposition 26.1. *Let $\mathcal{C}$ be a spherical fusion category, with dimension field $K = \mathbb{Q}(\dim X : X \in \text{Irr}(\mathcal{C}))$ and $G = \text{Gal}(K/\mathbb{Q})$. Write f_E for the formal codegrees of its fusion algebra. Then $f_E \in K$ and, for every character χ of G with conductor q ,*

$$\sum_E \left| \sum_{\sigma \in G} \chi(\sigma) \sigma(f_E^{-1}) \right|^2 \leq \frac{|G|}{q}.$$

Proof. Put $D = \dim \mathcal{C}$ and $s = S/D$ for the normalized matrix of $\mathcal{Z}(\mathcal{C})$. Let A_E be the constituent of $I(\mathbf{1})$ associated to E . It has dimension D/f_E and twist one, so $f_E \in K$. If A_{σ} corresponds to $\sigma(\dim)$, then $s_{A_{\sigma}, Y} = \sigma(\dim Y/D)$. Set $w_{\chi} = \sum_{\sigma} \chi(\sigma) e_{A_{\sigma}}$ and let P project onto the twist-one subspace. The cited local compression estimate gives $\|P s w_{\chi}\|^2 \leq |G|/q$. Its A_E coordinate is the inner sum in the assertion. Summing over these distinct simple coordinates proves the inequality; their multiplicities in $I(\mathbf{1})$ do not repeat coordinates. $\square$

Corollary 26.2. *Let $B = \sum_i N_i N_i^T$ be the regular Casimir of a fusion ring. Suppose the irreducible factors of $\det(xI - B)$ belonging to a fixed real quadratic field of discriminant q are $(x^2 - t_j x + n_j)^{m_j}$. A spherical categorification requires*

$$q \sum_j m_j \frac{t_j^2 - 4n_j}{n_j^2} \leq 1.$$

Proof. The Casimir eigenvalue attached to E is $\lambda_E = (\dim E) f_E$, with regular multiplicity $(\dim E)^2$. The quadratic field lies in K . If its index in K is e' , pull back its nontrivial character to G . Each conjugate pair of eigenvalues contributes $2e'^2 m_j (t_j^2 - 4n_j)/n_j^2$ to the preceding bound, whose right side is $2e'/q$. Cancelling and using $e' \geq 1$ proves the claim. $\square$

Proposition 26.3. *The rings $R8_2_318$ and $R8_2_735$ admit no pivotal or spherical categorification over $\mathbb{C}$.*

Proof. For both source tensors, direct multiplication gives

$$\det(xI - B) = (x - 16)^3(x - 8)(x^2 - 192x + 1024)(x^2 - 16x + 32).$$

Both quadratic factors belong to $\mathbb{Q}(\sqrt{2})$, of discriminant 8. Hence the necessary bound would read

$$8 \left(\frac{192^2 - 4 \cdot 1024}{1024^2} + \frac{16^2 - 4 \cdot 32}{32^2} \right) = \frac{5}{4} \leq 1,$$

a contradiction.

For completeness, the eight characters of each commutative ring have formal codegrees

$$16, 16, 16, 8, 8 + 4\sqrt{2}, 8 - 4\sqrt{2}, 96 + 64\sqrt{2}, 96 - 64\sqrt{2}.$$

Every character in the first six positions vanishes on at least one basis element. The remaining two are the Frobenius–Perron character and its quadratic conjugate, and are duality invariant. These statements follow by substitution in the source tensor; the complete exact character tables are supplied in the associated `rank8_full_I1_certificate.json`. Simple pivotal dimensions cannot vanish. Thus a pivotal structure would have one of the last two dimension characters and would be spherical, which has just been excluded. $\square$

These compression arguments alone give scoped exclusions. The independent module obstruction in Section 12 now excludes arbitrary complex categorifications of both rings.

An independent general proof for the four quadratic cases. The same estimate gives a second proof of Proposition 11.1, without a pseudo-unitary reduction. Each of its four rings has Casimir polynomial

$$(x - 8)^4(x^2 - 160x + 640)(x^2 - 32x + 128).$$

The sphericalization’s dimension field therefore contains both $\mathbb{Q}(\sqrt{10})$ and $\mathbb{Q}(\sqrt{2})$. Its index e over $\mathbb{Q}(\sqrt{10})$ is at least two. The first quadratic factor has quantity $Q = 9/4$ in Corollary 26.2. In the sphericalization’s positive-sign sector, formal codegrees are doubled, so compression requires $eQ/4 \leq 1$. But $eQ/4 \geq 9/8 > 1$, a contradiction.

27. THE CONDUCTOR-32 OBSTRUCTION

Theorem 27.1. *The ring $R8_2_857$ has no categorification over $\mathbb{C}$.*

Proof. Reorder the source basis as $(0, 1, 7, 2, 5, 4, 3, 6)$ and denote it by $X_0 = 1, X_1, \dots, X_7$. All objects are self-dual. Multiplication by X_1 is the symmetric tridiagonal matrix J_2 whose off-diagonal entries are one and whose diagonal is $(0, 1, 1, 1, 1, 1, 1, 2)$. If M_j denotes multiplication by X_j , then

$$M_0 = I, \quad M_1 = J_2, \quad M_{j+1} = (J_2 - I)M_j - M_{j-1} \quad (1 \leq j \leq 6).$$

These identities are direct consequences of the source tensor.

Put $\zeta = e^{2\pi i/32}$ and $L = \mathbb{Q}(\zeta + \zeta^{-1})$. The positive dimension character is $d_j = \sum_{r=-j}^j \zeta^r = \sin((2j+1)\pi/32)/\sin(\pi/32)$. Its eight Galois conjugates are distinct and give all ring characters. Their formal codegrees are the conjugates of

$$D = \sum_{j=0}^7 d_j^2 = 4 \csc^2(\pi/32), \quad \frac{1}{D} = \frac{2 - \zeta - \zeta^{-1}}{16}.$$

For any primitive even Dirichlet character χ of conductor 32, the reciprocal Fourier sum over $\text{Gal}(L/\mathbb{Q})$ is

$$\widehat{g}(\chi) = -\frac{\tau(\chi)}{16}, \quad |\widehat{g}(\chi)|^2 = \frac{32}{16^2} = \frac{1}{8}.$$

Here $\tau(\chi)$ is the primitive Gauss sum: summing $\zeta^a + \zeta^{-a}$ over representatives modulo sign gives precisely its full sum. The eight constituents of $I(\mathbf{1})$ thus violate Proposition 26.1: if the spherical dimension field has index e over L , their squared Fourier norm is e^2 , whereas the bound is $e/4$. Hence the ring has no spherical categorification.

Now suppose an arbitrary categorification exists, and take its canonical sphericalization with sign object ε of dimension -1 . In the $\varepsilon = 1$ sector the eight formal codegrees are $2\sigma(D)$. They belong

to the spherical dimension field K , so $L \subseteq K$. Put $e = [K : L]$. The same eight coordinates in the full induced-unit compression have squared norm $e^2/4$, with upper bound $e/4$. Thus $e = 1$ and $[K : \mathbb{Q}] = 8$.

Both lifts of every simple are self-dual, since duality preserves its nonzero spherical dimension while exchanging the lifts negates it. Choose the positive-second-indicator lifts and pass to the signed quotient $\varepsilon = -1$. Its constants satisfy

$$|T_{ij}^k| \leq N_{ij}^k, \quad T_{ij}^k \equiv N_{ij}^k \pmod{2}, \quad T_{ii}^j \equiv N_{ii}^j \pmod{4}.$$

For the last congruence, a negative-indicator lift can occur in $X_i^* \otimes X_i$ only with even multiplicity by [62, Theorem 3.9]. The signed difference is therefore the original multiplicity minus a multiple of four. Self-duality gives full symmetry of T . Consequently $T_{1j}^j = T_{jj}^1 = 1$ for $1 \leq j \leq 6$.

The off-diagonal entries of the signed multiplication matrix of X_1 are ± 1 . Successively switching the lifts of $X_2, \dots, X_7$ makes them all positive, while leaving the unit, X_1 , and all diagonal entries unchanged. Its matrix is now J_t , with diagonal $(0, 1, 1, 1, 1, 1, t)$ and $t \in \{-2, 0, 2\}$. The first seven rows give the same polynomial recurrence for all signed multiplication matrices. Thus $\dim X_1$ generates K .

If $t = -2$, the bottom-right entry of $M_2 = J_t^2 - J_t - I$ is 6, exceeding the original multiplicity $N_{27}^7 = 2$. If $t = 0$, then

$$\det(xI - J_0) = (x-1)(x+1)(x^2-2x-1)(x^4-4x^3+2x^2+4x-1).$$

The degree of $\dim X_1$ is at most four, contrary to $[K : \mathbb{Q}] = 8$. Finally, if $t = 2$, the recurrence makes every signed multiplication matrix equal to the original nonnegative one. The negative lift multiplicities all vanish, so the eight chosen lifts form a spherical fusion subcategory equivalent to the original categorification. This contradicts the spherical exclusion already proved. $\square$

The source-bound certificate verifies all 512 dimension-character products in $\mathbb{Q}(\zeta_{32})$, all eight exact Fourier norms, and the three displayed matrix branches. The argument is independent of any construction of the corresponding folded quantum-group datum.

27.1. A central-lift obstruction for the 2D2 rules.

Proposition 27.2 (No central invertible lift for the 2D2 rules). *Let $\mathcal{Q}$ be a pseudo-unitary fusion category with simple objects $1, f, r, s$ and fusion rules*

$$f^2 = 1, \quad fr = rf = s, \quad r^2 = s^2 = 1 + 2r + 2s, \quad rs = sr = f + 2r + 2s.$$

There is no invertible object $z \in Z(\mathcal{Q})$ whose underlying object is f .

Proof. Put $d = 2 + \sqrt{5}$, so $d^2 = 1 + 4d$. The displayed rules imply $\mathcal{Q}_{\text{ad}} = \mathcal{Q}$. Consequently the only central lift of the tensor unit is the tensor unit: such lifts are tensor automorphisms of the identity, equivalently characters of the universal grading group. Thus a hypothetical z satisfies $z^2 \cong 1$. In its pointed braided subcategory the positive ribbon twist $q = \theta_z$ belongs to $\{1, -1, i, -i\}$.

If $q = 1$, then $1 \oplus z$ is the regular algebra of a central Tannakian $\text{Rep}(C_2)$. Its module category has two simple free orbits, represented by $1, r$, and their rule is $\bar{r}^2 = 1 + 4\bar{r}$. This contradicts the rank-two fusion-category classification [23].

Suppose $q = \pm i$. Write I for induction to the center. The formal-codegree induction formula, as recalled in [30, Appendix A.2], gives the distinct constituents of $I(1)$ dimensions

$$1, \quad d^2, \quad 1 + 2d, \quad 1 + 2d.$$

Every constituent has twist 1. Each has tensor-unit multiplicity one after forgetting, and none contains f , because $FI(1)$ has no f -summand. The projection formula gives $I(f) \cong z \otimes I(1)$, disjoint from $I(1)$. Since $z^2 = 1$, the monodromy of z with any simple central object is ± 1 . Hence every constituent of $I(f)$ has twist $\pm i$.

In $I(r) \oplus I(s)$ the constituents belonging to $I(1)$ have total dimension

$$K = 4d^2 + 4(1 + 2d) = 8 + 24d.$$

Indeed their total multiplicities are, respectively, 0, 4, 2, 2. The constituents belonging to $I(f)$ have the same total dimension. Since

$$\dim(I(r) \oplus I(s)) = 2d(4 + 8d) = 16 + 72d,$$

the other constituents have total dimension $24d$. The first-indicator identity gives zero twist trace on $I(r) \oplus I(s)$. Its $I(1)$ part has real trace $8 + 24d$, its $I(f)$ part has purely imaginary trace, and the absolute real trace of the remaining part is at most $24d$. This is impossible.

Finally suppose $q = -1$. Let $\mathcal{I}$ be an Ising category with fermion ψ and noninvertible σ . The object $z \boxtimes \psi$ is a central boson in $\mathcal{Q} \boxtimes \mathcal{I}$. Condensing its regular algebra gives a pseudo-unitary fusion category $\mathcal{D}$ of rank six. All twelve simples before condensation have free orbits. Representatives give objects $1, f, u, r, s, v$ of dimensions

$$1, 1, \sqrt{2}, d, d, \sqrt{2}d,$$

with the above rules on $1, f, r, s$ and additional rules

$$\begin{aligned} fu = u, \quad fv = v, \quad u^2 = 1 + f, \quad ur = us = v, \\ uv = r + s, \quad rv = sv = u + 4v, \quad v^2 = 1 + f + 4r + 4s. \end{aligned}$$

We prove directly that these rules have no pseudo-unitary categorification.

Put $D = \dim(\mathcal{D}) = 8 + 16d$ and $T = Dd = 16 + 72d$. The six formal codegrees show that $I(1)$ has two constituents of each dimension $1, d^2, 2 + 4d$, all of twist 1, each with tensor-unit multiplicity one. Positivity and linear independence over $\mathbb{Q}(\sqrt{5})$ show that their restrictions have no u or v summands. The two constituents of dimension d^2 restrict to

$$1 + a_j r + (4 - a_j)s \quad (0 \leq a_j \leq 4),$$

and the two of dimension $2 + 4d$ restrict to

$$1 + f + b_j r + (4 - b_j)s \quad (0 \leq b_j \leq 4).$$

The r -coefficient of $FI(1)$ is 8, so if $a = a_1 + a_2$ then $b_1 + b_2 = 8 - a$. The total dimension K of the twist-one summands of $I(r)$ supplied by $I(1)$ is therefore

$$K = ad^2 + (8 - a)(2 + 4d) = 16 - a + 32d \geq 8 + 32d.$$

Let $R = T - K$ be the dimension of the other summands. The first two indicator identities [59], with $\nu_2(r) \leq 1$, imply

$$\sum_j w_j \cos t_j = -K, \quad \sum_j w_j \cos(2t_j) = D\nu_2(r) - K \leq D - K, \quad \sum_j w_j = R,$$

where $w_j > 0$ and e^{it_j} are their dimensions and twists. If $R = 0$ the first equation is already impossible; otherwise Cauchy–Schwarz gives

$$2K^2 \leq R(R + D - K), \quad 0 \leq T(T + D) - K(3T + D).$$

But the last expression is at most

$$T(T + D) - (8 + 32d)(3T + D) = -64 - 512d - 1088d^2 < 0.$$

This final contradiction excludes $q = -1$. □

Remark 27.3. The published Galois pseudo-unitarity reduction [25, Lemma 3.1] for categories having two Frobenius–Perron dimensions upgrades this statement to arbitrary complex categorifications of the displayed ring: a central lift survives Galois conjugation. This uses the corrected published formulation “Galois conjugate to a pseudo-unitary category”, not a claim that every categorification is itself pseudo-unitary.

Remark 27.4 (Scope of the inner-conjugation construction). Let $\mathcal{P} = \mathcal{Q}_{\text{pt}}$ and $\mathcal{D} = \mathcal{Z}_{\mathcal{P}}(\mathcal{Q})$. By [47, Remark 2.4(i)], this relative center is Morita equivalent to $\mathcal{P} \boxtimes \mathcal{Q}^{\text{rev}}$. Its invertibles are precisely those of $\mathcal{Z}(\mathcal{P})$, whose group is $C_2 \times C_2$ for either associator on $\mathcal{P} \simeq \text{Vec}_{C_2}^{\omega}$. Thus this natural inner-conjugation construction cannot realize the C_4 -based rings $R8_1_9$ or $R8_1_18$; no indicator assumption is needed.

The displayed rank-four rules have trivial universal grading, so the theorem implies $\text{Inv } \mathcal{Z}(\mathcal{Q}) = 1$. Morita invariance then gives $|\text{Inv } \mathcal{Z}(\mathcal{D})| = 4$. If $\mathcal{D}$ had the rules $R8_1_49$, its universal grading would also be trivial. The forgetful map on invertibles would therefore be injective, since its kernel is the group of tensor automorphisms of the identity, and hence bijective. Every invertible would have a central lift, contradicting the noncommuting invertible action in that ring. This rules out the proposed construction only. General exclusions are proved independently in Theorems 19.7 and 19.9 and Proposition 14.3.

28. MORITA RIGIDITY AND A REALIZED TWISTED PRECURSOR

The four open rings retain their general unresolved status. The following results distinguish an actual auxiliary construction from necessary module data and from a construction of a source ring.

28.1. Categorical Morita rigidity of $R8_3_928$.

Theorem 28.1. *Let $\mathcal{C}$ be a complex fusion category with based ring $R8_3_928$. Every indecomposable semisimple $\mathcal{C}$ -module category is equivalent to the regular module category. Consequently every categorical Morita partner is tensor equivalent to $\mathcal{C}$. No rank bound on module categories is assumed.*

Proof. In the source basis $g = X_7$ acts by $(0\ 7)(1\ 2)(3\ 4)(5\ 6)$. The established semionic obstruction forces the nontrivial pointed C_2 associator. This conclusion is invariant under the spherical/Galois reduction. The module pentagon therefore forbids a g -fixed module simple: such a stabilizer would trivialize that associator.

For a module simple m , write its internal end as $A = \sum_i a_i X_i$. The complete internal-end domain has the form $(1, a, a, b, b, c, c, 0)$, with $0 \leq a \leq 7$, $0 \leq b \leq 9$, $0 \leq c \leq 11$. These bounds follow from $a_i \leq [\text{FPdim}(X_i)]$; the exact dimension character and strict rational root bounds certify them. Testing all 960 vectors by entrywise and rational positive-semidefinite induction-Gram conditions leaves precisely

$$(1, 0, 0, 0, 0, 0, 0, 0), (1, 1, 1, 1, 1, 1, 1, 0), (1, 1, 1, 1, 1, 2, 2, 0), (1, 2, 2, 2, 2, 3, 3, 0).$$

This is a complete finite domain, not a search over a chosen module rank.

Let $B_{i,n} = [X_i m : n]$. Adjunction gives $(BB^t)_{ij} = \sum_k a_k N_{jk}^i$. Remove the columns of m, gm ; the others occur in g -pairs. Write u, d for the pair sums and differences on the three remaining row-pairs. Their Gram matrices are $P = \sum uu^t$ and $M = \sum dd^t$, where $M = I_3$ and

$$P \in \left\{ I_3, \begin{pmatrix} 9 & 10 & 14 \\ 10 & 13 & 18 \\ 14 & 18 & 25 \end{pmatrix}, \begin{pmatrix} 13 & 16 & 18 \\ 16 & 21 & 22 \\ 18 & 22 & 25 \end{pmatrix}, \begin{pmatrix} 13 & 18 & 20 \\ 18 & 25 & 28 \\ 20 & 28 & 33 \end{pmatrix} \right\}.$$

Every matrix has determinant one. Cauchy–Binet says that the squared integral 3×3 minors of a factor of P sum to one. There is a unique unimodular minor. All minors replacing one of its columns must vanish, forcing every extra column to be zero. Since m generates the module category, its columns are nonzero. Exactly three residual pairs remain, and the module category has rank eight.

The complete integral factorizations, with $u \equiv d \pmod{2}$ and $|d_i| \leq u_i$, give one factor for each internal end up to column permutation. The 48 signed permutation factors of $M = I_3$ exhaust its possibilities. Each resulting B is invertible, so the source actions are determined by conjugating the

regular matrices. Exact source-product and dual-transpose checks identify all four with the regular based module. This finite computation is replayed from the complete bounds.

The categorical conclusion needs one further step. Each such module contains a simple n with internal end 1. The internal-end equivalence then identifies the actual module category with $\mathcal{C}_1 = \mathcal{C}$. No uniqueness of module associators is inferred merely from its based action. Taking dual categories proves the Morita assertion. $\square$

There are exactly four connected separable algebra isomorphism classes within any hypothetical categorification: they are the internal matrix algebras of simples, modulo the four right- g orbits. Their four underlying objects above are distinct. Every dominant tensor functor from this category is an equivalence onto its fusion image. Indeed its target is a regular module category; if the target unit corresponds to X , the positive module dimension vector makes the simple corresponding to 1 have dimension $1/\text{FPdim}(X) \geq 1$. Hence X is invertible, the unit internal end is 1, and adjunction gives full faithfulness. These statements do not supply the hypothetical category.

28.2. A complex twisted $GH(C_6, 2)$ category. Put $u = 6 + \sqrt{37}$. The based ring with invertibles a_j , $j \in C_6$, and noninvertibles $a_j\sigma$ has relations

$$a_i a_j = a_{i+j}, \quad a_j \sigma = \sigma a_{-j}, \quad \sigma^2 = 1 + 2 \sum_{j \in C_6} a_j \sigma.$$

Its dimension $6(1 + u^2) = 444 + 72\sqrt{37}$ equals that of $R8_3_1768$. The completed rank-eight based-module route forces pointed associator class $3 \in H^3(C_6, \mathbb{C}^\times)$. Its existence is now established, without a unitarity assertion.

Proposition 28.2. *There is a complex fusion category with these rules and pointed associator class 3. This does not assert a categorification of $R8_3_1768$.*

Proof. Use the explicit Radchenko–Wheeler near-group realization associated to $\gamma = \begin{pmatrix} 145 & 12 \\ 12 & 1 \end{pmatrix}$, with group $G = C_{12}^2$ and multiplicity 144; see Radchenko–Wheeler, Section 5.3. Its bicharacter and Gaussian are

$$B((x, y), (a, b)) = e^{2\pi i(yb - xa)/12}, \quad q(x, y) = (-1)^{x+y} e^{2\pi i(y^2 - x^2)/24}.$$

These are the specified realization’s data, not parameters assigned to an arbitrary near-group category.

Let $H = \{(x, x + 6t) : x \in C_{12}, t \in C_2\}$ and use the cocycle $\omega((x, t), (a, s)) = (-1)^{ta}$. Its commutator is $B|_H$; the latter has radical $K = \{(2z, 2z) : z \in C_6\} = H^\perp$. On the algebraic twisted crossed product of the Leavitt realization, extend the source automorphisms and noninvertible endomorphism by

$$\beta_g(\lambda_h) = B(g, h)^{-1} \lambda_h, \quad \tau(\lambda_h) = q(h) U_h \lambda_h.$$

On the original algebra their restrictions are α_g, ρ . The original intertwiner identities verify multiplication and covariance: the cocycle occurs on both sides equally, and $q(h)^2 = 1$ on H . Coefficient comparison in the crossed-product direct sum gives $\text{End}(\tau) = \mathbb{C}^\omega[H]$ and six distinct invertible classes G/H . The original complete splitting intertwiners still decompose τ^2 ; its duality maps still satisfy both snake identities. Thus additive idempotent completion gives a rigid semisimple tensor category with simple unit and finitely many simples. These are the algebraic identities checked in the accompanying full construction; no C^* reconstruction or classification of arbitrary spherical categories is used.

Set $X = \lambda_{(1,1)}$ and $T = \lambda_{(0,6)}$. Then $X^{12} = T^2 = 1$ and $TX = -XT$. Hence $\mathbb{C}^\omega[H] \cong \bigoplus_{\eta \in C_6} M_2(\mathbb{C})$. The diagonal-average projector $p = 12^{-1} \sum_l X^l$ is primitive. Its corner $\sigma = (p, \tau)$ is simple; the six such central blocks are its invertible translates and $\tau = 2 \sum_j a_j \sigma$.

For clarity the crucial square multiplicity has an independent tensor proof. The diagonal subgroup $L = \{(x, x)\}$ is maximal isotropic and $q|_L = 1$. The algebraic isotropic orbifold identities give an

actual $GH(C_{12}, 1)$ category there. Inclusion of this crossed product into the twisted H crossed product preserves its complete basic Hom lists and all splitting intertwiners, hence extends to a tensor functor on idempotent completions. Its chosen primitive diagonal projector remains primitive. The old simple square is $\sigma_0^2 = 1 + \sum_{j \in C_{12}} b_j \sigma_0$. The functor identifies b_j with b_{j+6} and sends σ_0 to σ . Applying it proves the displayed multiplicity-two rule and self-duality. The explicit projector-rank calculation gives the same answer independently.

For $a = \beta_{(1,0)}$, one has $a^6 = \text{Ad}(\lambda_{(6,0)})$ and $a(\lambda_{(6,0)}) = -\lambda_{(6,0)}$. This cyclic obstruction is class 3 in $H^3(C_6, \mathbb{C}^\times)$, as required. $\square$

28.3. The $1 + \sigma$ construction route and its obstruction. The old based-module calculation has precisely two rank-eight completions for an algebra $1 + \sigma$. This does not prove that such an algebra exists in the category just constructed. In particular the canonical near-group algebra descends to the matrix algebra $\text{End}(\sigma)$, not to $1 + \sigma$.

More explicitly, write $Z = \lambda_{(0,6)}$ and take the normalized maps

$$r = \frac{1}{\sqrt{12}} \sum_l S_{(l,l)} \lambda_{(l,l)}, \quad t_0 = \frac{1}{\sqrt{12}} \sum_l T_{(l,l)} p, \quad t_1 = \frac{1}{\sqrt{12}} \sum_l T_{(l,l+6)} Z p.$$

Their primed algebraic duals satisfy $sr = 1$, $t'_i t_j = \delta_{ij} p$; t'_0, t'_1 span $\text{Hom}(\sigma^2, \sigma)$. After normalizing the unit-channel product to s , a multiplication on $1 + \sigma$ is specified by $t = b_0 t'_0 + b_1 t'_1$. Associativity is exactly

$$\begin{aligned} s(t \otimes 1) &= s(1 \otimes t), \\ (s \otimes 1) + t(t \otimes 1) &= (1 \otimes s) + t(1 \otimes t). \end{aligned}$$

The first equation is automatic here: cubic rotation is I_2 . The second is a quadratic morphism system in two variables, with coefficients from the explicit finite quantum dilogarithm. Separability must also be checked. The pairing obstruction in Section 29 proves that this route cannot produce the source ring, for any associator or algebra; solving this system would therefore not model *R8_3_1768*. The categorical dimension in the specified Stark realization is $6 - \sqrt{37}$, whereas its FP dimension is u ; positivity is not assumed in these algebraic equations.

28.4. Further route obstructions. A proper inclusion of fusion categories has FP index at least two. Indeed each indecomposable component over the subcategory has dimension at least its dimension: for a simple X , its internal end in the subcategory is a subobject of XX^* , and the internal-end module dimension formula gives that inequality. A proper inclusion has at least two components. The $SU(4)_8$ exceptional module-category index over either quartic target is $24 - 15\sqrt{2} + 12\sqrt{3} - 9\sqrt{6} \in (1, 2)$, excluding this route. The indicated F_{43}/G_{24} routes fail by dimension as well.

For *R8_3_1154*, put $d = 2 + \sqrt{5}$, $v = d^2$; its dimension is $60v$. Exact Weyl arithmetic excludes five entire families with arbitrary pointed factors and one connected étale condensation: $SU(4)_6$, $SU(5)_5$, $SU(6)_4$, $SU(3)_7 \boxtimes \text{Fib}$, and the rank-fourteen modular factor constructed in the rank-ten conformal model. In each case a rational required algebra dimension either violates algebraic integrality or exceeds the pointed dimension bound: every nonpointed simple has a strictly positive nonconstant coordinate in the exact positive dimension basis, so a rational-dimensional algebra must be pointed. The exact field enumerations and the all-pointed-factor argument are in the accompanying proof. These are obstructions to specified construction families, not general noncategorifiability theorems.

All new proofs, complete finite domains and independent audits are curated in `evidence/unresolved_retry_20261024/README.md`. The seven remaining open cases include these four rank-eight rings.

29. PAIRING OBSTRUCTIONS AND THE FOUR REMAINING RINGS

All four rings remain unresolved. Every application below uses the established general-complex existence reduction to a pseudo-unitary Galois representative; it does not assert unitarity of the original category.

29.1. Quaternionic invertibles and $R8_3_1768$. For a connected separable algebra in a pseudo-unitary category, its positive module trace supplies a symmetric separable Frobenius structure. The bimodule forgetful functor is pivotal Frobenius and preserves a chosen nondegenerate self-duality pairing with its pivotal sign. Consequently a negative selfdual bimodule can contain a positive selfdual source simple only with even multiplicity: its multiplicity space is alternating. This is pairing transport, not equality of numerical indicators. See Schaumann, Proposition 5.8 and Theorem 6.6 (<https://arxiv.org/abs/1206.5716>), and Mulevičius, Example 4.18 (<https://arxiv.org/abs/2206.02611>). The induced positive dimensions are $\dim U(Q)/\dim A$, which fixes the pivotal normalization.

Theorem 29.1. *No category with $GH(C_6, 2)$ fusion rules is categorically Morita equivalent to a categorification of $R8_3_1768$, for any associator or algebra.*

Proof. Put $u = 6 + \sqrt{37}$. The positive source second indicators are $(1, -1, 1, 1, 1, 1, 1)$ and its dimensions are

$$(1, 1, 5 + \sqrt{37}, 6 + \sqrt{37}, 7 + \sqrt{37}, 7 + \sqrt{37}, 6 + \sqrt{37}, 5 + \sqrt{37}).$$

Its weighted second-indicator total is $6u$. Such totals are Morita invariant by the center induction formula. A $GH(C_6, 2)$ category has six selfdual large simples of dimension u . Its pointed contribution is $1 + \nu_2(g^3)$. Comparing the irrational coefficient and then the rational part forces all six large indicators to be $+1$ and $\nu_2(g^3) = -1$. Thus this Morita dual would contain a negative-indicator invertible.

Choose any simple m in the connecting module and let A_m be its internal end. For a negative invertible Q in the dual, $\dim U_m(Q) = \dim A_m$. Pairing parity makes all six nonpointed source coefficients in $U_m(Q)$ even. Since each has $\sqrt{37}$ coefficient one, the square-root coefficient in $\dim A_m$ is even for every m .

The complete module filter, including every retained incomplete-action factor, has 21 possible induction factors. For a root algebra A and induction matrix B , the internal-end dimensions at every module simple are exactly

$$\dim A_m = \frac{(B_m \cdot \mathbf{d})^2}{\dim A}.$$

In 17 factors one of these dimensions has odd square-root coefficient. They cannot have such a dual. In each of the other four factors a simple has internal end of dimension one. Connectedness then makes that internal end the unit, and the internal-end equivalence identifies the module with the regular module. Its dual has rank eight, not twelve, a contradiction. The exact filter refactors all 61 predecessor candidates; the receipt lists all 17 odd witnesses and four unit witnesses. It does not discard the 15 factors with incomplete actions. $\square$

In particular no algebra $1 + g^j \sigma$ in any twisted higher-Haagerup precursor can model this source. The existence of the auxiliary category established above remains valid but supplies no model of $R8_3_1768$. The theorem is a construction-family exclusion, not noncategorifiability of the rank-eight ring itself.

29.2. Third rotations and $R8_3_1154$.

Proposition 29.2. *All six noninvertible third indicators equal two in a positive categorification of $R8_3_1154$. Each corresponding trivalent rotation is identity. All three necessary center patterns still survive.*

Proof. The actual rank-ten equivariantization reduction supplies lifts x, ux of X_4 and likewise X_5 , with u central of order two and $x^* = ux$. For odd n the two character subspaces of the cubic or higher vacuum space give $\nu_n(X_4) = \nu_n(x) + \nu_n(ux) = 2\nu_n(x)$, by spherical duality. Each lift's cubic vacuum space has dimension one. Thus each of $\nu_3(X_4), \nu_3(X_5)$ belongs to $\{2, 2\zeta_3, 2\zeta_3^2\}$, rather than an arbitrary sum of cube roots. Imposing these discrete possibilities on all three complete retained center systems forces both to equal two. Duality and the inner autoequivalence, followed by a second discrete check, force the other four noninvertible third indicators to equal two. The 33 excluded branches have exact integer separating certificates; all six retained branch records have exact rational feasibility witnesses. An independent matrix reconstruction checks every certificate. An order-three rotation on a two-dimensional space with trace two is identity. $\square$

The integral center is Vec even for an arbitrary complex categorification. Indeed an integral central simple forgets to $a1 + bg$; based centrality and the noncentrality of g force $b = 0$. Full adjoint closure makes a central object supported on the unit trivial. Hence no Morita representative has a nontrivial finite-group extension or equivariantization. The auxiliary rank-ten category has integral center double Semion, whose only nontrivial Tannakian descent returns to this source. Direct central condensations from $\text{Fib}^{\boxtimes 6}$, $GH(C_4, 1) \boxtimes \text{Fib}^{\boxtimes 3}$ and $GH(C_4, 1)^{\boxtimes 2}$, each with arbitrary pointed factors, are also excluded. Dimension and subgroup-index bounds eliminate the first two. In the last, the required index-three subgroup contains a central C_4 ; quotienting forces the impossible rank-two rule $x^2 = 1 + 4x$. These are specified central constructions, not all Morita constructions.

29.3. Simple centralizers for $R8_3_928$ and $R8_3_941$. Write $D = 144 + 96\sqrt{2} + 80\sqrt{3} + 56\sqrt{6}$. For either source, the only proper nontrivial subcategory is pointed of rank two. Any central fusion subcategory with a nonintegral object therefore forgets dominantly to the full source and has dimension at least D . Equality for a braided subcategory would make the noncommutative source braided, which is impossible.

For R928 its center is $S \boxtimes \mathcal{B}$, with S Semion and $\dim \mathcal{B} = D^2/2$. The only integral simple in $\mathcal{B}$ is its unit. A proper nontrivial subcategory and its centralizer would both have dimension at least D , contradicting their product $D^2/2$. Thus $\mathcal{B}$ is simple; its possible ranks are 17, 20, 23. The regular-module rigidity theorem and the correspondence between module categories and Lagrangian algebras imply that $I(1)$ is the unique Lagrangian algebra. Every connected etale algebra extends to one, by Witt triviality and lifting local-module algebras. The reverse subcategory correspondence then gives precisely three etale algebra classes in the whole center, in a chain $1 \subset A \subset I(1)$, with $\dim A = D/2$. The restriction from $\mathcal{B}$ is dominant, so its right adjoint gives that same unique nontrivial A in $\mathcal{B}$, and $\mathcal{B}_A \simeq \mathcal{C}$, $\mathcal{B}_A^0 \simeq S^{\text{rev}}$. These use DMNO, Lemma 3.5, Proposition 3.16, Corollary 3.30, Proposition 4.8, Theorem 4.10 and Propositions 5.4–5.8.

For R941 the fermion centralizer $\mathcal{E}$ has rank 22, dimension $D^2/2$, and integral part sVec . A splitting $\mathcal{E} = \mathcal{M} \boxtimes \text{sVec}$ would force the whole center to have rank 33 or 44, instead of 34, 37, 40, 43 in the complete retained list. If a proper nonpointed subcategory contains sVec , it and its centralizer both dominate the source. Their dimension product is D^2 , forcing both dimensions to be D , the impossible equality case above. A nonpointed subcategory not containing the fermion would generate $\mathcal{E}$ with sVec and be nondegenerate, giving the excluded splitting. Thus the only subcategories are Vec , sVec and $\mathcal{E}$.

Finally the conditional $1 + X$ Morita construction in the rank-six manuscript produces precisely R941; it is not forced by equal FP dimensions. Neither category nor that nonpointed algebra is constructed. Full proofs, complete finite certificates and audits for this section are curated in `evidence/frontier_20261027/README.md`.

30. REPRODUCTION AND MATHEMATICAL LIMITS

Run `python3 tools/verify_current_publication.py` from the repository root, and repeat with `-0`. The verifier checks the published documents and their website copies, the source identities,

the current decision ledger and the accepted finite certificate applications, and the complete current unresolved inventory. Run `python3 tools/verify_rigidity_20261017.py` for the inner-square and associated theoretical evidence, and repeat that command with `-0`. The current-publication verifier’s `-historical` option also replays the full chain of source-bound certificate applications in isolated inputs. The exact replay instructions and the scope of every recorded run appear in `docs/VERIFY.md`.

The historical rank-eight multiplicity-two full-certificate audit covers all 973 input tensors, including the 75 signed proof trees and four rank-seven subring proof transfers used by the corresponding base certificates. The theoretical applications and positive tensor comparisons are checked separately. All finite checks use integer or algebraic arithmetic. They validate their specified equations and source bindings; they do not independently prove categorical transport identities, cited theorems or census completeness. A timeout or an unfinished branch never constitutes an exclusion.

The cyclic induction and center STS arguments in Theorems 19.7 and 19.9, together with Proposition 14.3, complete the multiplicity-one classification. At multiplicity two the conclusions are 848 general exclusions and 29 unitary constructions. Theorem 20.1 excludes $R8_2_24$ by combining the saturated indicator signs, the actual inner square supplied by its invertible object, the common pivotal chart and tensor-functor rigidity. The source and weight checks are recorded in `evidence/rigidity_20261017/rank8/`.

The finite applications are conditional on the stated mathematical arguments, cited theorems and source census completeness. The multiplicity-three input is checked as described in Section 3; exhaustive enumeration is not independently rerun. This work does not classify categories up to tensor equivalence or claim that no further criterion is needed. Noncommutative Extended Haagerup models at rank eight have multiplicities four and seven [43]; they are genuine models outside the present census bound.

AI disclosure and verification status. GPT-6 Astra and GPT-6.1 Sol assisted with literature exploration, mathematical research, code development and manuscript preparation. The results, proofs and accompanying data are AI-generated and have not been independently verified by a human expert or formalized in Lean. They should be used with caution. Computational replay provides evidence for its explicit finite checks and does not replace mathematical review.

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