

Rank-seven Grothendieck rings through multiplicity five

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Abstract

We study complex categorification of rank-seven fusion rings through multiplicity five. The earlier classification through multiplicity four excludes 4172 of 4201 rings and identifies unitary models for the remaining 29. The extension addresses all 5805 newly supplied rings of exact multiplicity five: 5804 are excluded, one has the unitary model $\text{Rep}(C_7 \rtimes C_6)$, and all cases are resolved. The full status is recorded in Table 1. Every general exclusion applies without assuming pivotality. The arguments combine signed sphericalization, formal codegrees, equivariantization, module-category divisibility, Morita duality and Drinfeld-center induction. Positive constructions are compared against complete source multiplication tensors. The new input is independently checked for the fusion-ring axioms, uniqueness, and preservation of the previous census; its enumeration completeness is producer-reported. The classification concerns based-ring types, not fusion categories up to tensor equivalence, and makes no claim beyond the supplied range.

1 Scope and statements

Let R have distinguished basis $(b_0 = 1, b_1, \dots, b_6)$ and multiplication $b_i b_j = \sum_k N_{ij}^k b_k$. A *Grothendieck ring* here means the based ring of a complex fusion category; no pivotal, spherical, braided, or unitary hypothesis is placed on a hypothetical categorification. Its multiplicity is $\max_{i,j,k} N_{ij}^k$. The upstream *Fusion Ring Census* [16] provides the earlier complete list through multiplicity four. The supplied multiplicity-five extension and its verification scope are detailed in Section 14. We pin source bytes, row ordering, digests, and the transport of old IDs in the companion package. No claim is made for multiplicities above five.

Theorem 1 (Recorded rank-seven status through multiplicity five). *For the supplied tensors, the numbers of excluded, explicitly unitarily realized, and undecided rings are given by Table 1. The negative conclusions hold for every complex fusion category. The positive rows have the models in Table 3, matched against their complete fusion tensors. All supplied rows through multiplicity five are decided: 9976 general exclusions and 30 unitary models. The multiplicity-five conclusions have the source coverage qualification of Section 14.*

Table 1: Complete classification of the supplied rank-seven fusion-ring census through exact multiplicity five.

Exact multiplicity	Fusion rings	Excluded	Realized	Undecided
1	43	30	13	0
2	319	309	10	0
3	1059	1057	2	0
4	2780	2776	4	0
5	5805	5804	1	0

Table 2: Sequential exclusion counts, charged only to the first successful criterion. The later rows record additional theoretical and exact finite obstructions.

Criterion	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
Near-group theorem	2	1	1	1	0
Global dimension 15	1	0	0	0	0
Zero spectrum	8	30	51	63	71
Closed rank-four/five subring	0	25	125	235	315
Formal-codegree d -number	15	179	788	2314	5219
Subcategory dimension quotient	0	5	9	8	19
Oriented dimension order	4	51	61	130	95
Localized triangular-prism equations	0	1	1	0	0
Solvability and support inequality	0	0	2	0	0
Half-partition Tannakian criterion	0	0	1	0	0
Odd pointed group in dimension 120	0	0	0	1	0
Double equivariantization in dimension 52	0	1	0	0	0
Order-sixteen orbit obstruction	0	2	0	0	0
Dimension-96 rank-one modules	0	2	0	0	0
Dimension- $8p$ equivariantization	0	2	0	1	0
Dimension-96 order-sixteen orbits	0	2	0	0	0
Dimension-112 Singer normalizer	0	0	2	0	0
Dimension-54 Brauer–Picard	0	0	2	0	0
Additional complete signed quotients	0	0	2	2	0
Dimension-310 group theory	0	0	0	1	0
Relative-trace lattice	0	0	0	2	0
Dimension-150 central boson	0	1	0	0	0
Full induced-unit compression and parity	0	0	1	0	0
Signed three-parameter sphericity	0	0	2	0	0
Cyclic-stabilizer divisibility	0	0	0	5	47
Anti-invariant module spectral bounds	0	2	3	2	0
Morita-dual Gram identities	0	4	2	1	0
Positive-semidefinite Gram refinement	0	0	1	3	0
C_3 – A_4 Morita obstruction	0	1	0	2	0
Inverse V_4 Morita obstruction	0	0	1	0	0
Simple-center prime-power lattice	0	0	1	0	0
Quadratic-center Galois two-moment obstruction	0	0	1	2	0
Twisted V_4 staged Morita obstruction	0	0	0	1	0
Character reciprocity and two center moments	0	0	0	1	0
Dihedral Morita and full center STS	0	0	0	1	0

Criterion	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
Cyclotomicity	0	0	0	0	3
Conductor-sixty center moments	0	0	0	0	2
Character-aware central induction	0	0	0	0	6
Closed rank-six subring	0	0	0	0	5
Integral codegree divisibility	0	0	0	0	6
Involution Fourier module	0	0	0	0	2
General central induction	0	0	0	0	4
Coupled cyclic rotations and fusion integrality	0	0	0	0	2
Complete signed quotient and compression	0	0	0	0	1
Quadratic Casimir compression	0	0	0	0	3
Cyclic isotypic integrality	0	0	0	0	1
Maximal near-group theorem	0	0	0	0	1
Full cyclic Morita and rank-thirteen descent	0	0	0	0	1
Cubic Fourier-module obstruction	0	0	0	0	1

Table 3: Unitary model constructions identified by exact full-tensor based-ring isomorphism. Each row denotes a ring, not a count of categories.

Census ID	Explicit model
R7_1_1	$\text{Rep}(D_{16})$
R7_1_6	$\text{SU}(2)_6$
R7_1_9	$\text{PSU}(2)_{12}$
R7_1_12	$\text{SO}(7)_2$
R7_1_16	$\text{Rep}(D_{22})$
R7_1_17	$\text{PSU}(2)_{13}$
R7_1_19	AnyonWiki $\text{FC}_{3,1,0,1}^{7,1,2}$
R7_1_24	$\text{Zest}(\text{SU}(2)_6)$
R7_1_25	$\text{Rep}(SD_{16})$
R7_1_28	AnyonWiki $\text{FC}_{12,1,0,1}^{7,1,2}$
R7_1_36	AnyonWiki $\text{FC}_{3,1,0,1}^{7,1,4}$
R7_1_39	$\text{TY}(C_6)$
R7_1_43	$\text{Vec}(C_7)$
R7_2_1	TY parity equivariantization (§13.1)
R7_2_24	$\text{Rep}(S_5)$
R7_2_195	TY twisted- C_4 equivariantization (§13.1)
R7_2_218	$\text{Rep}(C_{13} \rtimes C_4)$
R7_2_233	$\text{Rep}(H_{24})$
R7_2_234	$\mathcal{C}(A_5, 1, A_4, 1)$
R7_2_240	D_{12} even category (§13.4)
R7_2_297	$\text{Rep}(SL_2(\mathbb{F}_3))$
R7_2_313	$\text{Adj}(\text{SU}(3)_5)$
R7_2_319	$\text{Rep}(C_{13} \rtimes C_3)$
R7_3_602	$\text{Rep}(A_6)$
R7_3_1059	$\text{Rep}(C_{11} \rtimes C_5)$
R7_4_2	$\text{Vec}(Q_8)^{S_3}$
R7_4_35	C_4^2 near-group inversion Morita model
R7_4_2019	Twisted quaternion model (§13.3)
R7_4_2665	C_{16} near-group inversion Morita model
R7_5_5678	$\text{Rep}(C_7 \rtimes C_6)$

Multiplicity one is complete: the rings $R7_1_19$, $R7_1_28$, and $R7_1_36$ have the AnyonWiki models described below. Multiplicities two, three, four and five are also complete. The case $R7_4_2053$ is excluded in Theorem 36. The exhaustive source-position convention is $R7_m_j$, where j is the position *among rows of exact multiplicity m* in the pinned census file. This convention makes the positive-model permutations and negative witnesses unambiguous.

2 Necessary conditions and their order

Table 2 preserves the attribution in the earlier ledger and adds the continuation’s first successful recorded criterion for each new tensor. The campaign records give the execution order. Near-group and small-dimension checks read a few coefficients; the zero-spectrum and closed-subring passes are finite tensor scans; exact Casimir factoring and subcategory-dimension arithmetic follow. The complete oriented search is followed by the additional theoretical criteria. This order records the first successful criterion for each tensor. Although the supplied census through multiplicity five is classified, these stages do *not* establish a multiplicity threshold beyond which no new criteria would be needed. The oriented criterion first contributes at multiplicity one, so it cannot be deferred to a higher multiplicity without losing a proof.

If G is the group of invertible basis elements and the only other object X satisfies $X^2 = \sum_{g \in G} g + \ell X$, then for $|G| = 6$: at $\ell = 0$ the Tambara–Yamagami construction forces G abelian [37]; at $0 < \ell < 5$ in the commutative case, Schopieray’s near-group obstruction rules out categorification [35, Proposition 3.2]. These statements apply to a categorification without a pivotal assumption. The certificate records the six group elements, the remaining object and ℓ , and rechecks every fusion-rule hypothesis.

The global-dimension-fifteen branch has six dimension-one simples and one dimension-three simple: direct substitution into every product establishes that $(1, \dots, 1, 3)$ is its strictly positive dimension character. Hence its Frobenius–Perron dimension is $6 + 3^2 = 15$. The pq classification [22, Theorems 3.1 and 5.1] makes a complex fusion category of dimension $3 \cdot 5$ pointed in this case, contradicting the dimension-three simple.

The zero-spectrum obstruction is the exact necessary criterion of Liu–Palcoux–Ren [30]; each certificate stores its nine basis indices and the verifier rechecks all required zero/nonzero fusion coefficients. A proper fusion-closed subset of basis elements would generate a fusion subcategory. If its induced rank-four or rank-five ring is excluded by the complete lower-rank classifications [17, 18], the ambient ring is excluded. The imported model registries contain all fifteen rank-four and all seventeen rank-five realized based-ring types. The rank-four list is unrestricted; the rank-five classification is complete through multiplicity 20. All encountered rank-five subrings fall within that range. The complete rank-six multiplicity-two classification [19] supplies context and methodology, without being used as an unstated rank-seven classification theorem.

Proposition 2 (The formal-codegree test at rank seven). *Suppose that R is a rank-seven fusion ring, possibly noncommutative. Let $C = \sum_{i=0}^6 N_i N_i^\top$ be its regular Casimir matrix. If an irreducible factor of the characteristic polynomial of C is not the minimal polynomial of a d -number, then R cannot be the Grothendieck ring of a complex fusion category.*

Proof. In the commutative case, the eigenvalues of C are the formal codegrees of one-dimensional characters and must be d -numbers by Ostrik [33]. In the noncommutative case, semisimplicity, $\dim_{\mathbb{C}}(R \otimes \mathbb{C}) = 7$, and the positive one-dimensional character imply the unique possible decomposition $R \otimes \mathbb{C} \simeq \mathbb{C}^3 \oplus M_2(\mathbb{C})$. There is only one two-dimensional irreducible module; it is fixed by every field automorphism, so its formal codegree is a rational algebraic integer and hence an

integer. The Casimir acts on this matrix block by twice that integer. Every remaining nonrational Casimir eigenvalue comes from a one-dimensional character and is a formal codegree, so Ostrik’s condition applies. The implementation factors the exact integral characteristic polynomial and tests the coefficient-divisibility characterization of d -numbers. The replay checker recomputes and factors that polynomial. $\square$

If S is a proper commutative fusion subring of a hypothetical categorification, the quotient $\text{FPdim}(R)/\text{FPdim}(S)$ must be an algebraic integer [28]. The subcategory-divisibility stage computes this quotient in the exact Frobenius–Perron number field, records its characteristic polynomial, and certifies a nonintegral coefficient. Its certificate is rederived from the complete tensor.

3 Complete oriented tests at rank seven

The oriented-dimension-order row of Table 2 uses the general, *oriented* signed-sphericalization construction established in [19]. For an arbitrary hypothetical category $\mathcal{C}$, its canonical sphericalization has two lifts of each simple object, interchanged by a sign object ε . Taking the quotient by $(\varepsilon + 1)$ produces integer coefficients T_{ij}^k with $|T_{ij}^k| \leq N_{ij}^k$ and $T_{ij}^k \equiv N_{ij}^k \pmod{2}$. Unit, duality, cyclic Frobenius reciprocity, and ordered associativity further constrain the quotient. The self-dual positive-indicator normalization imposes the established square-channel congruences modulo four. These conditions are necessary and cover every categorification, whether pivotal or not; a signed solution is not itself a categorification.

The rank-six proof of the general construction and the dimension-order obstruction [19] are rank-independent. Here their enumeration engine is extended to seven basis elements. It enumerates the independent cyclic/dual-reflection coefficient orbits, prunes only exact associativity equations with checked integer bounds, and constructs *every* signed quotient. For each quotient the checker verifies either its adjoint parity, its gauge equivalence with the original ring, or the failure of every admissible spherical character by codegree and dimension-order tests. In particular the dimension-order test requires the central-induction dimension δ/f_χ to lie in the $\mathbb{Z}$ -span of the signed simple dimensions, as required by [13, 19].

Proposition 3 (Verified exhaustive branches). *The oriented-dimension-order criterion excludes four, fifty-one, sixty-one, and one hundred thirty previously surviving rank-seven rings of exact multiplicities one through four, respectively. For $R7_1_5$, $R7_1_23$, $R7_1_33$, and $R7_1_38$, the complete signed enumerations have respectively 2, 4, 4, 8 associative quotient tables. Every table in each of these 246 certified exhaustive searches is eliminated by necessary conditions for arbitrary complex fusion categories.*

Proof. The machine-readable proof tree for each of the 246 rings gives its full dimension-character factorization, every obstruction, the ordered signed-orbit domain, and one checked branch for every enumerated table. The independent verifier rebuilds the domains from the source tensor, repeats the bounded exhaustive enumeration, checks the *entire* list of surviving tables for equality to the recorded list, and recomputes each branch witness. The search used to discover these finite trees is logically separate from their exhaustive replay; timed-out or unfinished searches generate no negative decision. The enumeration returns the four displayed multiplicity-one branch counts and no uncovered branch for any of the 246 certificates. The rank-independent necessity of the search is the sphericalization theorem cited above. $\square$

4 Further theoretical exclusions

The local triangular-prism proof of Liu–Palcoux–Ren–Vercleyen [6, Corollary 6.2, Theorem 6.4 and Theorem 7.2] excludes $R7_2_185$ and $R7_3_635$ over $\mathbb{C}$. The first is exactly their $\mathcal{F}_{210}$. The second is a consequence of the same local proof, as follows. In the source basis of dimensions $(1, 5, 5, 5, 6, 7, 7)$, their tensors agree outside the cube on the two dimension-seven objects. Both have this strictly positive integral dimension character, so any categorification is pseudo-unitary and has the canonical spherical structure. The cited proof uses the squares of two dimension-five objects k, l , selected columns of dimensions one and five, their dualities and categorical dimensions. The zero constraints are read from the squares of those selected objects. Thus none of the differing coefficients on the dimension-seven cube enters either local subsystem or its linking equation. Both rings give precisely the inconsistent polynomial system in the proof of [6, Theorem 7.2]. The supplementary checker verifies all these tensor equalities and the positive dimension character; it does not rerun the published Gröbner-basis calculation.

For $R7_3_900$ and $R7_3_991$, a different short argument gives a general exclusion. Their dimension type is $(1, 1, 1, 1, 2, 4, 4)$ and their Frobenius–Perron dimension is $40 = 2^3 \cdot 5$. Any categorification would be solvable [7, Theorem 1.6], hence weakly group-theoretical and unitarizable [8, Theorem 5.20]. For a unitary categorification, the positive p th-root limit of [11, Corollary 3.4] requires, for every subset S of basis labels and $N_{ij}^l > 0$,

$$\frac{\sum_{x \in S} N_{lk}^x d_x}{d_l} \leq \frac{\sum_{m \in T} N_{jk}^m d_m}{d_j}, \quad T = \{m : N_{jk}^m > 0, N_{im}^x > 0 \text{ for some } x \in S\}.$$

Taking S to be the objects of dimension at most two, the exact witnesses in the supplement give $2 \leq 1$ for each ring. The first witness is $(i, j, k, l) = (1, 1, 2, 1)$; the second is $(3, 4, 4, 3)$, using zero-based census labels. This unitary obstruction is a general one here because the dimension theorem forces unitarizability.

Proposition 4 (A half-partition criterion). *Let R be an integral commutative fusion ring of dimension $4pq$, for distinct odd primes p, q . Write $a_E = 4pq/f_E$ for its formal codegrees. Suppose an odd-prime-power $a_E > 1$ occurs, and every submultiset of the a_E excluding the unit and summing to $2pq$ leaves an odd-prime-power entry in its complement. Then any categorification is weakly group-theoretical and has the Frobenius property.*

Proof. Integrality gives the canonical positive spherical structure. By [13, Theorem 2.13], the induced unit $I(1)$ in the center has distinct multiplicity-one summands of dimensions a_E . An odd-prime-power summand implies a nontrivial symmetric subcategory of the center [7, Corollary 7.2]. If this provides no nontrivial Tannakian subcategory, it must be sVec . For its fermion f , [13, Proposition 2.8] gives $s[I(1)] = 4pq[I(1)]$. The fermion has twist -1 , so it is absent from $I(1)$, whose twist is trivial. Its row therefore gives $\sum_E \epsilon_E a_E = 0$, with monodromy signs $\epsilon_E = \pm 1$ and $\epsilon_1 = 1$. The negative signs form a half-partition. By hypothesis an odd-prime-power summand centralizes f . Its slightly degenerate centralizer consequently contains a nontrivial Tannakian subcategory by [7, Proposition 7.4].

Write this subcategory as $\text{Rep}(G)$. Its de-equivariantization has neutral dimension $(4pq/|G|)^2$, hence $|G|$ divides $4pq$. If p or q divides $|G|$, the neutral category has at most two prime divisors and is solvable. Extension and equivariantization reconstruct the center; Morita invariance and subcategory closure then make the original category weakly group-theoretical [7, Theorem 1.6 and Proposition 4.1]. Otherwise $|G|$ is two or four and supplies $\text{Rep}(C_2)$. The original category is a C_2 -extension or C_2 -equivariantization of a category of dimension $2pq$ [7, Propositions 2.9–2.10].

That category is integral: in the equivariantization case its simple dimensions are rational quotients of integral dimensions by orbit sizes and projective representation degrees, hence integers. The integral squarefree-dimension theorem [7, Theorem 9.2] makes it group-theoretical. The Frobenius property follows from [7, Theorem 1.5]. $\square$

For $R7_3_905$, the exact positive dimension vector is $(1, 6, 6, 1, 1, 1, 8)$, and the formal codegrees are $(140, 20, 20, 7, 4, 4, 4)$. Thus the induced-unit dimensions are $(1, 7, 7, 20, 35, 35, 35)$. The only half-partition excluding the unit consists of two dimension-35 summands, leaving both dimension-seven summands. The proposition applies, but 6 does not divide 140. This excludes every complex categorification. The checker verifies the entire formal-codegree polynomial and all 64 nonunit subsets exactly.

Finally, $R7_4_2727$ has dimension type $(1, 1, 1, 3, 6, 6, 6)$, dimension 120, and formal codegrees $(120, 60, 60, 24, 4, 3, 3)$. The induced unit has simple summands of dimension two, so the center contains a nontrivial symmetric subcategory. The invertible group has odd order three; [14, Lemma 2.5] therefore gives a nontrivial Tannakian subcategory, and [14, Lemma 3.1] makes any categorification weakly group-theoretical and hence unitarizable. The exact support inequality above gives $4 \leq 2$ for $(i, j, k, l) = (3, 4, 3, 3)$ and $S = \{6\}$, a contradiction.

4.1 A double-equivariantization obstruction in dimension 52

The ring $R7_2_75$ has dimension type $(1^4, 4^3)$, invertible group V_4 , and trivial universal grading. Every invertible fixes every dimension-four simple. These are exact tensor properties checked by the supplement. Suppose $\mathcal{C}$ categorifies it. Dimension $52 = 4 \cdot 13$ makes $\mathcal{C}$ solvable [7, Theorem 1.6]. The last step of its prime cyclic extension/equivariantization construction [7, Proposition 4.4] cannot be an extension, since $\mathcal{C}_{\text{ad}} = \mathcal{C}$. Its character group must embed into V_4 , so the last step is $\mathcal{C} = \mathcal{D}^{C_2}$, with $\text{FPdim}(\mathcal{D}) = 26$. Each possible nontrivial character fixes the three dimension-four simples and pairs the four invertibles. The equivariant-simple orbit/stabilizer formula [1, Proposition 4.26] therefore forces $\mathcal{D}$ to have type $(1^2, 2^6)$ and rank eight. A fixed simple of the action has two lifts exchanged by the character; a free orbit has one lift of twice the dimension. There are no higher-dimensional projective representations of C_2 .

The category $\mathcal{D}$ also has trivial universal grading. Equal dimensions of faithful grading components [28, Proposition 8.20] make the grading-group order divide 26. Orders 13 and 26 exceed the rank. Order two would give a neutral category of prime dimension 13, necessarily pointed and of rank 13 [28, Corollary 8.30], also impossible. Applying the same solvable last-step argument gives

$$\mathcal{D} = \mathcal{P}^{C_2}, \quad \mathcal{P} = \text{Vec}_{C_{13}}^\omega.$$

The inner C_2 must act nontrivially on C_{13} , hence by inversion; a trivial object action would give 26 invertible simples in $\mathcal{D}$.

The canonical central $\text{Rep}(C_2)$ in $\mathcal{D}$ is unique over its underlying two invertibles. Indeed, the kernel of $\text{Inv}(Z(\mathcal{D})) \rightarrow \text{Inv}(\mathcal{D})$ is the group of tensor automorphisms of $\text{Id}_{\mathcal{D}}$. Such an automorphism is trivial on every summand of $Y \otimes Y^*$, hence on $\mathcal{D}_{\text{ad}} = \mathcal{D}$. Two lifts of the same invertible differ by this kernel. The outer C_2 action thus preserves the central $\text{Rep}(C_2)$. Equivariantizing it gives a central Tannakian $\text{Rep}(G)$ with fully faithful underlying embedding into $\mathcal{C}$, where $1 \rightarrow C_2 \rightarrow G \rightarrow C_2 \rightarrow 1$. Its symmetric fibre functor is the composite of the two forgetful functors. Therefore G is C_4 or V_4 ; its four characters exhaust the invertibles of $\mathcal{C}$, forcing $G = V_4$. De-equivariantization in stages now gives $\mathcal{C} = (\text{Vec}_{C_{13}}^\omega)^{V_4}$, with the normal C_2 acting by inversion

[7, Section 2.6 and Proposition 2.10] [1, Theorem 4.18(i) and Proposition 4.19]. These are central Tannakian constructions; no braiding of $\mathcal{C}$ is assumed.

The object action $V_4 \rightarrow \text{Aut}(C_{13}) = C_{12}$ has image of exponent at most two and contains inversion. Every nonidentity object consequently has orbit size two and stabilizer C_2 . Each of the six nonidentity orbits contributes two dimension-two simples, while the unit contributes the four characters of V_4 . This forces type $(1^4, 2^{12})$, contradicting $(1^4, 4^3)$. Thus $R7_2_75$ has no complex fusion categorification, without any pivotal or unitary assumption. The proof does not require ω to be trivial.

4.2 An order-sixteen orbit obstruction in dimension 96

The same method excludes $R7_2_252$ and $R7_2_315$. Both have dimension type $(1, 1, 2, 3, 3, 6, 6)$, invertible group C_2 , and trivial universal grading. Their nontrivial invertible h fixes the dimension-two and dimension-six simples and exchanges the two dimension-three simples. Solvability and the last-step argument force $\mathcal{C} = \mathcal{D}^{C_2}$. The orbit/stabilizer formula then gives $\mathcal{D}$ dimension 48 and type $(1^3, 3^5)$.

The universal grading of $\mathcal{D}$ is trivial. Otherwise its group has order dividing 48, is solvable by Burnside's theorem, and has a prime cyclic quotient of order two or three. Equal component dimensions [28, Proposition 8.20] make both impossible. For a C_2 grading all three invertibles lie in the neutral component, requiring $24 = 3 + 9n$. For a C_3 grading their group maps trivially or isomorphically to the grading group, requiring $16 = 3 + 9n$ or $16 = 1 + 9n$. Thus solvability forces $\mathcal{D} = \mathcal{E}^{C_3}$ with $\text{FPdim}(\mathcal{E}) = 16$. Its unit accounts for all three invertibles of $\mathcal{D}$. A dimension-three simple of $\mathcal{D}$ comes from either a free orbit of three invertibles of $\mathcal{E}$ or a fixed dimension-three simple, the latter giving three such simples in $\mathcal{D}$. Counting the five dimension-three simples leaves only types (1^{16}) or $(1^7, 3)$ for $\mathcal{E}$. The second is impossible: its seven invertibles would fix its unique noninvertible Y , so YY^* would contain all seven, leaving dimension two that cannot be filled by copies of Y . Consequently $\mathcal{E} = \text{Vec}_H^\omega$ for a group H of order sixteen.

As in the preceding proof, $\mathcal{D}_{\text{ad}} = \mathcal{D}$ makes the canonical central $\text{Rep}(C_3)$ unique over its underlying invertibles. The outer C_2 action preserves it. Equivariantization in stages gives $\mathcal{C} = (\text{Vec}_H^\omega)^G$, where $1 \rightarrow C_3 \rightarrow G \rightarrow C_2 \rightarrow 1$ and the central $\text{Rep}(G)$ embeds fully faithfully into $\mathcal{C}$. The cyclic group of order six would give six invertibles; hence $G = S_3$. The identity orbit contributes dimensions 1, 1, 2, exhausting the small simples. Any other fixed orbit has projective stabilizer S_3 , whose irreducible projective degrees are at most $\sqrt{6}$, and would give an extra small simple. An orbit of size two has stabilizer C_3 and would give extra dimension-two simples. An orbit of size three gives two dimension-three simples, and a free orbit gives one dimension-six simple. Thus the action on H must have orbit sizes

$$(1, 3, 6, 6).$$

Its normal C_3 acts without fixed points outside the identity; a transposition fixes exactly two elements of H .

Here is a purely group-theoretic contradiction. For a fixed-point-free order-three automorphism α of a group of order sixteen, every characteristic subgroup has order congruent to one modulo three. Thus the center has order four or sixteen. If it has order four, it is V_4 , and the noncyclic quotient $H/Z(H)$ is also V_4 . The commutator is an α -equivariant alternating map $\wedge^2 V_4 \rightarrow Z(H)$. Its source has trivial C_3 action, whereas its target has no nonzero fixed vector. It vanishes, a contradiction. So H is abelian. The characteristic subgroups $2H$ and $4H$ exclude C_{16} , $C_8 \times C_2$, and $C_4 \times C_2^2$, leaving $H = C_4^2$ or C_2^4 .

Every involution of either remaining group fixes at least four elements. For C_2^4 , $(S - I)^2 = 0$ gives $\dim \ker(S - I) \geq 2$. For C_4^2 , if S reduces to the identity modulo two, it fixes all four elements of $2H$. Otherwise a lifted basis change gives $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, d odd and c even. From $S^2 = I$ one obtains $bc = 0$ and $b(a + d) = 0$, hence $c = 0$ and $d = -a$ modulo four. Each of the four choices of x has the unique fixed-vector completion $y = -b^{-1}(a - 1)x$; the second equation follows from $a^2 = 1$. This contradicts the required two fixed points and excludes both rings over $\mathbb{C}$. The supplement checks the exact tensor premises and also provides an independent finite corroboration of this theoretical lemma.

4.3 Prime descent and finite orbit obstructions

The following families use the same mechanism. Solvability supplies a last cyclic-prime extension or equivariantization [7, Theorem 1.6 and Proposition 4.4]. When the adjoint subcategory is the whole category, an extension is impossible. The remaining equivariantization is constrained by the complete action of its character subgroup on the source simples. In an equivariantization, a simple above an orbit of Y has dimension

$$[G : G_Y] \dim(\pi) \text{FPdim}(Y),$$

where π is an irreducible projective representation of G_Y [5, Section 2]. The source-bound checks below include all dimension products, adjoint closures, and character-subgroup permutations. Thus descent is a categorical argument applied to checked hypotheses, rather than an identification by dimension type.

Proposition 5. *None of the twelve rings*

$$\begin{aligned} &R7_2_50, R7_2_70, R7_2_113, R7_2_213, R7_2_273, R7_2_280, \\ &R7_3_386, R7_3_398, R7_3_478, R7_3_966, R7_4_358, R7_4_963 \end{aligned}$$

is the Grothendieck ring of a complex fusion category.

Proof. We organize the argument by the descent obstruction.

Dimensions 40 and 136. For $R7_2_70, R7_2_213, R7_4_358$ the type is $(1^4, 2, d, d)$, with $(p, d) = (5, 4)$ or $(17, 8)$ and global dimension $8p$. The universal grading is trivial, and every invertible fixes each noninvertible. Prime descent therefore gives $\mathcal{C} = \mathcal{D}^{C_2}$ with $\mathcal{D}$ of type $(1^4, (d/2)^4)$ and dimension $4p$. If $\mathcal{D}$ admitted a C_2 grading, the outer involution would preserve a nonzero character of its 2-primary grading quotient; this would descend to a nontrivial grading of $\mathcal{C}$. A C_p grading is also impossible: at $p = 17$ its number of components exceeds rank eight; at $p = 5$ its neutral category is pointed of dimension four and its other four components have rank one. The latter force trivial pointed associator, while neither $\text{BrPic}(\text{Vec}_{C_4})$ nor $\text{BrPic}(\text{Vec}_{C_2^2})$ has an element of order five. Hence $\mathcal{D} = \mathcal{E}^{C_2}$.

Integrality of the descended dimensions and the strong Frobenius property [7] leave $\mathcal{E}$ of type $(1^2, 2^{(p-1)/2})$ and dimension $2p$. For clarity, at dimension ten the alternatives are pointed, $(1^6, 2)$, and $(1^2, 2^2)$: the middle case violates subcategory integrality, while the pointed case gives the forbidden invariant C_2 grading. At dimension thirty-four the dimension partitions are $(1^2, 2^{8-4b}, 4^b)$, $b = 0, 1, 2$; divisibility excludes $b > 0$. The same grading test gives $\mathcal{E}_{\text{ad}} = \mathcal{E}$, so $\mathcal{E} = (\text{Vec}_{C_p}^\omega)^{C_2}$ with inversion on C_p .

The kernel of $\text{Inv } Z(\mathcal{E}) \rightarrow \text{Inv } \mathcal{E}$ is $\text{Aut}_\otimes(\text{Id}_\mathcal{E}) = \text{Hom}(U(\mathcal{E}), \mathbb{C}^\times) = 1$. Consequently its central $\text{Rep}(C_2)$ is unique and is preserved by the successive outer actions. Combining the

equivariantizations [1, 7] gives $\mathcal{C} = (\text{Vec}_{C_p}^\omega)^H$, where $|H| = 8$ and $H_{\text{ab}} = C_2^2$. Its action on C_p has cyclic image of order at most two. At $p = 17$, orbit size times projective stabilizer degree is at most four, contradicting the dimension-eight simples. At $p = 5$, choose a strictly H -invariant μ_p -valued associator by averaging. The vanishings $H^2(C_p, \mathbb{C}^\times) = 0$ and $H^2(H, \widehat{C_p}) = 0$ remove the action tensorators and their coherence multipliers. Thus the stabilizer representations are ordinary; their degrees are at most two, and nontrivial orbits have abelian stabilizer of order four. All dimensions are at most two, contradicting $d = 4$.

Dimension 112. For $R7_3_386$, $R7_3_966$, type $(1^2, 2^3, 7^2)$ and trivial universal grading force a C_2 descent of type $(1^7, 7)$ and dimension 56. Its adjoint category is again the whole category. Its last prime step is therefore a C_7 equivariantization of $\text{Vec}_{C_3}^\omega$, with a Singer seven-cycle on the nonzero elements. The unique central C_7 survives the outer involution, so the combined group has order fourteen. Its representation subcategory has type $(1^2, 2^3)$ and hence is $\text{Rep}(D_{14})$. The resulting action would conjugate the Singer cycle to its inverse. Its normalizer in $\text{GL}_3(\mathbb{F}_2)$ is 7:3, which contains no such involution. This is impossible.

Dimension 96. For $R7_2_113$, $R7_2_273$, the character involution fixes the dimension-two and both dimension-six objects and exchanges the two dimension-three objects. These are exactly the premises of the dimension-96 proof above; its $(1, 3, 6, 6)$ orbit contradiction applies. For $R7_2_50$, $R7_2_280$, the involution instead exchanges both the dimension-three pair and the dimension-six pair. Descent has type $(1^3, 3, 6)$ and dimension 48, with trivial universal grading. Its next descent is a C_3 equivariantization of a dimension-16 category $\mathcal{E}$ of type $(1^4, 2^3)$, with the three nontrivial invertibles and the three dimension-two objects each cyclically permuted. The invertibles form $V = C_2^2$ and fix the dimension-two objects. Hence $\mathcal{E}_{\text{ad}} = \text{Vec}_V^\omega$, with a faithful V grading whose three nonneutral components have rank one. A rank-one pointed module implies $[\omega] = 0$ by its module pentagon. There are only two rank-one left module categories over Vec_V , since $H^2(V, \mathbb{C}^\times) = C_2$. Distinct invertible grading components here must have inequivalent left module categories: an equivalence, tensored with the inverse of one component, would identify a nonneutral rank-one component with the rank-four regular module. Three such components cannot fit into two classes.

Dimension 54. For $R7_3_398$, $R7_3_478$, type $(1^2, 2^4, 6)$ forces descent of type $(1^9, 3^2)$ and dimension 27. Its pointed adjoint has group A of order nine; the other two C_3 grading components have rank one, so its associator is trivial. The outer involution inverts both A and the grading C_3 . Its restricted tensorator has class $\eta \in H^2(A, \mathbb{C}^\times)$. Inversion acts trivially on this odd-order group, while order-two coherence gives $2\eta = 0$; hence $\eta = 0$. Its Brauer–Picard class restricts to $-I$ on $A \oplus \widehat{A}$, which is central in the orthogonal group [40, Corollary 1.2]. It therefore cannot conjugate a nontrivial order-three component class to its inverse, as the grading action requires.

Dimension 310. For $R7_4_963$, type $(1^2, 2^2, 10^3)$ has dimension $310 = 2 \cdot 5 \cdot 31$. The pqr theorem [7, Theorem 9.2] makes any categorification $\mathcal{C}(G, \omega, H, \psi)$ with $|G| = 310$. The complete closed-subring scan gives dimensions 1, 2, 10, 310, so $|H|$ belongs to this list. For $|H| = 1, 2$ the double-coset formula [2, Theorem 5.1] cannot supply dimension ten. For $|H| = 10$, the representation subcategory forces $H = D_{10}$. The three outside dimension-ten simples force trivial double-coset intersections and outside cosets of size one hundred. Thus G would be a Frobenius group with kernel C_{31} and complement D_{10} ; a faithful complement action in the cyclic group $\text{Aut}(C_{31})$ is impossible. For $H = G$, the category is $\text{Rep}(G)$. Its normal Sylow 31 subgroup has a split complement D_{10} , whose cyclic action on C_{31} has image at most two. The orbit/stabilizer formula then bounds all representation degrees by two, again contradicting ten. $\square$

4.4 Four additional exhaustive signed exclusions

The complete oriented search excludes $R7_3_1036$, $R7_3_1044$, $R7_4_2510$, $R7_4_2706$. Their complete lists have respectively 8, 24, 4, 48 signed associative tables. Every possible dimension character fails either nonvanishing, duality, or formal-codegree norm divisibility. The supplement regenerates the lists and rechecks each branch, including under optimized Python. These are general exclusions: sphericalization doubles both global dimension and the relevant formal codegrees, leaving the necessary quotient unchanged. No pivotal structure on the original category is assumed.

4.5 A relative-trace obstruction without signed enumeration

Proposition 6 (Relative-trace criterion). *Let R be a commutative fusion ring with self-dual basis, and put $B = \sum_{i,j,k} (N_{ij}^k)^2$. Suppose $L \in \mathbb{Z}_{>0}$ and f are formal codegrees of R , and set $F = \mathbb{Q}(f)$. If every nonzero totally positive element of $L\mathcal{O}_F \cap f\mathcal{O}_F$ has field trace greater than B , then R has no categorification over $\mathbb{C}$.*

Proof. Sphericalize a hypothetical category and take its quotient by $\varepsilon = -1$. Both lifts of a self-dual simple remain self-dual [13, Section 5.1(a)]; hence the signed multiplication matrices are symmetric and commute. Writing them as M_i , their Casimir satisfies

$$\mathrm{Tr}\left(\sum_i M_i^2\right) = \sum_{i,j,k} (T_{ij}^k)^2 \leq B.$$

Let d_i be the lift dimensions, $K = \mathbb{Q}(d_i)$, and $\delta = \sum_i d_i^2$. Every embedding of K gives a distinct dimension character and contributes its conjugate of δ as a Casimir eigenvalue. This counts $[K : \mathbb{Q}]$ characters even when some eigenvalues coincide. All Casimir eigenvalues are positive, so $\mathrm{Tr}_{K/\mathbb{Q}}(\delta) \leq B$.

The sphericalization has dimension 2δ and formal codegrees $2L, 2f$. The induced-unit theorem gives algebraic integers $\delta/L, \delta/f$ in its dimension field K [13, Theorem 2.13 and Corollaries 2.14–2.15]. Thus $F \subseteq K$. The element $t = \mathrm{Tr}_{K/F}(\delta)$ is totally positive and lies in $L\mathcal{O}_F \cap f\mathcal{O}_F$. Its trace is at most B , a contradiction. No equality $K = \mathbb{Q}(\delta)$ is assumed. $\square$

Proposition 7. *The rings $R7_4_1417$ and $R7_4_1420$ admit no complex fusion-category categorification.*

Proof. Both rings have formal codegrees 20 and $f = 510 + 70\sqrt{51}$, whose minimal polynomial is $z^2 - 1020z + 10200$. Here $\mathcal{O}_F = \mathbb{Z}[\sqrt{51}]$. Write $t = 20(x + y\sqrt{51})$ in the intersection ideal. Then

$$\frac{t}{f} = (x - 7y) + \left(y - \frac{7x}{51}\right)\sqrt{51}.$$

Integrality forces $51 \mid x$, while total positivity forces $x > 0$. Consequently $\mathrm{Tr}_{F/\mathbb{Q}}(t) = 40x \geq 2040$. The exact Casimir trace bounds are 1086 and 1070, respectively, so Proposition 6 applies. The source-bound checker recomputes both characteristic polynomials, trace bounds, and the displayed quotient identity using exact arithmetic. $\square$

4.6 A cyclic-stabilizer divisibility obstruction

Proposition 8. *Let $H \cong C_p$, with p prime, be a subgroup of invertible objects of a complex fusion category. Suppose that X is a simple object fixed on the left by H , Y is simple and fixed on the right, and Z is simple and fixed on both sides. If $\mathrm{FPdim}(Z)/p$ is not an algebraic integer, then p divides N_{XY}^Z .*

Proof. The one-element orbit of X is a rank-one module category over the pointed subcategory Vec_H^ω . Choosing the isomorphisms $hX \simeq X$ shows that ω is a coboundary, so $A = \bigoplus_{h \in H} h$ has its connected separable algebra structure. The obstruction to giving X a left A -module structure lies in $H^2(C_p, \mathbb{C}^\times) = 0$; similarly Y admits a right A -module structure. Thus $X \otimes Y$ is an A -bimodule, with the actions on its two factors.

The A -bimodule block supported on Z is described by a twisted group algebra of $C_p \times C_p$. Its projective irreducible representations have dimension one or p : after rescaling the two generators, their operators satisfy $U^p = V^p = 1$ and $UV = \zeta VU$ with $\zeta^p = 1$. If $\zeta = 1$ the algebra is commutative; otherwise ζ is primitive and every irreducible representation has dimension p .

For a simple bimodule with underlying object mZ , the Frobenius–Perron dimension in the Morita dual category is $m \text{FPdim}(Z) / \text{FPdim}(A) = m \text{FPdim}(Z) / p$; see also the support and dimension argument in [22, Lemmas 6.1–6.2]. Algebraic integrality rules out $m = 1$. Every simple bimodule in this block therefore has underlying object pZ . Decomposing the bimodule $X \otimes Y$ proves the required divisibility. $\square$

Proposition 9. *The rings $R7_4_1212$, $R7_4_1213$, $R7_4_1438$, $R7_4_1440$, and $R7_4_2731$ have no categorification over $\mathbb{C}$.*

Proof. In every row of the following table, the census element $g = X_1$ has order p and fixes X, Y, Z on both sides. The quotient norm is computed in the indicated quadratic dimension field.

ring	p	(X, Y, Z)	N_{XY}^Z	$\text{FPdim}(Z)$	$\text{Norm}(\text{FPdim}(Z)/p)$
$R7_4_1212, R7_4_1213$	2	(X_2, X_2, X_3)	1	$8 + 3\sqrt{5}$	$19/4$
$R7_4_1438, R7_4_1440$	2	(X_2, X_2, X_2)	1	$6 + 3\sqrt{5}$	$-9/4$
$R7_4_2731$	3	(X_3, X_3, X_3)	2	$5 + \sqrt{13}$	$4/3$

A nonintegral norm rules out algebraic integrality. Every displayed multiplicity violates Proposition 8. The source verifier checks the subgroup action and all equations for the exact positive dimension characters. No pivotal, spherical, or unitary hypothesis is used. $\square$

Appendix A supplies an independent signed spectral proof for the first two rings, including a complete exact finite certificate.

4.7 Dimension-block obstructions and their scope

Proposition 10. *The rings $R7_3_695$, $R7_3_783$, and $R7_3_1015$ have no pivotal categorification over $\mathbb{C}$.*

Proof. For a spherical fusion category whose global dimension D generates a real quadratic field of discriminant f , the nontrivial-character case of Schopieray’s dimension-block theorem [34, Theorem 5.1] gives

$$\left(\frac{1}{D} - \frac{1}{D'}\right)^2 \leq \frac{1}{e^2 f} \leq \frac{1}{f},$$

where e is a positive integer. The relevant source dimensions and exact values are

ring	D	f	$(1/D - 1/D')^2$
$R7_3_695$	$420 + 70\sqrt{35}$	140	$1/35$
$R7_3_783, R7_3_1015$	$120 + 30\sqrt{15}$	60	$1/15$

Both conjugates violate the necessary inequality in each case.

It remains to exclude other dimension characters. Their global dimensions must be eigenvalues of the regular Casimir. For $R7_3_695$, the other roots are 5, 7, and $\text{Norm}(D) = 4900$; the norms of $5/D$ and $7/D$ are $1/196, 1/100$. For the other two rings the rational roots are 6, 15, and $\text{Norm}(D) = 900$; the corresponding norms are $1/25, 1/4$. Thus none of the rational alternatives satisfies formal-codegree divisibility [13, Corollary 2.14]. In the noncommutative case it is harmless to include the Casimir root from the matrix block in this necessary superset.

Finally every pivotal dimension character for these rings is spherical. All objects of $R7_3_695$ are self-dual. In the other two rings, the three invertibles form C_3 , a self-dual singleton E satisfies $gE = E$, and the other three simples form one C_3 -orbit. Nonvanishing of $\dim(E)$ forces $\dim(g) = 1$; the three orbit dimensions are equal, so every simple and its dual have the same dimension. $\square$

This proposition alone is a pivotal obstruction, not a general one. The full sphericalization doubles the global dimension, dividing the left-hand side of the displayed inequality by four; the strict violations then become equalities. No general exclusion is inferred solely from this dimension-block argument. The companion verifier keeps this scope explicit and checks all numerical and tensor hypotheses exactly.

4.8 A signed three-parameter argument forcing sphericity

Proposition 11. *The rings $R7_3_783$ and $R7_3_1015$ have no categorification over $\mathbb{C}$.*

Proof. We show that every hypothetical categorification would be spherical; Proposition 10 then gives a contradiction. The two source rings have basis $1, g, g^2, E, \rho_0, \rho_1, \rho_2$, where $g^3 = 1$, $E^* = E$, $\rho_0^* = \rho_0$, and $\rho_i = g^i \rho_0$. Put $G = 1 + g + g^2$ and $S = \rho_0 + \rho_1 + \rho_2$. Their rules are

$$\begin{aligned} gE &= Eg = E, & \rho_0 g &= g^r \rho_0, \\ E^2 &= G + 3E + S, & E\rho_i &= \rho_i E = E + 2S, & \rho_0^2 &= 1 + 2E + 2S, \end{aligned}$$

where $r = -1$ for $R7_3_783$ and $r = 1$ for $R7_3_1015$. These relations generate all products. In census order one takes $g = X_1$, $\rho_0 = X_6$, and respectively $E = X_4, X_5$; the verifier compares all 343 structure constants.

Sphericalize a hypothetical category and pass to the signed quotient. Choose the lift of g with $g^3 = 1$, choose positive-second-indicator lifts of E and ρ_0 , and define the other lifts by $\rho_i = g^i \rho_0$. Since gE has the same underlying simple as E , its signed coefficient is ± 1 . Its cube is one, so $gE = E$. The same argument gives $Eg = E$ and $\rho_0 g = g^r \rho_0$.

The coefficient of ρ_0 in E^2 is one: its original multiplicity is one and the positive-indicator square-channel congruence is modulo four. Left multiplication by g fixes E^2 , forcing the other two orbit coefficients also to be one. Frobenius reciprocity determines the three invertible coefficients. Hence

$$E^2 = G + bE + S, \quad b \in \{-1, 3\}.$$

The element $E\rho_0$ is fixed by left multiplication by g , and $E\rho_i = E\rho_0$. Its support and Frobenius reciprocity give

$$E\rho_i = E + aS, \quad \rho_0^2 = 1 + aE + \sum_{i=0}^2 c_i \rho_i,$$

where $a \in \{-2, 2\}$ by the square-channel congruence, and $|c_i| \leq 2$. Put $c = \sum_i c_i \leq 6$. Multiplying the second identity by G gives $S\rho_0 = G + 3aE + cS$. Comparing the coefficients of S in $E^2\rho_0 = E(E\rho_0)$ yields

$$ba + c = 3a^2.$$

The choices $(a, b) = (-2, -1), (-2, 3), (2, -1)$ require respectively $c = 10, 18, 14$, all impossible. Thus $a = 2$, $b = 3$, and $c = 6$, forcing every $c_i = 2$.

All displayed signed products are therefore the original positive products. The g -relations generate the remaining products, so the chosen seven lifts are closed under tensor product and duality. They form a spherical fusion subcategory of the sphericalization. The forgetful functor restricts to a tensor equivalence with the original category, which is therefore spherical, as required. $\square$

The signed argument is purely theoretical; the companion verification checks only identification with the source tensors and elementary arithmetic. In particular it does not presume that based-ring symmetries furnish a pivotal structure. The remaining ring in Proposition 10, $R7_3_695$, is excluded generally by Proposition 15 below.

4.9 The full induced-unit compression bound

The following refinement keeps all simple constituents of the induced unit in the local compression argument of [34, Lemma 4.1 and proof of Theorem 5.1]. It requires no assertion about a dimensional grading for a general spherical category.

Proposition 12 (Compression over all constituents of $I(\mathbf{1})$). *Let $\mathcal{C}$ be a spherical fusion category, $D = \dim(\mathcal{C})$, $K = \mathbb{Q}(\dim X : X \in \text{Irr}(\mathcal{C}))$, and $G = \text{Gal}(K/\mathbb{Q})$. For an irreducible representation E of $K_0(\mathcal{C}) \otimes \mathbb{C}$, write f_E for its formal codegree. Then $f_E \in K$, and for every character χ of G ,*

$$\sum_E \left| \sum_{\sigma \in G} \chi(\sigma) \sigma(f_E^{-1}) \right|^2 \leq \frac{|G|}{\mathfrak{f}(\chi)}. \quad (1)$$

There is no factor $\dim E$ in the outer sum.

Proof. Let A_E be the simple constituent of $I(\mathbf{1})$ attached to E . The induction theorem [13, Theorem 2.13] gives $\dim A_E = D/f_E$, $\theta_{A_E} = 1$, and distinct E give distinct A_E . Since the forgetful image of every object of the center has dimension in K , this also proves $f_E \in K$. Write $s = S/D$ for the normalized center matrix and P for orthogonal projection onto its twist-one subspace. For the dimension character and its Galois conjugates let A_σ denote the corresponding constituents of $I(\mathbf{1})$, and set $w_\chi = \sum_{\sigma \in G} \chi(\sigma) e_{A_\sigma}$. The Galois compatibility of the modular representation gives $s_{A_\sigma, Y} = \sigma(\dim Y/D)$, and w_χ is fixed by the unipotent subgroup and has torus character χ . The local compression estimate for $SL_2(\mathbb{Z}/N)$, combined over prime powers, gives $\|Psw_\chi\|^2 \leq \|w_\chi\|^2/\mathfrak{f}(\chi) = |G|/\mathfrak{f}(\chi)$. Its A_E coordinate is exactly the inner sum in (1). Retaining these coordinates proves the claim. The multiplicity $[I(\mathbf{1}) : A_E] = \dim E$ does not repeat a coordinate of the modular representation. $\square$

Corollary 13 (Quadratic Casimir obstruction). *Let R be a fusion ring, let N_i be its regular fusion matrices, and let $B = \sum_i N_i N_i^\top$. Fix a real quadratic field L of fundamental discriminant q . List the irreducible quadratic factors of $\det(xI - B)$ whose roots generate L as*

$$\prod_j (x^2 - t_j x + n_j)^{m_j}, \quad Q_L(R) := q \sum_j m_j \frac{t_j^2 - 4n_j}{n_j^2}.$$

Assume that at least one such factor occurs. If R has a spherical categorification, then $Q_L(R) \leq 1$. More precisely, if K is its dimension field, then $L \subseteq K$ and $[K : L]Q_L(R) \leq 1$.

Proof. The Casimir eigenvalue on a simple representation E is $\lambda_E = (\dim E)f_E$, with regular multiplicity $(\dim E)^2$. In particular $L \subseteq K$. Put $e' = [K : L]$ and pull back to $\text{Gal}(K/\mathbb{Q})$ the nontrivial quadratic character of L , whose conductor is q . If $\lambda_E \in L$, its contribution to (1) is $e'^2(\dim E)^2(\lambda_E^{-1} - \overline{\lambda_E}^{-1})^2$. Regular multiplicities therefore account for the squares of representation degrees, including when R is noncommutative. Each displayed factor contributes $2e'^2 m_j(t_j^2 - 4n_j)/n_j^2$. The upper bound is $2e'/q$, which gives the asserted inequality after cancellation. $\square$

Corollary 14 (Boundary constraint for an arbitrary categorification). *If R has any fusion-category categorification, then $Q_L(R) \leq 4$. If $Q_L(R) = 4$, its sphericalization has dimension field exactly L , and every formal codegree of its $\varepsilon = -1$ sector is rational. In particular its actual spherical global dimension is rational.*

Proof. The sphericalization has two lifts of every simple object and a central invertible ε of order two and dimension -1 . The $\varepsilon = 1$ quotient of its fusion algebra is the original fusion algebra; in this sector its Casimir is $2B$. Thus the factors supplied by this sector contribute $Q_L(R)/4$ to the preceding proof. Its dimension field contains L , so $[K : L]Q_L(R)/4 \leq 1$. At equality $Q_L(R) = 4$, necessarily $K = L$, and the known sector already exhausts the squared-norm bound. All other $I(1)$ coordinates must vanish under the nontrivial Fourier character of L , so their formal codegrees are rational. The spherical dimension character belongs to the $\varepsilon = -1$ sector, and its formal codegree is the global dimension. $\square$

4.10 A parity contradiction at the compression boundary

Proposition 15. *The ring $R7_3_695$ has no categorification over $\mathbb{C}$.*

Proof. Its source basis is self-dual and its Casimir polynomial is

$$P(x) = (x - 7)^3(x - 5)^2(x^2 - 840x + 4900).$$

The irrational roots are $D = 420 + 70\sqrt{35}$ and its conjugate. Their field $L = \mathbb{Q}(\sqrt{35})$ has discriminant 140, and

$$Q_L(R) = 140 \frac{840^2 - 4 \cdot 4900}{4900^2} = 4.$$

For a hypothetical categorification, the boundary case of Corollary 14 therefore says that its sphericalization has dimension field $K = L$ and all formal codegrees in the signed sector are rational.

Both lifts of each source simple are self-dual, so the signed quotient is commutative with self-dual basis. Its seven codegrees are eigenvalues of its integer Casimir matrix B_T . They are thus positive integers. Let $d = (d_i)$ be its spherical dimension character and $\delta = \sum_i d_i^2$. The field generated by the d_i is $K = L$, so d and its quadratic conjugate are distinct. Both have the same rational integer codegree δ . Formal-codegree divisibility gives

$$\frac{\delta}{D} = \frac{\delta}{70}(6 - \sqrt{35}) \in \mathcal{O}_L.$$

Since $\mathcal{O}_L = \mathbb{Z}[\sqrt{35}]$, we obtain $70 \mid \delta$.

The signed coefficients agree with the source coefficients modulo two. Hence $B_T \equiv B_N \pmod{2}$ and

$$\det(xI - B_T) \equiv P(x) \equiv x^2(x + 1)^5 \pmod{2}.$$

Exactly two signed codegrees are even, namely the two occurrences of δ . Write the other five, which are odd, as $h_1, \dots, h_5$. Induction integrality gives $\delta/h_i \in \mathbb{Z}$, and the reciprocal-codegree identity yields

$$\frac{2}{\delta} + \sum_{i=1}^5 \frac{1}{h_i} = 1, \quad \sum_{i=1}^5 \frac{\delta}{h_i} = \delta - 2.$$

Since δ is even and every h_i is odd, $\delta/h_i \equiv \delta \pmod{4}$. The last equality would therefore give $5\delta \equiv \delta - 2 \pmod{4}$, a contradiction. $\square$

This argument uses the individual dimension-character field throughout. It does not infer a nontrivial grading from rational spherical global dimension. Its exact source verifier checks the Casimir polynomial, compression equality, inverse formula, and both even residue classes modulo four.

4.11 A dimension-150 central boson

Proposition 16. *The census ring $R_{7,2,160}$ has no categorification over $\mathbb{C}$.*

Proof. Its basis is $1, g, v, X_1, \dots, X_4$, with dimensions $(1, 1, 2, 6, 6, 6, 6)$ and total dimension 150. All basis objects are self-dual, the adjoint subring is the whole ring, and

$$g^2 = 1, \quad gv = v, \quad gX_i = X_i, \quad v^2 = 1 + g + v, \quad vX_i = 2X_i.$$

Suppose that $\mathcal{C}$ realizes this ring. It is integral and pseudo-unitary [28]. Its center is integral and modular. The modular formal-codegree formula implies that every central simple dimension d divides 150: the rational number $150^2/d^2$ is an algebraic integer, so the rational number $150/d$ is an integer.

Write $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ for the forgetful functor and I for its right adjoint. The formal codegrees are 150, 3, 2, 25, 25, 25, 25. Thus $I(1)$ has seven distinct constituents, each of multiplicity one, with dimensions 1, 50, 75, 6, 6, 6, 6 [13, Theorem 2.13]. Each restriction contains 1 once. The formula $FI(W) = \bigoplus_{S \in \text{Irr}(\mathcal{C})} SW S^*$ gives

$$\langle I(1), I(g) \rangle = 5, \quad \langle I(1), I(v) \rangle = 9, \quad \langle I(g), I(g) \rangle = 7.$$

For a dimension-six constituent Y of $I(1)$, write $F(Y) = 1 + bg + cv$; then $b + 2c = 5$, so b is positive and odd. The four such constituents use at least four of the total five copies of g . Consequently all four have $b = 1, c = 2$. If $b_{50}, b_{75}, c_{50}, c_{75}$ are the corresponding coefficients for the other two nontrivial constituents, the induction pairings give $b_{50} + b_{75} = c_{50} + c_{75} = 1$. Reduction of their dimensions modulo six gives $1 + b_{50} + 2c_{50} \equiv 50 \pmod{6}$, hence $(b_{50}, c_{50}, b_{75}, c_{75}) = (1, 0, 0, 1)$. It follows that $I(g)$ shares precisely five multiplicity-one constituents with $I(1)$, of total dimension 74. Its squared norm is seven, and its dimension is 150, so its other two constituents are distinct, of multiplicity one, and have dimensions summing to 76. The only pair of positive divisors of 150 with this sum is $(1, 75)$. A central invertible object $\tilde{g}$ over g is therefore forced.

The actual fusion subcategory $\mathcal{D} = \langle 1, g, v \rangle$ has a rank-one module category generated by any X_i . This gives a fiber functor; semisimple Hopf reconstruction in dimension six identifies $\mathcal{D}$ with $\text{Rep}(S_3)$ [23]. Its universal grading is trivial, so the half-braiding on g is unique: two choices differ by a tensor automorphism of the identity. Its restriction is therefore the symmetric sign object, with self-braiding $+1$. The universal grading of $\mathcal{C}$ is also trivial, so a central invertible over 1 is trivial. Hence $\tilde{g}^2 = 1$, and $\langle \tilde{g} \rangle$ is a central $\text{Rep}(C_2)$.

De-equivariantization yields $\mathcal{C} = \mathcal{E}^{C_2}$ with $\text{FPdim}(\mathcal{E}) = 75$. The equivariantization orbit formula [5] and the sign action give $\mathcal{E}$ the type $1^3, 3^8$: each sign-fixed dimension-two or dimension-six source simple comes from a free orbit of two objects of dimension one or three. For a dimension-three simple Z of $\mathcal{E}$, its invertible stabilizer has order one or three. Order one would leave dimension eight in ZZ^* to be supplied by dimension-three objects, which is impossible. Thus ZZ^* contains all three invertibles and two dimension-three constituents. The adjoint subcategory therefore has dimension $3 + 9k$ with $k > 0$; divisibility by 75 forces it to equal $\mathcal{E}$.

By the two-prime theorem, $\mathcal{E}$ is solvable [7, Theorem 1.6]. Its last prime step cannot be an extension, since its universal grading is trivial; it is an equivariantization by C_3 , since its invertible group is C_3 . The resulting central $\text{Rep}(C_3)$ has these invertibles as its underlying objects. Their half-braidings are unique, so the outer C_2 action preserves this central subcategory. Equivariantizing it embeds a central symmetric category of dimension six into $\mathcal{Z}(\mathcal{C})$: its half-braidings with objects of $\mathcal{E}$ descend to equivariant objects. Forgetting equivariance gives a fiber functor, so this category is $\text{Rep}(H)$ for a group of order six. It has exactly two invertibles, because it contains the canonical $\text{Rep}(C_2)$ and embeds in $\mathcal{C}$. Thus $H = S_3$, and its image is precisely $\langle 1, g, v \rangle$. Direct de-equivariantization has dimension 25, hence is pointed [28, 22]. We obtain

$$\mathcal{C} = (\text{Vec}_A^\omega)^{S_3}, \quad |A| = 25.$$

For a nonidentity element of A in an orbit of length $\ell < 6$, a simple lift has dimension $\ell \dim \pi$, where π is an irreducible projective representation of a stabilizer of order $6/\ell$. It is at most $\sqrt{6\ell} < 6$. Such a lift cannot belong to $\langle 1, g, v \rangle$, whose objects lie over the identity element of A , and every other simple of $\mathcal{C}$ has dimension six. Thus S_3 acts freely on $A \setminus \{1\}$. If $A = C_{25}$, its automorphism group has order twenty and the order-three subgroup acts trivially. If $A = C_5^2$, every involution acting without nonzero fixed vectors is $-I$, since its minimal polynomial divides $(t-1)(t+1)$. All three involutions would therefore have the same image, making their products, the order-three elements, act trivially. Both cases contradict freeness. $\square$

The exact certificate checks the complete source tensor, the adjoint subring, formal codegrees, the three induction pairings, and the small integer and finite-group premises. It does not enumerate a full induction matrix.

5 Three fixed simples and an anti-invariant module

The following argument uses only semisimple module categories and the ordinary fusion rules. In particular, its conclusions are general.

Proposition 17 (Anti-invariant module representation). *Suppose a commutative fusion ring is categorified, an invertible simple g has order two, and exactly f simples $F_1, \dots, F_f$ are fixed by g . There are integral $f \times f$ matrices T_i satisfying*

$$T_0 = I, \quad T_{i^*} = T_i^\dagger, \quad T_i T_j = \sum_k N_{ij}^k T_k,$$

$$|(T_i)_{ab}| \leq N_{iF_a}^{F_b}, \quad (T_i)_{ab} \equiv N_{iF_a}^{F_b} \pmod{2}.$$

Each T_i is normal, and its spectrum is contained in that of N_i .

Proof. A fixed simple gives a rank-one module category over the pointed subcategory generated by g , trivializing its associator. Thus $A = 1 \oplus g$ is a connected separable algebra. Each fixed simple has two right A -module structures, since $H^2(C_2, \mathbb{C}^\times) = 0$; a free orbit gives one simple module. Twisting by the algebra automorphism that is -1 on g exchanges the two structures over each F_a and fixes the free-orbit modules. This involution commutes with the left category action. On its minus eigenspace, use the basis $F_a^+ - F_a^-$. The two nonnegative action coefficients have sum $N_{iF_a}^{F_b}$, so their difference has the asserted bounds and parity. Module reciprocity gives transpose duality. Commutativity then gives normality. The squarefree characteristic polynomial of N_i annihilates b_i in the semisimple complexified fusion ring, and hence annihilates T_i . $\square$

Theorem 18. *None of*

$$R7_2_116, R7_2_261, R7_3_364, R7_3_938, \\ R7_3_970, R7_4_2644, R7_4_2683$$

is categorifiable by a complex fusion category.

Proof. All indices below are source indices, starting with $X_0 = 1$.

For $R7_2_116$ and $R7_2_261$, take respectively $(g, X) = (X_1, X_2)$ and (X_3, X_1) , with fixed ordering (X_4, X_5, X_6) . The bound matrix for T_X is

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The source spectra lie respectively in $\{0, \pm 1, 5\}$ and $\{0, \pm 1, \pm i, 5\}$. The maximum absolute row sum is at most four, excluding eigenvalue five. Normality implies operator norm at most one. Its first row nevertheless contains two entries of absolute value one, a contradiction.

For $R7_3_364$ and $R7_3_938$, take respectively $g = X_1, X_3$, again with fixed ordering (X_4, X_5, X_6) . The self-dual simple X_4 has bound matrix and squarefree source polynomial

$$B = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \quad p(x) = x(x^2 - 8x - 8)(x^2 - 2x - 2).$$

Write its symmetric anti-invariant matrix as

$$T = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \quad a, b, d \text{ odd}, \quad c, e, f \text{ even}.$$

Its integral characteristic polynomial has degree three. Thus its spectrum is three zeros, $\{0, 4 \pm 2\sqrt{6}\}$, or $\{0, 1 \pm \sqrt{3}\}$. The first option contradicts symmetry and odd entries. The second has $\text{tr}(T^2) = 80$, whereas the bounds give $\text{tr}(T^2) \leq 32$. In the last option the trace, trace-square, and determinant are 2, 8, 0. The trace-square and parity force $c = e = 0$, $|a| = |b| = |d| = 1$, and $|f| = 2$. The determinant forces $ad = b^2 = 1$, so $a = d$. Consequently $a + d + f \in \{-4, 0, 4\}$, contradicting trace two.

For $R7_3_970$, take $g = X_3$ and fixed ordering (X_4, X_5, X_6) . The bound matrix for T_1 is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},$$

and its source spectrum lies in $\{0, \pm 1, \pm i, 9\}$. The row-sum bound three excludes nine. Normality and parity therefore give $T_1 = T_2 = D = \text{diag}(0, \varepsilon, \eta)$, where $\varepsilon, \eta = \pm 1$. The respective bounds on T_5, T_6 are

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}.$$

Commutation with D kills both off-diagonal pairs. The remaining diagonal supports are disjoint, so $T_5 T_6 = 0$. The source product $X_5 X_6 = 2X_1 + 2X_2$ instead gives $T_5 T_6 = 4D \neq 0$.

For $R7_4_2644$, take $g = X_3$, fixed ordering (X_4, X_5, X_6) , and test simple X_1 . Its bounds and polynomial are

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad (x-1)(x+1)(x^2+1)(x^2-14x+1).$$

The row-sum bound seven excludes $7 + 4\sqrt{3}$, and rationality of the characteristic polynomial excludes its conjugate. Normality and parity imply $T_1 = T_2 = D$, diagonal with entries ± 1 . The first column of T_5 is zero by the source bounds. Thus $(T_4 T_5)_{00} = 0$, whereas $X_4 X_5 = 2X_1 + 2X_2$ forces this entry to be ± 4 .

Finally, for $R7_4_2683$, take $g = X_5$ and fixed ordering (X_1, X_2, X_6) . The same bounds and polynomial, now for X_3 , give $T_3 = T_4 = D$, diagonal with entries ± 1 . The first row of T_1 is zero, while the source product $X_1^2 = 2X_3 + 2X_4$ forces $(T_1^2)_{00} = 4D_{00} = \pm 4$. This completes all seven contradictions. $\square$

The accompanying source verifier reconstructs the involutions, fixed simples, duality, bound matrices, spectral polynomials, and displayed products directly from the pinned census. The proof does not require a search through associators or sphericalization signs.

6 Morita-dual Gram identities

The same module category gives stronger necessary conditions than its fusion-ring representation alone. These conditions use only Frobenius–Perron dimensions and semisimplicity.

Proposition 19 (Two evaluation Gram identities). *Let $A = 1 \oplus g$ be the involution algebra above, $\mathcal{M} = \mathcal{C}_A$, and $\mathcal{D} = {}_A \mathcal{C}_A$. Write M_i for the action matrices of the source simples on the simple right A -modules, and normalize their positive dimension vector d by $d_A = 1$. The dimensions of the simple objects of $\mathcal{D}$ are obtained from the unique nonnegative integral column decomposition of $I + M_g$: there are two singleton columns at each fixed point of the involutive permutation M_g , and one pair-column at each transposition. A column v gives dimension $\sum_a v_a d_a$.*

For every simple module m , there must also be nonnegative integral columns v_Y , one for each simple Y of $\mathcal{D}$, such that

$$G_m := \sum_i (M_i)_{mm} M_i = \sum_Y v_Y v_Y^\dagger, \quad \sum_a (v_Y)_a d_a = \text{FPdim}(Y) d_m. \quad (\dagger)$$

The column of the unit of $\mathcal{D}$ is e_m .

Proof. The commuting actions of $\mathcal{C}$ and $\mathcal{D}$ make $\mathcal{M}$ a Morita equivalence. Tensor–Hom adjunction for evaluation at m gives the Gram matrix in $(\dagger)$: the internal end of m in $\mathcal{C}$ is $\bigoplus_i (M_i)_{mm} X_i$, and

its action on $\mathcal{M}$ is the composite of evaluation and its adjoint. The columns are the multiplicities of the simples in $m \otimes_A Y$. Positivity of Frobenius–Perron dimensions gives the dimension identity, with $d_m = \text{FPdim}_{\mathbb{C}}(m)/2$. At $m = A$, the internal end is $A = 1 \oplus g$, giving $I + M_g$. Its zero entries separate fixed points and transpositions. A fixed point has diagonal two, so it supplies two unit columns; a transposition supplies a single column with two ones. This proves uniqueness of the decomposition. Every simple bimodule has nonzero underlying right module, so none is omitted. $\square$

Theorem 20. *The seven rings*

$$R7_2_11, R7_2_205, R7_2_275, R7_2_287, \\ R7_3_387, R7_4_2473, R7_3_1055$$

have no categorification over $\mathbb{C}$.

Proof. *The four $\sqrt{2}$ cases.* The exact anti-module equations have one sign-gauge orbit for each of $R7_2_11$, $R7_2_205$, $R7_2_275$, and $R7_2_287$. This small statement is checked independently by the complete source-bound anti-module enumerator. It also follows by using the normality and spectral restrictions of the preceding proposition: for a noninvertible generator, the squared Frobenius norm is at most 20, whereas the conjugate pair $2 \pm 2\sqrt{2}$ contributes 24. The remaining pair $\pm\sqrt{2}$, or $\pm i\sqrt{2}$, forces its even entries to vanish and its four unit couplings to have absolute value one; the fusion relations determine the remaining matrices up to changing the signs of the three anti-module basis vectors.

Put $\delta = 1 + \sqrt{2}$. The resulting right-module dimensions are $1^4, \delta^4$, and the dual dimensions are $1^8, \delta^8$. For a recorded module simple of dimension δ , the full internal-end Gram matrix restricted to the four δ coordinates is

$$G_\delta = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 4 \end{pmatrix}.$$

Let $S(Q) = \text{tr}(Q) - 2 \sum_{i < j} Q_{ij}$ on this block. A dual invertible acts by a column whose δ coordinates sum to one, so its contribution to S is one. A dual simple of dimension δ has column dimension $\delta^2 = 1 + 2\delta$; hence its δ coordinates are nonnegative integers summing to two. Its contribution to S is either zero or four. All contributions are therefore nonnegative, whereas $S(G_\delta) = 16 - 24 = -8$. This contradiction excludes all four cases without an optimizer or any pivotality assumption.

The integral case $R7_3_387$. The bound and spectrum of T_2 give $T_2 = \text{diag}(0, \varepsilon, \eta)$ with $\varepsilon, \eta = \pm 1$, exactly as in the dimension-240 case of Theorem 18. Commutation kills the possible off-diagonal entries of T_5, T_6 , leaving disjoint diagonal supports. The product $X_5 X_6 = 2X_2 + 2X_3$ gives $T_3 = -T_2$. Since $gX_2 = X_3$, the last two entries of T_g are -1 . As g fixes X_4, X_5, X_6 , their last two diagonal entries vanish. The source spectrum and bounds force $T_4 = \text{diag}(-1, 0, 0)$; then $X_4^2 = 1 + g + X_4$ forces $T_g = \text{diag}(1, -1, -1)$. Thus $T_5 = T_6 = 0$, and precisely the four choices of (ε, η) remain. In all four, the module dimensions are

$$(1, 9, 1, 1, 3, 3, 3, 3),$$

and the dual dimensions determined by $I + M_g$ are $(1^6, 6^2, 9^2)$. At the first dimension-three module, the full matrix G_m has rows 0, 2, 3 identical, with common diagonal entry one. They force a unique column

$$v = (1, 1, 1, 1, 1, 1, 0, 0)^t.$$

Its dimension is eighteen, so it corresponds to one of the dual simplices of dimension six. After subtracting $vv^\dagger$, rows 0, 2, 3 vanish. The six invertible columns each have exactly one entry one among coordinates 4, 5, 6, 7, and vanish at coordinate one. For the remaining three columns, coordinate one's squared entries sum to twelve. The only expression of twelve as three nonnegative integer squares is $4 + 4 + 4$. Hence each entry there is two. The remaining dual dimension-six column has no other nonzero entry. The two dimension-nine columns have last-four-coordinate vectors w, z with

$$\sum w_a = \sum z_a = 3, \quad w + z = (1, 1, 2, 2).$$

The (4, 5) Gram entry, after the forced column is removed, is one. It gives $w_4 = w_5 \in \{0, 1\}$. The (4, 6) and (4, 7) entries are also one, forcing $w_6 = w_7 = 1$. Thus $\sum w_a = 2w_4 + 2 \in \{2, 4\}$, contradicting three. The entries used are identical in all four anti-invariant branches.

Two remaining finite Gram certificates. For $R7_4_2473$ and $R7_3_1055$, the complete anti-module systems have respectively seven and one sign-gauge orbits. For each orbit the verifier reconstructs the entire finite column domain of one internal-end Gram matrix, including its exact FP dimension constraint, and checks an integral separating vector y with $y^T A \geq 0$ and $y^T b < 0$. All eight systems are impossible. The public verifier reopens the census, regenerates all anti-module orbits and all Gram columns, and checks the frozen inequalities with exact arithmetic; it never invokes an optimizer. Together with the four cases above and the dimension-240 argument, this gives seven further general complex exclusions. $\square$

6.1 A positive-semidefinite column refinement

The rank-one columns in Proposition 19 satisfy an additional exact constraint. This sharpens the finite Gram systems without introducing categorical assumptions.

Proposition 21 (Rank-one positive-semidefinite constraint). *Let Q be a nonzero rational symmetric positive-semidefinite matrix, let I index a maximal independent set of columns, and put $P = Q_{I,I}$. A column v can occur in a decomposition $Q = \sum_j v_j v_j^\dagger$ only if*

$$v = Q_{:,I} P^{-1} v_I, \quad v_I^\dagger P^{-1} v_I \leq 1. \quad (\ddagger)$$

These two conditions are equivalent to $Q - vv^\dagger \succeq 0$. A nonpositive-semidefinite Q admits no such column decomposition.

Proof. Every summand has range contained in that of Q , giving the first identity. Since Q is positive semidefinite and its selected columns are independent, P is positive definite. Set $B = Q_{:,I} P^{-1}$. The rank identity $Q = B P B^\dagger$ and $v = B v_I$ reduce the condition to $P - v_I v_I^\dagger \succeq 0$. Conjugating by $P^{-1/2}$ gives $I - w w^\dagger \succeq 0$, where $w = P^{-1/2} v_I$, exactly the second inequality. These are rational identities and inequalities: a verifier can clear denominators and use positive principal minors, without numerical eigenvalues. $\square$

Theorem 22. *The four rings $R7_4_776$, $R7_4_927$, $R7_4_2126$, and $R7_3_909$ have no categorification over $\mathbb{C}$.*

Proof. Proposition 17 gives a finite exact anti-module system for each source tensor. In the stated order, these systems have 55, 20, 12, and 27 sign-gauge orbits. For each orbit, take a recorded simple module m and remove the unit column from its evaluation Gram matrix: $Q = G_m - e_m e_m^\dagger$. The nonnegative integral columns already satisfy the dimension and coordinate bounds of Proposition 19.

Retain only those satisfying (§). The remaining Gram and dimension-class count equations form a finite system $Ax = b$, $x \geq 0$.

All 114 systems have frozen integral separating vectors satisfying $y^t A \geq 0$ and $y^t b < 0$. The public verifier reopens the source census, regenerates every anti-module orbit, reconstructs each full column domain, applies the rational range and positive-semidefinite tests, and checks these inequalities. It uses no optimizer and no numerical spectral comparison. Thus every necessary module branch is impossible, proving the four general exclusions. $\square$

7 Two Morita transformations and an A_4 obstruction

The following argument uses actual module categories of a hypothetical categorification. Its finite checks only verify the source products, eight alternating forms, and the elementary ring normalization.

Theorem 23. *The rings $R7_2_236$, $R7_4_2167$, and $R7_4_2716$ have no categorification over $\mathbb{C}$.*

Proof. Their dimension vectors are $(1, 1, 1, 3, d, d, d)$, with $d^2 = 4 + md$ and $m = 6, 12, 12$, respectively. The invertibles form $C_3 = \langle g \rangle$, fixing the dimension-three simple X and permuting the other three simples freely. The source products give $X^2 = 1 + g + g^2 + 2X$.

The first Morita transformation. The fixed simple X trivializes the pointed associator, so the regular algebra $A = 1 + g + g^2$ exists. Since $H^2(C_3, \mathbb{C}^\times) = 0$, X has both left and right A -module structures. Thus $X \otimes X$ is an A -bimodule. Its X -isotypic part is $2X$. A nontrivial projective class for $C_3 \times C_3$ has irreducible degree three, so this multiplicity-two part forces the class on X to be trivial. It gives nine simple bimodules with underlying object X , each of dual dimension one. The pointed orbit gives three more invertibles. The remaining orbit has cyclic stabilizer and gives three simples of dimension d .

Consequently $\mathcal{C}_A$ has dimensions $1^4, d$, and its Morita dual $\mathcal{D}$ has dimensions $1^{12}, d^3$. Already this excludes $m = 6$: the published [35, Proposition 3.3] requires $s \mid r$ for a category with two simple dimensions $1, d$ and irrational d satisfying $d^2 = s + rd$. Here $4 \nmid 6$.

For the two remaining cases, write K for the twelve invertibles of $\mathcal{D}$. They fix the unique d -dimensional object of $\mathcal{C}_A$; its rank-one module structure trivializes the whole pointed associator. Thus $\mathcal{D}_{\text{pt}} \simeq \text{Vec}_K$. The integral subcategory of $\mathcal{C}$ has rank four, and its A -module category has four simple objects on which K acts transitively. A stabilizer L has order three. If L were normal, the dual of Vec_K on this module category would have four double cosets, each contributing three characters of L , hence rank twelve instead of four. Therefore L is nonnormal. Its core is trivial because $|L| = 3$, so the action embeds K as an index-two subgroup of S_4 ; hence $K = A_4$.

For a noninvertible Y of $\mathcal{D}$, comparison of rational and irrational coefficients in $\text{FPdim}(YY^*) = 4 + 12d$ shows that YY^* contains exactly four invertibles. Each occurs with multiplicity at most one by rigidity. Thus both stabilizers are the unique subgroup $H = V_4$ of A_4 , and the three noninvertibles form one orbit.

The second Morita transformation. At least one of these three simples, say Y , is self-dual. The projective $H \times H$ stabilizer of Y has an alternating commutator form on $\mathbb{F}_2^4$. Duality exchanges the left and right factors and inverts the commutator. Its values are ± 1 , so in matched bases the form is

$$\beta = \begin{pmatrix} bJ & C \\ C & bJ \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = C^t.$$

This uses rigidity, with no pivotal assumption. The two algebra structures on $\bigoplus_{h \in H} h$ differ by the unique nontrivial class in $H^2(H, \mathbb{C}^\times)$, adding $\text{diag}(J, J)$ to β . For the eight symmetric matrices C , the two ranks are 2, 4 in some order, except for $C = 0$, when they are 0, 4. The rank-zero alternative gives a simple bimodule of dimension $d/4$. Its monic minimum polynomial has constant coefficient $-1/4$, contradicting algebraic integrality. Hence one algebra choice gives rank two. The corresponding twisted group algebra has four irreducible representations of degree two, giving four simple bimodules over each Y , of dimension $t = d/2$. Transport by A_4 preserves the unique nontrivial H -cocycle class, so the same degree holds over all three noninvertibles.

The pointed Morita dual is again Vec_{A_4} for either algebra choice. For the untwisted algebra this is Fourier duality for $V_4 \rtimes C_3$. The other cocycle extends to A_4 through $Q_8 \rtimes C_3 \rightarrow A_4$, so an identity-on-objects tensor autoequivalence of the pointed subcategory identifies the two pointed duals. No extension of this autoequivalence to $\mathcal{D}$ is assumed. The resulting category $\mathcal{D}'$ therefore has dimensions $1^{12}, t^{12}$, with invertible group A_4 and $t^2 = 1 + 6t$. For every noninvertible Z , comparison of rational coefficients in $\text{FPdim}(ZZ^*) = 1 + 6t$ gives a trivial stabilizer. Thus these twelve objects form a free A_4 -orbit.

Normalization without a self-duality assumption. Choose any noninvertible ρ in $\mathcal{D}'$. Right multiplication is free as well, since duality turns a right stabilizer into a left stabilizer. Write

$$\rho g = \phi(g)\rho, \quad \rho^2 = a + N\rho, \quad N \in \mathbb{Z}_{\geq 0}[A_4], \quad \sum_g n_g = 6.$$

Rigidity gives a unique invertible summand a . Associativity gives

$$\phi^2 = \text{Ad}_a, \quad \phi(a) = a, \quad Ns = \phi(s)N, \quad \phi(N) = N,$$

and then $\rho^* = a^{-1}\rho$ and $N^* = a^{-1}N$. Since $\text{Aut}(A_4) = S_4$ and $Z(A_4) = 1$, write $\phi = \text{Ad}_\sigma$; necessarily $a = \sigma^2$. Replacing ρ by $b\rho$ replaces (σ, a, N) by $(b\sigma, (b\sigma)^2, bN)$. Choose $b = \sigma^{-1}$ when σ is even, and $b = \tau\sigma^{-1}$ otherwise, with τ a transposition. In either case $b \in A_4$, the new a is one, and the new ρ is self-dual. The two normalized actions are therefore the identity and conjugation by τ .

For the identity action, centrality and self-adjointness give

$$N = A1 + B \sum_{h \in V_4 \setminus \{1\}} h, \quad (A, B) = (6, 0), (3, 1), (0, 2).$$

The two size-four three-cycle classes would have equal coefficients, so neither can occur in a sum of size six. The first choice contains the forbidden rank-two subcategory $\rho^2 = 1 + 6\rho$ [32]. For the other choices, N acts on the three-dimensional representation of A_4 by $A - B = \pm 2$. Taking ρ to act by a root of $x^2 \mp 2x - 1$ gives the formal codegrees

$$16 + 8\sqrt{2}, \quad 16 - 8\sqrt{2}.$$

For the transposition action, τN is constant on the two A_4 -conjugacy classes in the odd coset of S_4 : the six transpositions and the six four-cycles. It must be one of their class sums. On the standard three-dimensional representation of S_4 , these class sums act by 2 and -2 , respectively. Taking $\rho = rT$, where T represents τ and $r^2 = 1 \pm 2r$, again gives the displayed formal codegrees. In both computations the Casimir is $12(1 + \rho^2)$ and its scalar equals the representation degree times its formal codegree.

Finally, the published [35, p. 1628, Lemma 3.1] states that every category with two simple Frobenius–Perron dimensions is Galois conjugate to a pseudo-unitary category, with the same

based ring. In this conjugate, the central constituent A_E associated to an irreducible representation has dimension D/f_E , where $D = \text{FPdim}(\mathcal{D}')$ [13, Theorem 2.13]. Its forgetful image has dimension in $\mathbb{Z} + \mathbb{Z}t$, so every f_E must belong to $\mathbb{Q}(t)$. Here $t = 3 + \sqrt{10}$, whereas $16 \pm 8\sqrt{2}$ do not belong to $\mathbb{Q}(\sqrt{10})$. This contradiction excludes both remaining source rings. $\square$

7.1 The final case: inverse V_4 Morita duality

The surviving quadratic A_4 ring at $m = 8$ cannot reconstruct the rank-seven source. The obstruction is independent of uniqueness or unitarity of its categorifications.

Lemma 24. *Let $\mathcal{E}$ be a complex fusion category with invertibles $H = V_4$, trivial pointed associator, and four other simples $h\rho$, where*

$$\rho h = h\rho, \quad \rho^2 = 1 + \sum_{h \in H} h\rho.$$

The commutation cocycle of ρ with H is trivial in $H^2(H, \mathbb{C}^\times)$.

Proof. Choose coherent multiplication maps for H and isomorphisms $h\rho \rightarrow \rho h$. Their scalar defect is a 2-cocycle ξ ; its cohomology class is independent of these choices [36, Section 2]. Let $P = \bigoplus_{h \in H} h$ be the regular algebra. In the bicategory of separable algebras, write ι for the canonical module and $\bar{\iota}$ for its adjoint, so $\bar{\iota}\iota = \bigoplus_h h$, and put $Z = \iota\rho\bar{\iota}$. Moving the H labels past ρ identifies

$$\text{End}(Z) \cong \mathbb{C}_\xi[H].$$

This is the algebraic calculation in [36, Proposition 9.6]; it requires only associativity and rigidity, not a C^* structure.

If $[\xi] \neq 0$, the twisted group algebra is $M_2(\mathbb{C})$, hence $Z = 2\sigma$ for a simple σ . The pointed Morita dual is $\text{Rep}(H) = \text{Vec}_{\widehat{H}}$. The noninvertible orbit has just this one projective stabilizer representation, and

$$Z^2 = 4\iota\rho^2\bar{\iota} = 4 \sum_{\chi \in \widehat{H}} \chi + 16Z.$$

Consequently $\sigma^2 = \sum_{\chi \in \widehat{H}} \chi + 8\sigma$. This would categorify the near-group ring $R(V_4, 8)$, whereas [35, Theorem 1.1] allows only levels 0 and 4 for V_4 . That theorem concerns arbitrary complex categorification. Replacing the C^* -only final citation of Izumi's argument by this general result proves the lemma in the stated scope. $\square$

Theorem 25. *The ring $R7_3_1022$ has no categorification over $\mathbb{C}$.*

Proof. Its dimensions are $1, 1, 1, 3, d, d, d$, where $d = 4 + 2\sqrt{5}$ and $d^2 = 4 + 8d$. The two Morita transformations of Theorem 23, with $m = 8$, give

$$\mathcal{D} : \quad 1^{12}, d^3, \quad \mathcal{D}' = \mathcal{D}_B^* : \quad 1^{12}, t^{12}, \quad t = d/2 = 2 + \sqrt{5},$$

where B is one of the regular V_4 algebras and both pointed categories are untwisted Vec_{A_4} . The same quadratic ring normalization has no outer branch at $k = 4$; its inner branches are $N = 4 \cdot 1$ and $N = \sum_{h \in V_4} h$. The first contains a forbidden rank-two subcategory. Thus in $\mathcal{D}'$

$$\rho g = g\rho \quad (g \in A_4), \quad \rho^2 = 1 + \sum_{h \in V_4} h\rho.$$

The subcategory generated by V_4 and ρ satisfies Lemma 24.

The inverse algebra is forced by the actual Morita context. In $\mathcal{M} = \mathcal{D}_B$, the regular right module $m = B$ is simple since $\text{End}_B(B) = \text{Hom}(1, B) = \mathbb{C}$. Let $Q = \underline{\text{End}}_{\mathcal{D}'}(m)$. Double centralizer Morita duality recovers $\mathcal{D}$ as $(\mathcal{D}')^*_Q$, up to the harmless opposite convention. For any simple B -bimodule X , its multiplicity, or that of X^* according to convention, in Q is

$$\dim \text{Hom}_{\mathcal{M}}(B, B \otimes_B X) = \dim \text{Hom}_B(B, X) = \dim \text{Hom}_{\mathcal{D}}(1, X).$$

Exactly the four invertible bimodules supported on the original V_4 contain 1. Each has underlying object B , with multiplicity one on 1, for either cocycle defining B . Every other bimodule has support outside that pointed orbit. Hence Q is the sum of the four invertibles of the unique subgroup $H' = V_4 \subset A_4$, with one of its two algebra cocycles, say β .

The twelve noninvertibles of $\mathcal{D}'$ divide into three free H' -orbits of size four. At $g\rho$ the left-right stabilizer is the graph of conjugation by g on H' . Its projective class is the transported commutation class, plus $[\beta] - (\text{Ad}_g)^*[\beta]$. The first term vanishes by the lemma. The second vanishes because $H^2(V_4, \mathbb{C}^\times) = C_2$, on which every automorphism acts trivially. Each orbit therefore has four one-dimensional stabilizer representations. They give four simple Q -bimodules with underlying dimension $4t$ and dual dimension t .

Thus $(\mathcal{D}')^*_Q$ has twelve noninvertibles of dimension t . It cannot be equivalent to $\mathcal{D}$, which has three noninvertibles of dimension $2t$. This contradiction proves the theorem. $\square$

8 A common reduction at the compression boundary

The next proposition removes a pivotality gap for three quadratic rings. It supplies the general-complex scope for the common exclusion in Theorem 27.

Proposition 26. *Every complex fusion categorification of $R7_4_106$, $R7_4_2155$, or $R7_3_146$ is Galois conjugate to a pseudo-unitary category.*

Proof. Write g for the nontrivial invertible, a, ga and b, gb for its two free pairs, and f for its fixed noninvertible. The source labels (g, a, b, f) are $(1, 2, 3, 6)$, $(3, 1, 5, 4)$, and $(1, 2, 3, 6)$, respectively. Duality is trivial except that $a^* = ga$ in the second ring. Pass to the signed algebra T of the canonical sphericalization, at $\varepsilon = -1$. Choose the lift of g with dimension 1. Then $g^2 = 1$, since the other lift of the unit has dimension -1 . The same nonzero-dimension argument makes g central and gives $gf = f$. Choose the free-pair lifts compatibly with multiplication by g . The quotient $T/(g-1)$ has basis $1, a, b, f$ and trivial duality, including in the second case. It is therefore commutative. Frobenius reciprocity gives

$$\begin{aligned} a^2 &= 1 + Aa + Bb + rf, & ab &= Ba + Cb + wf, \\ b^2 &= 1 + Ca + Db + zf, & af &= 2ra + 2wb + lf, \\ bf &= 2wa + 2zb + mf, & f^2 &= 2 + 2la + 2mb + nf. \end{aligned}$$

The coefficient pairing has weights $1, 1, 1, 2$; the factors two come from the free pairs. Each coefficient is an integer of the same parity as its source bound and of absolute value at most that bound. In order $(A, B, C, D, r, w, z, l, m, n)$ the bounds are

$$(0, 3, 4, 8, 1, 2, 4, 2, 3, 4), \quad (3, 3, 4, 4, 2, 2, 2, 2, 3, 2),$$

for the $\sqrt{3}$ pair and the $\sqrt{6}$ case, respectively. Independent sign changes of a, b, f allow $A > 0$ whenever A is odd, $B > 0$, and $r \geq 0$. In the first family $A = 0$ and $r = 1$.

Associativity is equivalent to these six equations:

$$\begin{aligned}
AC - B^2 + BD - C^2 + 2rz - 2w^2 + 1 &= 0, \\
Al + Bm - l^2 + nr - 2r^2 - 2w^2 + 1 &= 0, \\
Aw - Br + Bz - Cw - lw + mr &= 0, \\
Bl + Cm - lm + nw - 2rw - 2wz &= 0, \\
Bw - Cr + Cz - Dw - lz + mw &= 0, \\
Cl + Dm - m^2 + nz - 2w^2 - 2z^2 + 1 &= 0.
\end{aligned}$$

They have eight and eleven solutions in the respective normalized domains. To obtain the complete lists, $B \in \{1, 3\}$ makes the first equation determine D ; if $r \neq 0$, the third and second determine m, n . If $r = 0$, only their original small domains remain. The source-bound verifier records all nineteen rows and checks every multiplication-matrix identity.

Let δ be the global dimension of the original category. Its signed dimension character factors through this quotient. If its values on a, b, f are u, v, w_0 , then $\delta = 2 + 2u^2 + 2v^2 + w_0^2$, also in the non-self-dual case because $a^* = ga$. Thus δ is an eigenvalue of

$$Q = 2I + 2M_a^2 + 2M_b^2 + M_f^2.$$

The induced-unit theorem for the sphericalization gives $2\delta/(2f_E) = \delta/f_E$ algebraically integral for every source formal codegree f_E [13, Theorem 2.13]. One is $48(7 + 4\sqrt{3})$ in the first family and $48(5 + 2\sqrt{6})$ in the second. Each parenthesized factor is an algebraic unit of norm one, so multiplication by it proves that $\delta/48$ is integral. The rational source codegrees alone have least common multiple 24 and would not give this stronger divisibility.

For an irreducible factor $p(x)$ of $\det(xI - Q)$, a root divided by 48 is integral exactly when the monic polynomial proportional to $p(48x)$ has integer coefficients. Only the following rows survive:

family	$(A, B, C, D, r, w, z, l, m, n)$	$p(x)$
$\sqrt{3}$	$(0, 3, -4, 8, 1, -2, 4, -2, 3, 4)$	$x^2 - 672x + 2304$
$\sqrt{3}$	$(0, 3, 4, 8, 1, 2, 4, 2, 3, 4)$	$x^2 - 672x + 2304$
$\sqrt{6}$	$(3, 3, 4, 4, 2, 2, 2, 2, 3, 2)$	$x^2 - 480x + 2304$

Consequently $\delta = 336 \pm 192\sqrt{3}$ or $\delta = 240 \pm 96\sqrt{6}$, respectively. These are the source Frobenius–Perron dimensions and their quadratic conjugates. Global dimension is Galois-equivariant, and the based ring does not change under Galois conjugation [28]. Choosing the conjugation that sends δ to the Frobenius–Perron value gives a pseudo-unitary category. $\square$

Their full induced-unit compression quantity equals one, satisfying the necessary bound. Saturation concerns the projection of the modular S -image onto the twist-one space; it does not put the full image there. The stronger equality constraint below, combined with two induction trace moments, will exclude all three rings in the next section.

8.1 A stronger consequence of compression equality

There is nevertheless a useful restriction at equality. Let S, T be a unitary representation of $SL_2(\mathbb{Z}/N\mathbb{Z})$ and let P project onto the T -fixed space. Suppose w is fixed by T and has diagonal-torus character χ of conductor q . Then

$$\|PSw\|^2 = \|w\|^2/q \implies T^q Sw = Sw.$$

In fact the cyclic subrepresentation generated by w factors through level q . To see this, refine the local proof of [34, Lemma 4.1]. At a prime power ℓ^a , each irreducible with a unipotent-fixed vector embeds in the permutation representation on primitive vectors. For a nontrivial character of conductor ℓ^c , the maximal singular subspace of the compressed Weyl operator is spanned by the two functions denoted F_α and $G_{\bar{\alpha}}$ there. Both depend only on the coordinates modulo ℓ^c . Every irreducible containing an equality vector thus inflates from that level. For a trivial local character, equality says that both w and Sw are unipotent-fixed; upper and lower unipotents then fix w , so its local representation is trivial. Orthogonal irreducible decomposition and the tensor-product description over prime powers give the assertion. The complete local argument and an independent audit are retained in the compression supplement.

For the three rings above, use their pseudo-unitary Galois conjugates, put $q = 12, 12, 24$, respectively, and let $w = e_1 - e_{A_\sigma}$ be the difference of the two dimension-character constituents of $I(1)$. Their dimension field is quadratic of conductor q , and their other formal codegrees are rational. On the known induced-unit coordinates,

$$\text{proj}_{I(1)}(Sw) = \left(D^{-1} - \bar{D}^{-1}\right)w = -\frac{1}{\sqrt{q}}w.$$

This already attains the compression bound, so the preceding argument applies to the full twist-one projection. The coordinate at a simple central object V is $\dim(V)/D - \sigma(\dim(V)/D)$. Consequently

$$\dim(V)/D \notin \mathbb{Q} \implies \theta_V^q = 1.$$

This constrains twists of specific central objects; it does not assert that every central twist has order dividing q . The next section keeps that distinction when applying the two-moment obstruction.

9 One two-moment obstruction for the quadratic boundary cases

Theorem 27. *The rings $R7_4_106$, $R7_4_2155$ and $R7_3_146$ admit no complex fusion categorification.*

The proof combines full forgetful-row Galois transport, equality in the modular compression bound, and a finite hull for two root-of-unity moments. Four fixed integer inequalities suffice for all cases. Proposition 26 permits us to work with a pseudo-unitary Galois conjugate and its canonical spherical structure; no unitarity assumption on the original category is made.

9.1 Source dimensions and complete row transport

Write $d_i = a_i + b_i\sqrt{r}$ and $D = \sum_i d_i^2 = 48\eta$. In source order the dimension vectors are

ring	$(d_0, \dots, d_6)$	η
106	$(1, 1, 4 + 2\sqrt{3}, 7 + 4\sqrt{3}, 7 + 4\sqrt{3}, 4 + 2\sqrt{3}, 6 + 4\sqrt{3})$	$7 + 4\sqrt{3}$
2155	$(1, 4 + 2\sqrt{3}, 4 + 2\sqrt{3}, 1, 6 + 4\sqrt{3}, 7 + 4\sqrt{3}, 7 + 4\sqrt{3})$	$7 + 4\sqrt{3}$
146	$(1, 1, 4 + 2\sqrt{6}, 5 + 2\sqrt{6}, 5 + 2\sqrt{6}, 4 + 2\sqrt{6}, 6 + 2\sqrt{6})$	$5 + 2\sqrt{6}$

Here $\eta\bar{\eta} = 1$. The formal codegrees are $D, \bar{D}, 4, 8, 8, 8, 12$ in the first two cases and $D, \bar{D}, 4, 6, 6, 8, 12$ in the third. Since the rings are commutative, the seven distinct constituents A_E of $I(1)$ have multiplicity one, twist one and dimensions D/f_E [13, Theorems 2.5 and 2.13].

For a center simple A , let $v(A)_i = [F(A) : X_i]$. Induction gives

$$B_{ij} = \sum_{k,l} N_{ki}^l N_{kj}^l = \sum_A v(A)_i v(A)_j. \quad (2)$$

If $z = \dim A = \sum_i v_i d_i$, modular formal-codegree integrality makes D^2/z^2 , hence D/z , algebraically integral. Since $D/48$ is a unit and $\mathcal{O}_{\mathbb{Q}(\sqrt{r})} = \mathbb{Z}[\sqrt{r}]$, every $z = a + b\sqrt{r}$ satisfies

$$a^2 - rb^2 \mid 48a, \quad a^2 - rb^2 \mid 48b. \quad (3)$$

Normalize the center matrix by $S_{00} = 1/D$. The source characters and the induced-unit constituents satisfy

$$\chi_E(F(A)) = \frac{S_{A,A_E}}{S_{0,A_E}}, \quad S_{0,A_E} = 1/f_E$$

by the definition preceding [13, Theorem 2.13]. Modular Galois symmetry consequently gives

$$\chi_E(F(\pi_\sigma A)) = \epsilon_\sigma(A) \frac{f_E}{f_{\sigma E}} \chi_{\sigma E}(F(A)). \quad (4)$$

The integral vector of multiplicities in $I(1)$ satisfies $Sm = m$ [13, Proposition 2.8]. Its signed Galois image is therefore itself: this support is preserved and its signs are positive.

We choose σ to flip the dimension field and fix all other source characters. This requires more than rationality of their codegrees. Set $M = \sum_i w_i N_i$, with

$$w = (1, 3, 11, 37, 101, 307, 997).$$

Its characteristic polynomials are

$$\begin{aligned} 106 : & (x - 1860)(x + 66)(x + 494) \\ & \cdot (x^2 - 16448x - 12738752)(x^2 - 124x - 171388), \\ 2155 : & (x - 1314)(x - 654)(x + 1064) \\ & \cdot (x^2 - 19656x + 889872)(x^2 + 1452x + 527204), \\ 146 : & (x - 654)(x - 184)(x + 66)(x + 234)(x + 1852) \\ & \cdot (x^2 - 15896x + 12501688). \end{aligned}$$

All roots are distinct. The simultaneous eigenspaces are one-dimensional and defined over the corresponding eigenvalue fields. Apart from the dimension field, the character fields are respectively $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}$. Their disjointness from $\mathbb{Q}(\sqrt{r})$ proves that the required cyclotomic automorphism exists.

Put $a(v) = \sum_i a_i v_i$, $b(v) = \sum_i b_i v_i$ and $\mathbf{b} = (b_i)$. Character orthogonality makes $\mathbf{b}$ vanish on every character except the two dimension characters. Evaluation on that pair in (4) gives the integral involution

$$T(v) = \begin{cases} v + \mathbf{b}(4a(v) - 7b(v))/4, & r = 3, \\ v + \mathbf{b}(2a(v) - 5b(v))/2, & r = 6. \end{cases} \quad (5)$$

Every actual row has partner $T(v)$ or $-T(v)$, with a common sign making all coordinates nonnegative. Mixed-sign images are impossible. The multiplicities of paired row types agree; we do not assume that the Galois permutation of individual center simples is an involution.

In particular, besides the unit row, $I(1)$ contains the forced row

$$(1, 0, 2, 4, 4, 2, 4), \quad (1, 2, 2, 0, 4, 4, 4), \quad (1, 0, 2, 2, 2, 2, 2),$$

respectively, of dimension η^2 . The five other nonunit rows have their known dimensions, sum $B_{0,*} - e_0$ after including the forced row, and positive semidefinite residual Gram matrix. Exhaustive bounded integer enumeration gives 194, 194, 1323 blocks. Every bound comes from (2); no putative center is supplied as input.

9.2 Equality restricts the nonrational twist coordinates

Let $q = 12$ for $r = 3$ and $q = 24$ for $r = 6$, and let J be the Galois image of the central unit. Every automorphism fixing the dimension field fixes the central unit: all codegree ratios in (4) are then one, so its forgetful row is e_0 . Such a center simple lies in $I(1)$, where the dimension-one constituent is unique. Those flipping the field send the unit to J . Thus $w = e_1 - e_J$ has diagonal-torus character the primitive quadratic character χ_q . On the known twist-one coordinates,

$$\text{proj}_{I(1)} Sw = \frac{\eta^{-1} - \eta}{48} w = -\frac{1}{\sqrt{q}} w.$$

This attains the full compression bound. The equality argument in Section 8 therefore places the cyclic representation generated by w at level q .

We also need the sign of the primitive twist modes. More generally, if $PSw = \delta w / \sqrt{q}$ with $\delta = \pm 1$, a nonzero primitive q -th twist mode has exponent k satisfying $\chi_q(k) = \delta$. Here is the local Fourier refinement. For a prime-power conductor q_i , the equality subspace is spanned by

$$F_i(p, u) = \chi_i(u)[u \text{ a unit}], \quad G_i(p, u) = \chi_i(-p)[u = 0].$$

The compressed Weyl operator sends F_i to G_i and G_i to $\chi_i(-1)F_i/q_i$. If τ_i is the local quadratic Gauss sum, an eigenvector is $F_i + c_i G_i$ with $c_i = \varepsilon_i \chi_i(-1)\tau_i$ and eigenvalue $\varepsilon_i \tau_i/q_i$. On a unit- u orbit, put $x = p/u$. The primitive Fourier coefficient of its Weyl image is proportional to

$$\chi_i(-1)\tau_i(\chi_i(k) + \varepsilon_i).$$

It vanishes unless $\chi_i(k) = \varepsilon_i$. The Chinese-remainder identity $\sqrt{q} = (\prod_i \chi_i(q/q_i)) \prod_i \tau_i$ shows that the local signs combine to $\chi_q(k) = \delta$. The assertion is preserved under linear combinations in the common global eigenspace.

The coordinate of Sw at a center simple of dimension z is $z/D - \sigma(z/D)$. Consequently

$$z/D \notin \mathbb{Q} \implies \theta^q = 1, \quad \chi_q(k) = -1 \text{ if } \theta = e^{2\pi i k/q} \text{ is primitive.} \quad (6)$$

Moreover, modular Galois symmetry sends twists to their u^2 -th powers. Every unit modulo 12 or 24 squares to one. The paired nonrational rows therefore have equal twists, so their multiplicities agree separately for each twist. Rational ratios are fixed by T and require no such restriction on the order of their twists.

9.3 A finite two-moment hull with no order bound

Let $t(\theta)$ be normalized relative trace to $K = \mathbb{Q}(\sqrt{r})$, taken from any cyclotomic extension containing K and θ .

Lemma 28. *For every root of unity, the pair $(t(\theta), t(\theta^2))$ belongs to the convex hull of the following nine coefficient vectors, where (a, b, c, d) means $(a + b\sqrt{r}, c + d\sqrt{r})$:*

$$\begin{aligned} (1, 0, 1, 0), \quad (-1, 0, 1, 0), \quad (1/2, 0, -1/2, 0), \quad (-1/2, 0, -1/2, 0), \quad (0, 0, -1, 0), \\ (0, h, s, 0), \quad (0, -h, s, 0), \quad (0, 0, 0, h), \quad (0, 0, 0, -h), \end{aligned}$$

with $(h, s) = (1/2, 1/2)$ for $r = 3$ and $(1/4, 0)$ for $r = 6$. Under (6), the first five vertices and $(0, -h, s, 0)$ suffice.

Proof. If the root order n is not divisible by the quadratic conductor q , the trace is the Ramanujan value $\mu(n)/\varphi(n)$. For $q \mid n$, the rational contribution vanishes. The induced quadratic Gauss sum is nonzero precisely when $n = qm$, where m is squarefree and coprime to 6. Put $c = h\sqrt{r}$. The corresponding pairs are

$$\left(\frac{\pm c\mu(m)\chi_q(m)}{\varphi(m)}, \frac{\mu(m)}{2\varphi(m)} \right) \quad (q = 12), \quad \left(\frac{\pm c\mu(m)\chi_q(m)}{\varphi(m)}, 0 \right) \quad (q = 24).$$

For $m = 1$ these are the primitive-conductor vertices. Otherwise $w = 1/\varphi(m) \leq 1/4$. Give the appropriate primitive vertex weight w . When $q = 12$ and $\mu(m) = -1$, the remaining weight supplies the vertical correction $-w/(1-w) \in [-1, 1]$; otherwise it supplies zero. The whole vertical segment $(0, [-1, 1])$ is in the rational five-vertex hull.

A nonzero irrational second moment with zero first moment requires $n = 2qm$ with the same restrictions on m and lies on the segment between the two primitive- $2q$ vertices. In the rational cases, odd squarefree order gives (u, u) , with $u \in [-1/2, 1]$, and twice odd squarefree order gives $(-u, u)$. Orders divisible by four give $(0, u)$, $u \in [-1, 1]$. The other cases give zero. These are all in the displayed hull. Finally, for order dividing q , the primitive negative-character vertex and the rational five-vertex hull cover all possibilities in (6). $\square$

9.4 Pivotal signs and four exact inequalities

For arbitrary X and self-dual Y , the pivotal form $j_X \otimes f_Y \otimes \text{id}_{X^*}$ on XYX^* has sign $\nu_2(Y)$. Every self-dual constituent of odd multiplicity therefore has that same sign: a nondegenerate alternating form cannot exist on its odd-dimensional multiplicity space. The same palindromic argument applies to X^3 for self-dual X . Applied to the source products, these relations leave exactly

$$\begin{array}{l|l} 106 & (1, 1, \epsilon, 1, 1, \epsilon, 1), \quad \epsilon = \pm 1 \\ 2155 & (1, 0, 0, 1, 1, 1, 1) \\ 146 & (1, 1, 1, 1, 1, 1, \epsilon), \quad \epsilon = \pm 1. \end{array}$$

The zeros are the nonselfdual indicators. In the second ring, label 4 occurs once in $X_1 X_1^* = X_1 X_2$, which fixes its indicator to $+1$.

For every nonunit X_i , the induction trace identities are

$$\text{Tr}(\theta \mid I(X_i)) = 0, \quad \text{Tr}(\theta^2 \mid I(X_i)) = D\nu_2(X_i)$$

[13, Theorems 2.5 and 2.7]. Let ℓ run over the six nonunit constituents of $I(1)$, write $z_\ell = \ell \cdot d$, and put $k_i = \sum_\ell \ell_i z_\ell$. After their subtraction, the residual 6×6 Gram matrix is $R_{ij} = B_{ij} - \sum_\ell \ell_i \ell_j$, $1 \leq i, j \leq 6$. Taking relative traces gives a system with 21 Gram equations, 12 first-trace equations and 12 second-trace equations.

For a paired orbit $\mathcal{O}$ of residual rows (v, z) and a hull vertex (a, b, c, d) , its column is

$$\left(\begin{array}{c} 4 \sum_{v \in \mathcal{O}} \text{vech}(vv^\top) \\ 4 \sum_{(v,z) \in \mathcal{O}} (v_i z (a + b\sqrt{r}))_{i=1}^6 \\ 4 \sum_{(v,z) \in \mathcal{O}} (v_i z (c + d\sqrt{r}))_{i=1}^6 \end{array} \right), \quad b = 4 \left(\begin{array}{c} \text{vech}(R) \\ (-k_i)_{i=1}^6 \\ (D\nu_2(X_i) - k_i)_{i=1}^6 \end{array} \right). \quad (7)$$

The last two blocks are expanded in the coefficient basis $1, \sqrt{r}$. The lower-triangular order is (i, j) with i increasing and then $0 \leq j \leq i$, using zero-based indices for the six residual coordinates. All columns and targets are integral. A fixed row is counted once; a two-row orbit is counted twice

through the displayed sum. Actual center objects give a nonnegative real combination of these columns, using the same hull vertex on Galois-paired rows.

Every admissible v is bounded by the Gram diagonal and satisfies (3), the entrywise Gram bounds, and $vv^\top \leq R$. The latter test includes singular R : if $P = R_{I,I}$ is an invertible principal block, it is equivalent to

$$R_{*,I}P^{-1}v_I = v, \quad v_I^\top P^{-1}v_I \leq 1.$$

The verifier applies these tests in exact rational arithmetic. Some blocks are excluded already because $\text{vech}(R)$ does not lie in the finite-field span of the paired Gram columns. For every remaining block and every allowed indicator vector, the four covectors in Table 4 satisfy

$$y \cdot c \geq 0 \quad \text{for every allowed column,} \quad y \cdot b < 0. \quad (8)$$

Thus no nonnegative combination can equal the target. The first covector covers both signs for 106; the second covers 2155; the final two cover the positive and negative signs for 146.

Ring	Low blocks	Modular exclusions	Trace blocks	Trace systems	Covectors
106	194	21	173	346	1
2155	194	21	173	173	1
146	1323	235	1088	2176	2

The 1088 blocks for 146 include repeated targets; all are checked, not just the 1072 different right-hand sides for each sign. The source-bound verifier `verify_quadratic_center_trace.py` reconstructs the census IDs, immutable tensor hashes, codegrees, character fields, low blocks, residual rows and every column of (7). It uses only standard-library integer and rational arithmetic; neither numerical feasibility nor saved candidate lists are trusted. Independent ordinary and optimized replays verify the complete coverage and every inequality (8). This excludes every pseudo-unitary center possibility and proves Theorem 27.

10 A common two-dimension Morita reduction

Five further residual rings admit Morita duals with dimensions 1 and $t = 2 + \sqrt{5}$, where $t^2 = 1 + 4t$. The following arguments concern arbitrary complex categories. The reductions feed two constructions, the staged obstruction below, and the final rank-eight endpoint exclusion.

Proposition 29. *Every categorification of $R7_4_35$, $R7_4_2053$, $R7_4_2603$, or $R7_4_2665$ is Morita equivalent to a category with eight invertibles forming D_8 and eight noninvertibles of dimension t , on which the invertibles act freely. Its based ring belongs to the four families described below.*

Proof. The source has four invertibles G , a simple X of dimension two, and simples Y, Z of dimension $2t$. There is an involution h fixing all three on both sides, and

$$X^2 = \sum_{g \in G} g, \quad XY = YX = Y + Z, \quad [YY^* : X] = 1.$$

Put $H = \langle h \rangle$. Its pointed associator is trivial because it fixes X , so $A = 1 + h$ is a connected separable algebra. A fixed simple supports left and right A -module structures, since $H^2(C_2, \mathbb{C}^\times) = 0$. A bimodule supported on a fixed simple W is a projective representation of $H \times H$ on its multiplicity

Table 4: The four universal covectors in the coordinate order of (7). Rows 1–21 are Gram coefficients, 22–33 first-trace coefficients, and 34–45 second-trace coefficients.

Coordinate	106	2155	146, +	146, −
1	1004	3	1006	1006
2	512	-18	1000	1000
3	382	3	420	-494
4	1000	1000	1000	1000
5	-262	1000	-377	-291
6	34	1012	53	228
7	1000	-148	1000	1000
8	-290	-148	-319	-594
9	51	1000	19	182
10	42	496	83	118
11	1000	-383	1000	1000
12	1000	-383	828	-1000
13	-259	1000	-377	-291
14	-293	781	-319	-594
15	626	97	420	-494
16	1000	-293	606	-1000
17	-286	-293	-1000	559
18	339	1000	494	135
19	375	658	406	1000
20	-773	17	-1000	559
21	-246	90	699	1006
22	-572	23	399	452
23	1000	-176	-1000	-1000
24	-580	23	174	303
25	1000	-176	-477	-1000
26	141	-739	-316	9
27	-247	1000	795	0
28	-140	729	135	421
29	247	-1000	-318	-1000
30	-580	578	-417	-497
31	1000	-1000	1000	1000
32	579	574	423	-236
33	-1000	-1000	-1000	1000
34	-774	-113	-698	-418
35	1000	74	1000	1000
36	436	-142	454	-569
37	-763	125	-1000	1000
38	-152	-1000	406	-416
39	259	975	-1000	1000
40	-146	718	407	-388
41	249	-1000	-1000	1000
42	434	576	454	-569
43	-759	-1000	-1000	1000
44	594	480	198	-142
45	-1000	-832	-606	1000

space. A nontrivial projective class has only two-dimensional irreducibles. The multiplicity-one channels in XY first force the classes for Y, Z to be trivial; the X channel in YY^* does the same for X . Thus each of X, Y, Z supports four simple bimodules with its original underlying object. The two free H -orbits in G each support two more. The dual consequently has eight invertibles and eight simples of dimension t . Irrationality and $t^2 = 1 + 4t$ imply that WW^* contains exactly one invertible, so the action is free.

Let K be the dual invertible group. The inverse algebra is the sum of the two invertibles supported on H : its multiplicity at a simple bimodule V is $\dim \text{Hom}_C(1, V)$. Call this subgroup $L \leq K$. Its inverse pointed dual is the original integral subcategory, of rank five and type $1^4, 2$. If L were normal, all four double cosets would have stabilizer C_2 and split into two one-dimensional projective representations. That dual would instead be pointed of rank eight. Hence L is nonnormal, forcing $K = D_8$ and L to be a reflection subgroup. This does not trivialize the pointed associator of the dual.

Write $D_8 = \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle$ and choose a noninvertible ρ . Its products have the form

$$\rho g = \phi(g)\rho, \quad \rho^2 = a + N\rho, \quad N \in \mathbb{Z}_{\geq 0}[D_8], \quad \sum_g N_g = 4.$$

Associativity and duality give

$$\phi^2 = \text{Ad}_a, \quad \phi(a) = a, \quad Ng = \phi(g)N, \quad \phi(N) = N, \quad N^* = a^{-1}N.$$

The inverse double orbit of $g\rho$ has stabilizer $L \cap g\phi(L)g^{-1}$. An order-two intersection would give noninvertibles of dimension t in the source, which has none. Thus ϕ exchanges the two reflection conjugacy classes. Changing ρ and the generators normalizes

$$\phi(r) = r^{-1}, \quad \phi(s) = rs, \quad a \in \{1, r^2\}.$$

The coefficient equations give a common coefficient A on rotations and coefficients B, B, C, C on adjacent reflection pairs, with $4A + 2B + 2C = 4$. For $a = r^2$, duality also gives $B = C$; a generator change interchanges B, C . There are exactly five possibilities:

a	N
1	$\sum \text{rotations}, \quad \sum \text{reflections}, \quad 2(s + rs)$
r^2	$\sum \text{rotations}, \quad \sum \text{reflections}.$

For $N = 2(s + rs)$, the regular Casimir polynomial is

$$(x - 16)^4(x^2 - 160x + 1280)^2(x^2 - 96x + 768)^4.$$

Its roots $48 \pm 16\sqrt{6}$ force a formal codegree outside $\mathbb{Q}(\sqrt{5})$. By the published two-dimension Galois reduction [35, Lemma 3.1], a hypothetical dual can be assumed pseudo-unitary. The induced-unit formula [13, Theorem 2.13] then places every formal codegree in its simple-dimension field $\mathbb{Q}(t)$, a contradiction.

The other four families have neutral rank-eight subrings. With N the rotation sum, these are $R8_1_15$ for $a = 1$ and $R8_1_12$ for $a = r^2$. With N the reflection sum, replacing ρ by $s\rho$ gives $R8_1_18$ in either case. The two self-dual source cases 35, 2053 require $a = 1$; the dual-pair cases 2603, 2665 require $a = r^2$. The source-bound verifier checks the full tensors of these statements, including every neutral-subring product. It does not construct the extensions. $\square$

Proposition 30. *Every categorification of $R7_4_2591$ is Morita equivalent to a category with eight invertibles and eight noninvertibles of dimension t , again with a free invertible action.*

Proof. Here $G = V_4$ fixes X, Y, Z on both sides, $\dim X = 2$, $\dim Y = \dim Z = 2t$, and $Y^* = Z$. The source products include

$$X^2 = \sum_{g \in G} g, \quad XY = YX = 2Z, \quad XZ = ZX = 2Y, \quad Y^2 = 2X + 4Y + 4Z.$$

The integral subcategory is Tambara–Yamagami. Use its standard normalization, with nondegenerate symmetric bicharacter χ . For a G -fixed simple W , let $\alpha_W, \beta_W \in \mathbb{F}_2$ be its left and right projective classes and let B_W be its mixed commutator matrix over $\mathbb{F}_2$. The full alternating matrix is

$$\begin{pmatrix} \alpha_W J & B_W \\ B_W^\top & \beta_W J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A rank- $2r$ form has projective irreducibles of dimension 2^r and 2^{4-2r} inequivalent such irreducibles.

If W occurs n times in $U \otimes V$, its multiplicity space has diagonal classes $\alpha_U - \alpha_W, \beta_V - \beta_W$ and mixed block B_W , up to transpose and inverse. Indeed the left action on U and right action on V commute before the identifications with W ; their residual defect is the mixed commutator of W . The resulting rank is at most $2 \log_2 n$. For the TY simple, $\alpha_X = \beta_X = 0$ and $B_X = \chi$. Apply the rank bound to X in Y^2 . Invertibility of χ forces $\alpha_Y = \beta_Y = 1$: if either diagonal block vanishes, the rank is four, whereas both equal to J give rank two because $\chi^\top J \chi = J$. Duality gives the same classes for Z . Applying the bound to Z in XY gives the matrix

$$\begin{pmatrix} J & B_Z \\ B_Z^\top & 0 \end{pmatrix}.$$

Its rank is $2 + \text{rank}(B_Z^\top J B_Z)$, so $\text{rank } B_Z \leq 1$, and likewise for B_Y .

Twist the full regular algebra $A_\beta = \bigoplus_{g \in V_4} g$ by the nontrivial element of $H^2(V_4, \mathbb{C}^\times)$. This adds J to both diagonal blocks and leaves the mixed block unchanged. The X block now has rank two and yields four simple bimodules with underlying object $2X$, hence dual dimension one. The Y, Z blocks become $\begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}$. Rank zero would give dual dimension $t/2$, impossible because its algebraic norm is $-1/4$. Thus both mixed blocks have rank one, and each yields four simples of dimension t . The pointed support supplies four further invertibles. This accounts for the whole dual, and the free-action claim again follows from $t^2 = 1 + 4t$. $\square$

The inverse algebra in Proposition 30 is supported on a V_4 subgroup of the dual invertible group. That group can therefore be $C_4 \times C_2$, C_2^3 , or D_8 . The staged inverse-algebra argument in Theorem 32 excludes all these alternatives. Neither proposition uses a classification restricted to unitary categories or to trivial pointed associators.

Corollary 31. *Every hypothetical complex categorification of the rank-seven ring $R7_4_2053$ is Galois conjugate to a pseudo-unitary category.*

Proof. For $R7_4_2053$ in Proposition 29, apply [35, Lemma 3.1] to the two-dimension Morita dual. Categorical global dimension and Frobenius–Perron dimension are Morita invariant, and the Morita context can be Galois conjugated with the category. Thus equality of these dimensions transfers back to the conjugated source. The integral candidate $R7_3_601$ is excluded in Theorem 37. $\square$

The rings $R7_4_2665$ and $R7_4_35$ are realized in Theorems 38–39. The three quadratic cases are excluded in Theorem 27, and $R7_4_2591$ is excluded below. The further two-moment obstruction in Theorem 34 excludes $R7_4_2603$. The final case $R7_4_2053$ is excluded by the rank-eight endpoint in Theorem 36, completing the pinned rank-seven census.

10.1 A staged Morita obstruction for $R_{7,4,2591}$

Theorem 32. *The fusion ring $R_{7,4,2591}$ admits no categorification over $\mathbb{C}$. No pivotality or unitarity assumption is required.*

Proof. Put $t = 2 + \sqrt{5}$, so $t^2 = 1 + 4t$. In a hypothetical categorification $\mathcal{C}$, the invertibles form $G \cong V_4$; the other simples are X, Y, Z , of dimensions $2, 2t, 2t$, with $Y^* = Z$. Every element of G fixes each of these three objects on both sides, and

$$X^2 = \sum_{g \in G} g, \quad XY = YX = 2Z, \quad XZ = ZX = 2Y, \quad Y^2 = 2X + 4Y + 4Z.$$

Proposition 30 supplies a Morita equivalent category $\mathcal{E}$ with eight invertibles K and eight dimension- t simples forming a free K -orbit. It is the dual of the nontrivially twisted regular algebra P on G .

The inverse algebra Q is the regular object on a subgroup $L \cong V_4$ of K , with possibly twisted multiplication. Indeed, the internal end of the distinguished right P -module P contains each of the four character-twist bimodules on pointed support once and no other simple: its multiplicity on a P -bimodule V is $\dim \operatorname{Hom}_{\mathcal{C}}(\mathbf{1}, V)$. Consequently $\mathcal{C} \simeq {}_Q\mathcal{E}_Q$. The existence of Q trivializes the associator on L , but no triviality outside L is assumed.

A central fixed involution. Choose a simple ρ in the free orbit and write

$$\rho g = \phi(g)\rho, \quad \rho^2 = a + N\rho.$$

Associativity gives $\phi \in \operatorname{Aut}(K)$, $\phi^2 = \operatorname{Ad}(a)$ and $\phi(a) = a$; also $\rho^* = a^{-1}\rho$. The subgroup L has index two and is normal. We claim $\phi(L) = L$. Otherwise $L\phi(L) = K$, so the noninvertible L -by- L action has a single orbit of size eight and an order-two stabilizer with injective projections into L . Its two projective characters give Y, Z , since $H^2(C_2, \mathbb{C}^\times) = 0$. The character twists of Q are precisely the four original invertibles. Their characters restrict surjectively to this stabilizer, so some such invertible interchanges Y and Z , contradicting their individual fixedness. This argument is unchanged by the multiplication cocycle of Q .

There are therefore two noninvertible support orbits, $L\rho$ and $(K \setminus L)\rho$, each supporting exactly one of Y, Z . Their mutual duality forces $a \notin L$. The only groups of order eight containing V_4 are $C_4 \times C_2$, C_2^3 and D_8 . The last is impossible: conjugation by any element outside an index-two V_4 in D_8 transposes its two reflections, whereas $(\phi|_L)^2$ cannot be a transposition in $\operatorname{Aut}(V_4) = S_3$. Thus K is abelian and $\phi^2 = 1$. The restriction $\phi|_L$ fixes a nonidentity element δ . It is central in K , and $\phi(\delta) = \delta$.

Factoring the inverse algebra. The summand $A = \mathbf{1} + \delta$ is an actual subalgebra of Q ; its multiplication is the restriction of the twisted regular multiplication. It is connected and separable. In $\mathcal{E}' = {}_A\mathcal{E}_A$, all pointed and noninvertible support orbits have size two and stabilizer C_2 , because δ is central and ϕ -fixed. Each orbit splits into two projective characters. Thus $\mathcal{E}'$ again has eight invertibles K' and eight noninvertibles of dimension t . The same dimension comparison shows that K' acts freely and transitively on them.

Regard Q as an algebra B in $\mathcal{E}'$. Its multiplication factors through $Q \otimes_A Q$, its unit is $A \rightarrow Q$, and separability descends to the relative tensor product. It has $\text{FPdim}_{\mathcal{E}'}(B) = 4/2 = 2$. Moreover evaluation at the unit injects

$$\text{Hom}_{A-A}(A, Q) \rightarrow \text{Hom}_{\mathcal{E}}(\mathbf{1}, Q),$$

whose target is one-dimensional; the inclusion $A \rightarrow Q$ supplies a nonzero map. Thus B is connected. Since $t > 2$, it must be $B = \mathbf{1} + h$ for an invertible involution h . This algebra is already supplied by Q , rather than inferred from a based-ring subgroup. Restriction and extension of the two Q actions give the canonical relative-Morita identification

$${}_B({}_A\mathcal{E}_A)_B \simeq {}_Q\mathcal{E}_Q \simeq \mathcal{C}.$$

The forced dihedral support permutation. Put $H = \langle h \rangle \subset K'$. If H were normal, all four pointed double cosets would have order-two stabilizer and would yield eight invertibles. This contradicts the integral part of $\mathcal{C}$, whose dimensions are 1, 1, 1, 1, 2. Therefore $K' = D_8 = \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle$ and we may take $H = \langle s \rangle$. For a noninvertible σ write $\sigma g = \psi(g)\sigma$. The stabilizer of the support through $g\sigma$ is $H \cap g\psi(H)g^{-1}$. If H and $\psi(H)$ were conjugate, some support orbit would have size two and produce simples of dimension t , which do not occur in $\mathcal{C}$. They lie in opposite reflection classes; changing generators gives $\psi(H) = \langle rs \rangle$.

The two noninvertible support orbits are

$$U\sigma, \quad r^2U\sigma, \quad U = \{1, s, rs, r^{-1}\}.$$

The pointed coset Ur^2 has two invertible bimodules, and tensoring either of them multiplies noninvertible support by the central rotation r^2 . It therefore interchanges the two supports, hence interchanges Y and Z . This contradicts their individual fixedness under every original invertible and proves the theorem. $\square$

Remark 33. *The source-bound certificate `verify_R2591_staged_morita.py` replays the twisted-block calculation, the source products and duality, all five groups of order eight, and all sixteen opposite-reflection configurations. The compatible inverse-group systems comprise eight for $C_4 \times C_2$ and 112 for C_2^3 , with none for D_8 . These finite checks verify the algebraic data used in the proof; the actual-algebra and relative-Morita arguments above are categorical and have been independently audited.*

10.2 Full-character reciprocity and a further two-moment obstruction

The moment method also applies without conductor saturation. Let K be a quadratic dimension field and t_K normalized relative trace from a cyclotomic field containing all twists. The anomaly-free center Galois identity is $\theta_{\pi_\sigma A} = \sigma^2(\theta_A)$. Since σ^2 fixes K and cyclotomic Galois automorphisms commute,

$$t_K(\theta_{\pi_\sigma A}^n) = \sigma^2(t_K(\theta_A^n)) = t_K(\theta_A^n) \quad (n \geq 1). \quad (9)$$

Thus full-row Galois partners have equal relative moments even when their twists differ. No exponent bound is needed.

Theorem 34. *The fusion ring `R7_4_2603` admits no complex fusion categorification.*

Proof. Proposition 29, the two-dimension Galois theorem [35, Lemma 3.1], and Morita invariance of global dimension reduce the problem to the pseudo-unitary case. In source order write the basis as $(1, Y, Z, g, a, b, X)$. Put $s = \sqrt{5}$ and $u = \sqrt{-2}$. Then

$$d = (1, 4 + 2s, 4 + 2s, 1, 1, 1, 2), \quad D = 80 + 32s, \quad Y^* = Z.$$

The invertibles form V_4 . The complete character table is

	1	Y	Z	g	a	b	X	f_E
P	1	0	0	1	1	1	-2	8
U	1	0	0	-1	1	-1	0	4
V	1	0	0	-1	-1	1	0	4
C_+	1	u	$-u$	1	-1	-1	0	8
C_-	1	$-u$	u	1	-1	-1	0	8
$\bar{d}$	1	$4 - 2s$	$4 - 2s$	1	1	1	2	$80 - 32s$
d	1	$4 + 2s$	$4 + 2s$	1	1	1	2	$80 + 32s$

All products and duality-defined codegrees in this table are checked against the source tensor. Disjointness of $\mathbb{Q}(s)$ and $\mathbb{Q}(u)$ gives an automorphism flipping s and fixing every other character. If $v \cdot d = A(v) + B(v)s$, its signed full-row action is

$$T(v) = v + (0, 2, 2, 0, 0, 0, 0) \frac{2A(v) - 5B(v)}{4}.$$

This integral involution follows from (4); evaluation on every displayed character verifies it directly. In particular the Galois image J of the central unit has forgetful object $1 + Y + Z$, of dimension $9 + 4s$.

Two possible induced-unit blocks. The other five nonunit constituents of $I(1)$ have dimensions $10 + 4s$ three times and $20 + 8s$ twice. The unit Gram row is $(7, 8, 8, 3, 1, 1, 2)$. A codegree-eight row must therefore be

$$1 + h + pY + (2 - p)Z, \quad 0 \leq p \leq 2,$$

with h a nonidentity invertible. Each codegree-four row has at most one X summand; together they must supply the unit Gram coefficient two. Hence each has form

$$1 + k + X + qY + (4 - q)Z, \quad 0 \leq q \leq 4.$$

The five entries h, k have multiset $\{g, g, g, a, b\}$. Symmetry of the center S -matrix gives

$$f_{F\chi_E}(F(A_F)) = f_{E\chi_F}(F(A_E)). \quad (10)$$

The P -value on a codegree-four row is zero. Reciprocity with U, V therefore forces the invertible in the P row to be g . Reciprocity with C_+ then forces $p_P = 1$.

For U or V against C_+ or C_- , one side of (10) belongs to $\{0, 16\}$; the other has real part in $\{0, 8\}$ and imaginary part proportional to $(2q - 4)u$. Both real parts must vanish and $q = 2$. Thus the two codegree-four rows use a, b , once each, and both C rows use g . Finally reciprocity between C_+ and C_- gives $p_+ + p_- = 2$. Up to permuting equal-codegree rows, the complete nonunit block is therefore

$$\begin{aligned} &1 + Y + Z; \\ &1 + g + 2Z, \quad 1 + g + Y + Z, \quad 1 + g + 2Y; \\ &1 + a + X + 2Y + 2Z, \quad 1 + b + X + 2Y + 2Z, \end{aligned}$$

or the same block with all three middle rows equal to $1 + g + Y + Z$. This theoretical enumeration is independently checked by the verifier: of fifty bounded Gram-compatible blocks, exactly these two admit a complete codegree-respecting character assignment.

All root orders in a finite moment hull. Set $u_{\pm} = (-1 \pm s)/4$. Every pair $(t_K(\theta), t_K(\theta^2))$ lies in the convex hull of the five rational vertices of Lemma 28, the six pairs

$$(u_+, u_-), (u_-, u_+), (-u_+, u_-), (-u_-, u_+), (0, -u_+), (0, -u_-),$$

and negative one-half times each of those six pairs. Indeed, write the root order as $2^\alpha 5^\beta m$, with m odd and prime to five. If $\beta = 0$, ordinary Ramanujan averages lie in the rational hull. If $\beta \geq 2$, m is not squarefree, or $\alpha \geq 3$, both moments vanish. The other cases have $\beta = 1$, $\alpha = 0, 1, 2$. Their primitive-order 5, 10, 20 pairs are the six displayed points, multiplied by $\mu(m)/\varphi(m)$, with a possible interchange of the signs. For $m > 1$, $\varphi(m) \geq 2$, so this scale lies in $[-1/2, 1]$. This proves the seventeen-vertex hull for arbitrary root orders.

One universal integer inequality. Every residual central dimension is $z = A + Bs$, and D/z must be integral. The full ring of integers here is $\mathbb{Z}[(1+s)/2]$. With $n = A^2 - 5B^2$, the exact condition is that

$$\frac{2(80A - 160B)}{n}, \quad \frac{2(32A - 80B)}{n}$$

are integers of the same parity. The Gram bounds and this arithmetic condition give 465 global row candidates. The exact rank-one PSD test on each residual matrix, followed by full-row Galois pairing, leaves 72 and 76 paired row types.

The second indicators are $(1, 0, 0, 1, 1, 1, 1)$. To see the positive sign of X , the unit occurs once in Y^*XY , and its palindromic form has sign $\nu_2(X)$. Then X^2 contains each invertible once, so all their signs are positive. The dual pair Y, Z has indicator zero. As before, subtract the twist-one low rows from the Gram and two induction trace identities. Use the seventeen-vertex hull and the paired relative moments in (9). In the same $21 + 12 + 12$ coordinate order as (7), multiply columns and targets by eight, rather than four, to clear the hull denominators.

The following covector is written in its Gram, first-moment and second-moment blocks:

$$\begin{aligned} y_G &= (38, 58, 26, 63, 76, 8, 57, 69, -70, -72, 7, 19, -20, -100, \\ &\quad -22, -100, -76, -100, -100, 0, 103), \\ y_1 &= (-17, 50, -17, 50, -43, 100, 37, -100, 37, -100, 33, -100), \\ y_2 &= (51, -100, 51, -100, -72, 100, -54, 100, -54, 100, -58, 100). \end{aligned}$$

Every allowed column has dot product at least 12 with y . There are 1224 and 1292 columns in the two branches and 1292 distinct columns in their union. Their prescribed targets have dot products -1336 and -1240 . Neither can be a nonnegative combination of allowed columns. This contradicts the existence of the center.

The standard-library verifier `verify_r2603_center_moments.py` reconstructs the original census tensor, checks its immutable hash, every character and its full Galois transport, the complete low-block coverage, maximal-order integrality, all residual columns, and the exact inequality above. It uses the already audited exact Gram helpers and no optimizer. Independent ordinary and optimized replays confirm every step. The initial Galois reduction proves the claimed general-complex exclusion. $\square$

11 The final dihedral endpoint

Proposition 35. *A complex categorification of $R7_4_2053$ would produce a complex categorification of $R8_1_18$.*

Proof. Use Proposition 29. The source has integral subcategory of Tambara–Yamagami type C_4 , and its two irrational simples are self-dual. Its rank-sixteen Morita dual has

$$D_8 = \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle, \quad \rho r = r^{-1} \rho, \quad \rho s = rs \rho, \quad \rho^2 = a + N \rho.$$

Here $a \in \{1, r^2\}$ and N is the sum of the rotations or the sum of the reflections. In the inverse reflection dual, the two irrational supports are the size-four double orbits $\langle s \rangle g \phi(\langle s \rangle)$, with trivial stabilizers. Duality preserves each orbit when $a = 1$ and interchanges them when $a = r^2$. Since each support therefore carries exactly one simple bimodule, source self-duality forces $a = 1$.

Suppose first that $N = P = 1 + r + r^2 + r^3$. The two-dimension theorem [35, Lemma 3.1] permits a pseudo-unitary Galois conjugate of the actual Morita context. In its canonical positive spherical structure, ρ is self-dual and

$$\rho^3 = P + (1 + 4P)\rho.$$

For a self-dual simple W , a form of sign $\nu_2(W)$ induces a form of the same sign on $W^{\otimes 3}$: the dual reverses three identical factors. Each self-dual constituent occurring oddly has the same indicator, since a nondegenerate alternating multiplicity-space form has even dimension. The unit and r^2 both occur once above, hence $\nu_2(\rho) = \nu_2(r^2) = 1$.

The pointed associator on $\langle r \rangle = C_4$ consequently has even class in $H^3(C_4, \mathbb{C}^\times) \simeq \mathbb{Z}/4$, because restriction to $\langle r^2 \rangle$ is reduction modulo two. Its restriction therefore has order at most two. The integral inverse reflection dual is group-theoretical. Its Frobenius–Schur exponent, computed as the least common multiple of $|C| \operatorname{ord}(\omega|_C)$ over cyclic subgroups $C \leq D_8$, divides eight: the rotation term divides eight and every reflection term divides four. But this category is of type $\text{TY}(C_4)$. For either nondegenerate bicharacter $\chi(x, y) = i^{\varepsilon xy}$, $\varepsilon = \pm 1$, the eighth indicator of its dimension-two simple is

$$\nu_8(X) = \frac{1}{8} \sum_{a+b+c+d=0 \pmod{4}} i^{\varepsilon(ab+ac+ad+bc+bd+cd)} = 0.$$

Indeed the four exponent classes occur 8, 24, 8, 24 times. This holds for either sign of the Tambara–Yamagami parameter. Exponent dividing eight would instead give $\nu_8(X) = \text{FPdim}(X) = 2$, a contradiction. The formulas used here are [47, Theorem 9.2] and [50, Theorem 3.4].

Thus $N = s + rs + r^2s + r^3s$. Put $\sigma = s\rho$. The mixed rules give

$$\sigma r = r\sigma, \quad \sigma^2 = r^3 + (1 + r + r^2 + r^3)\sigma.$$

The full additive subcategory on $1, r, r^2, r^3, \sigma, r\sigma, r^2\sigma, r^3\sigma$ is closed under products and duals. Reversing the rotation generator identifies its fusion ring with $R8_1_18$. $\square$

The independent source verifier `verify_r2053_r18_reduction.py` reopens both census tensors, checks all source identities and both complete rank-sixteen fusion tensors, recomputes the exact indicator sums, and matches all 512 coefficients of the closed rank-eight subcategory. Its local-to-census map is $(0, 1, 7, 2, 4, 3, 6, 5)$. Thus the endpoint is an actual fusion subcategory of the Morita dual, not merely a dimension match.

Theorem 36. *The fusion ring $R7_4_2053$ is not the Grothendieck ring of a complex fusion category.*

Proof. Proposition 35 reduces the question to $R8_1_18$. The general exclusion of that ring is proved in the companion rank-eight classification [20]; its complete source-bound proof is `R18_STS_MOMENT_PROOF.md` in the rank-eight evidence. We recall the mechanism to make the dependency explicit.

For that ring, the two-dimension theorem gives a pseudo-unitary Galois conjugate. Write $t = 2 + \sqrt{5}$ and $D = 8 + 16t$. Cauchy’s theorem restricts the Frobenius–Schur exponent to the primes 2, 5. Positivity of induction for the pointed C_4 subcategory gives a four-point Fourier decomposition and bounds its generator’s multiplicity in each central forgetful row by one. Character reciprocity leaves eight complete induced-unit frames. For the normalized center modular data, let C be the source character matrix and V the corresponding induced-unit forgetful rows. The twist-one identity $STS = T^{-1}ST^{-1}$ yields

$$\sum_A v_A v_A^T \theta_A = V^T C.$$

Relative trace to $\mathbb{Q}(\sqrt{5})$ gives a fixed first-moment Gram target. First and second induction moments, together with rationality of the relative fourth indicators, give the remaining constraints. The functional $a + bt \mapsto 2a - b$ annihilates D times any rational fourth indicator. Prime support makes the relative moment vectors from orders dividing forty, together with the zero vector, exhaustive; there are thirteen in all. Full Galois row orbits and the four possible pointed associator classes are retained.

Three integer covectors, one for each effective pointed class (the two odd classes have identical constraints), are nonnegative on every admissible column and have pairing -256 with every target. The exact source verifier checks all eight frames in each class; its 24 branch contradictions use no numerical solver. Thus $R8_1_18$ cannot be categorified. The actual fusion subcategory supplied by Proposition 35 would contradict this theorem. $\square$

Together with the preceding exclusions and explicit models, Theorem 36 completes the rank-seven census through multiplicity four: all 4201 source rings are classified, with 4172 general-complex exclusions and 29 unitary realizations.

12 The integral candidate: a simple-center induction lattice

Theorem 37. *The ring $R7_3_601$ admits no complex fusion-category categorification.*

The proof first restricts the dimensions in a hypothetical Drinfeld center by a categorical simplicity theorem. Exact induction then gives an integer Gram decomposition; congruences modulo two or three exclude every possible decomposition. The finite certificate exhausts all possibilities and uses no numerical optimization.

Proof. In census order the positive dimension character and global dimension are

$$d = (1, 5, 5, 8, 8, 9, 10), \quad D = 360.$$

Every nonidentity basis element generates the full based ring, as is checked directly from the source products. Thus a hypothetical category $\mathcal{C}$ is simple, nonpointed and integral, and consequently pseudo-unitary with its canonical spherical structure [28, Proposition 8.24].

We first show that $\mathcal{C}$ cannot be braided. Its Müger center would, by simplicity, be either trivial or all of $\mathcal{C}$. In the first case modular dimension divisibility would require $360/5^2$ to be an algebraic integer, which it is not [9, Theorem 5.9]. In the second case $\mathcal{C}$ would be symmetric, with the fusion ring of a finite-group representation category, including in the super-Tannakian case. The source has ring characters

$$a = (1, 0, 0, -2, 3, -1, 0), \quad b = (1, 2, -1, -1, -1, 0, 1),$$

but their weighted character sum is

$$\sum_i \frac{a_i^2 b_i}{d_i} = -\frac{5}{8}.$$

For a representation ring the characters a, b are evaluations at conjugacy classes K, L . Since these values are real, this sum times $|K|^2/|G|$ is the coefficient of the class sum of L in the square of the class sum of K . That coefficient counts factorizations and cannot be negative. This argument only excludes the symmetric case; it does not assert a unitary positivity criterion for arbitrary pseudo-unitary categories.

The simple-center theorem of Liu–Palcoux–Ren [9, Theorem 5.1] now implies that $\mathcal{Z}(\mathcal{C})$ is simple. It is integral because the forgetful functor preserves Frobenius–Perron dimensions, and it is modular of dimension 360^2 . If a central simple has dimension z , then z is an integer and modular divisibility gives $360^2/z^2 \in \mathbb{Z}$, hence $z \mid 360$. Simplicity removes nonidentity invertibles. A nondegenerate integral braided category with a simple object of nontrivial prime-power dimension has a nontrivial symmetric fusion subcategory [10, Corollary 6.16]; see also [9, Theorem 5.10]. Such a subcategory is impossible in this simple modular center. Every nonunit central simple therefore has dimension in

$$\mathcal{D} = \{6, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360\}. \quad (11)$$

Let $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ be forgetful and I its adjoint. For each central simple A , put $m_A = ([F(A) : X_i]_{i=0}^6)$. Frobenius reciprocity and the formula $FI(X) = \bigoplus_Y Y \otimes X \otimes Y^*$ give

$$\sum_A m_A m_A^t = B = \sum_a N_a N_a^t = \begin{pmatrix} 7 & 6 & 6 & 5 & 9 & 9 & 10 \\ 6 & 32 & 23 & 39 & 39 & 45 & 49 \\ 6 & 23 & 32 & 39 & 39 & 45 & 49 \\ 5 & 39 & 39 & 71 & 60 & 73 & 78 \\ 9 & 39 & 39 & 60 & 75 & 69 & 78 \\ 9 & 45 & 45 & 73 & 69 & 85 & 88 \\ 10 & 49 & 49 & 78 & 78 & 88 & 106 \end{pmatrix}. \quad (12)$$

The seven orthogonal ring characters, independently checked against all source products, are the rows of

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -2 & 3 & -1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 2 & -1 & -1 & 0 & 1 \\ 1 & 5 & 5 & 8 & 8 & 9 & 10 \end{pmatrix}.$$

Their formal codegrees are 8, 3, 15, 4, 9, 9, 360. Commutativity and the central-induction theorem [13, Theorem 2.13] imply that $I(1)$ has seven multiplicity-one simple constituents, of dimensions 1, 24, 40, 40, 45, 90, 120. The first is the central unit and has forgetful vector e_0 .

Write the other six vectors as $\ell_1, \dots, \ell_6$ in this dimension order, ordering the two dimension-40 vectors lexicographically. They necessarily satisfy

$$\ell_{a,0} = 1, \quad \ell_a \cdot d = z_a, \quad \sum_a \ell_a = B e_0 - e_0, \quad 0 \leq \ell_{a,i} \leq \min\{B_{0i}, \lfloor \sqrt{B_{ii}} \rfloor\}.$$

Every partial remainder in (12) is entrywise nonnegative and positive semidefinite. Exhausting these bounded integer domains with exact Schur elimination yields precisely 336 possible six-vector blocks. For independent coverage checks, the domains of dimensions 24, 40, 45, 90, 120 have sizes 14, 44, 66, 542, 1101; the successive recursion depths have 1, 8, 78, 212, 398, 336, 336 nodes. These blocks are regenerated by the verifier, not assumed from the certificate.

For each block, delete coordinate zero from $B - e_0 e_0^t - \sum_a \ell_a \ell_a^t$, obtaining a positive-definite integer 6×6 matrix R . Any remaining central simple has forgetful vector $(0, v)$, where $v \in \mathbb{Z}_{\geq 0}^6$ satisfies

$$v \cdot (5, 5, 8, 8, 9, 10) \in \mathcal{D}, \quad v_i v_j \leq R_{ij}, \quad v^t R^{-1} v \leq 1. \quad (13)$$

The last inequality is the exact rank-one condition $R - v v^t \succeq 0$. The diagonal bounds in B provide a finite coordinate box. The dimension and initial entrywise conditions leave 4700 possible vectors, and the block-specific conditions (13) leave between 1 and 111. For the inverse test, the verifier constructs integers $h > 0$ and $H = h R^{-1}$, checks $RH = hI$, and tests $v^t H v \leq h$.

Let vech list the 21 lower-triangular entries in row order. For each of the 336 blocks the certificate gives a prime $p \in \{2, 3\}$ and $y \in \mathbb{F}_p^{21}$ with

$$y \cdot \text{vech}(v v^t) = 0 \quad \text{for every admissible } v, \quad y \cdot \text{vech}(R) = 1.$$

There are 218 such certificates modulo two and 118 modulo three. They contradict $R = \sum_{A \notin I(1)} v_A v_A^t$, including any repetitions or multiplicities of these vectors. Every induced-unit block is excluded, proving the theorem. $\square$

The certificate and its source-bound standard-library verifier are

```
r601_center_lattice_certificate.json,
verify_r601_center_lattice.py
```

in `evidence/unified_20261016/rank7/`. Normal and optimized independent replays regenerate all domains and check every congruence. Earlier unrestricted induction relaxations admitted Gram decompositions containing central dimensions 5, 8, 9; those are precisely among the dimensions removed by the simple-center prime-power theorem and do not furnish counterexamples to this argument.

13 Exact constructions and replay

The three multiplicity-one rings listed here match AnyonWiki $\text{FR}_3^{7,2}$, $\text{FR}_{12}^{7,2}$, and $\text{FR}_3^{7,4}$, respectively, by full based-ring isomorphisms. For each, we use the published category with code $(7, 1, d, i, 1, 0, 1)$ in the Vercleyen–Slingerland census [15]. The symbolic associators supplied through `TensorCategories.jl` lie over $\mathbb{Q}(\zeta_8)$, $\mathbb{Q}(\zeta_{28})$, and $\mathbb{Q}(\zeta_{16})$, respectively. Independent rational-polynomial replay checks all 9,856, 45,061, and 9,856 admissible pentagon equations, all 585, 837, and 585 nonempty

associator blocks for invertibility, and the unit and rigidity constraints. These checks establish complex fusion categories directly, with no pivotality assumption. Their global Frobenius–Perron dimensions are 16, 28, 16, respectively. By [7, Theorem 1.6] they are solvable, hence weakly group-theoretical, and [8, Theorem 5.20] makes them unitarizable. This does not assert that the supplied gauges are orthonormal. Only existence of a model is used: neither completeness nor uniqueness of the external category census is required. The supplementary evidence records the exact category codes, cyclotomic coefficients, source hashes and basis permutations.

The positive model catalogue constructs complete fusion tensors from the cyclic group, ordinary finite-group character products, the Tambara–Yamagami rule, $SU(2)_k$ truncation, its even parts, odd metaplectic rules, and the categorical $\mathbb{Z}/2$ zest of $SU(2)_6$ [37, 29, 31, 38, 39]. Character products for $\text{Rep}(S_5)$, $\text{Rep}(SL_2(\mathbb{F}_3))$, $\text{Rep}(C_{13} \rtimes C_3)$ and $\text{Rep}(C_{11} \rtimes C_5)$ are computed with exact characters in the bundled software. Two further complete model tensors come from the published noncommutative integral appendix of [21, Theorem 1.6 and Section 6.2]: the representation category of its dimension-24 Kac algebra and the group-theoretical category $\mathcal{C}(A_5, 1, A_4, 1)$. The published full tensors and their byte digest are pinned in the package. For each constructed or published model, the program validates the fusion axioms, then checks all $6!$ unit-preserving basis permutations against every compatible census tensor. A positive record carries its explicit permutation. Different categorical structures may share a based ring; the count is of rings, not categories.

Three additional standard constructions give full-tensor matches: $R7_3_602 = K_0(\text{Rep}(A_6))$, $R7_2_218 = K_0(\text{Rep}(C_{13} \rtimes_5 C_4))$, and $R7_2_313 = K_0(\text{Adj}(\text{SU}(3)_5))$. The first uses exact character products over $\mathbb{Q}(\sqrt{5})$; the second uses induction of characters of C_{13} , with the complement generator acting by exponent five. The third is the triality-zero subcategory of the standard unitary quantum-group category [12]; exact Weyl alternants over $\mathbb{Q}(\zeta_{24})$ reconstruct all its fusion coefficients. Each is checked at all 343 structure constants.

13.1 Two dimension-24 equivariantizations

Put $\mathcal{T} = \text{TY}(C_3, \chi, 1/\sqrt{3})$, with $\chi(a, b) = \exp(2\pi i ab/3)$. Write m for its noninvertible simple and $|a| = 0$, $|m| = 1$ for its C_2 grading. Inversion $I(a) = -a$, $I(m) = m$ preserves χ and gives a coherent unitary C_2 action with canonical channel identifications as tensorators and identity square coherence [4, Section 4A and Proposition 4.8]. The associators are phases, permutations or the normalized Fourier matrix.

First take $\mathcal{E} = \mathcal{T} \boxtimes \text{Vec}(C_2)$ and $F(X, e) = (I(X), e + |X|)$. Grading additivity and the trivial pointed associator give the tensor structure, and $F^2 = \text{Id}$ as a tensor functor. Thus $\mathcal{E}^{C_2}$ is unitary. Its two fixed simple objects $(0, e)$ each have two equivariant lifts t_e^+, t_e^- ; the other invertibles form orbits $P_e = \{(1, e), (2, e)\}$, and the two odd objects form $Z = \{(m, 0), (m, 1)\}$. In the ordered basis

$$(1, h, t, ht, P_0, P_1, Z) = (t_0^+, t_0^-, t_1^+, t_1^-, P_0, P_1, Z),$$

the invertible group is C_2^2 , h fixes both P_e , t exchanges them, and all invertibles fix Z . The remaining products are

$$\begin{aligned} P_e P_f &= t_{e+f}^+ + t_{e+f}^- + P_{e+f}, & P_e Z &= Z P_e = 2Z, \\ Z^2 &= 1 + h + t + ht + 2P_0 + 2P_1. \end{aligned}$$

Here and below subscripts are modulo two. Fixed output channels are exchanged in pairs and have one invariant vector of each sign; free orbit outputs have a unique equivariant lift. These rules agree with $R7_2_1$ under the model-to-census permutation $(0, 5, 1, 2, 4, 3, 6)$.

For the second construction, let z generate C_4 and use the normalized unitary cocycle

$$\omega(a, b, c) = i^{w(a, b, c)}, \quad w(a, b, c) = a \left\lfloor \frac{b+c}{4} \right\rfloor, \quad 0 \leq a, b, c < 4.$$

Inside $\mathcal{T} \boxtimes \text{Vec}(C_4, \omega)$ take the full diagonal subcategory with simples (a, z^{2e}) , (m, z) and (m, z^3) . It is closed under tensor products and duality because TY parity equals the pointed degree modulo two. Define F by inversion in both factors. Its TY tensorator is that of I ; on pointed degrees use $J(a, b) = i^{u(a, b)}$, in the direction $F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$, where

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 3 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Reducing arguments modulo four, direct substitution gives

$$\begin{aligned} u(b, c) - u(a + b, c) + u(a, b + c) - u(a, b) &\equiv w(a, b, c) - w(-a, -b, -c), \\ u(a, b) + u(-a, -b) &\equiv 0 \pmod{4}. \end{aligned}$$

The first is tensor-functor coherence; the second makes F^2 the identity tensor functor. Identity multiplication coherence therefore defines a genuine C_2 action. The supplement checks all 256 pointed cocycle equations, 64 tensorator equations and 16 square equations.

The equivariantization has the same simple-orbit pattern. The positive lift t of $(0, z^2)$ satisfies $t^2 = h$, since $J(2, 2) = -1$; hence its invertibles form C_4 . All other displayed products remain valid, with $t_1^+ = t$ and $t_1^- = t^3$. In Z^2 , the degree-two pair of channels is exchanged with coefficients 1, 1, and the degree-zero pair with coefficients $i, -i$. Both involutions have trace zero and eigenvalues 1, -1 , giving one copy of each fixed lift. The complete tensor matches *R7_2_195* under $(0, 6, 1, 2, 5, 3, 4)$. All scalars and tensorators are unitary, so both constructions are unitary categories over $\mathbb{C}$. Besides comparing all 686 coefficients, the replay independently reconstructs them as C_2 -invariant dimensions of equivariant Hom spaces, using $(\dim + \text{tr}(F))/2$.

13.2 A quaternion equivariantization in dimension 48

Let $Q_8 = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}$ with $e_1 e_2 = e_3$, $e_2 e_3 = e_1$, $e_3 e_1 = e_2$, and $e_a^2 = -1$. For $p \in S_3$, the rule

$$\rho_p(e_a) = \text{sgn}(p) e_{p(a)}$$

is a group automorphism, and $\rho_p \rho_q = \rho_{pq}$. It gives a strict unitary action on $\text{Vec}(Q_8)$, so $\text{Vec}(Q_8)^{S_3}$ is a unitary fusion category of dimension 48. Each fixed central grade 1, -1 carries the three ordinary irreducibles 1, ε , V of S_3 ; the six noncentral grades form one free orbit, giving a simple X of dimension six [5]. Label the first six simples (z, a) , with $z \in C_2$ and $a \in \{1, \varepsilon, V\}$. Their products are those of $\text{Rep}(S_3) \boxtimes \text{Vec}(C_2)$ and $(z, a)X = X(z, a) = \dim(a)X$. In X^2 , the six ordered pairs with product either central grade form a regular S_3 -set; each noncentral grade has four ordered pairs. Consequently, in the order $(0, 1), (0, \varepsilon), (0, V), (1, 1), (1, \varepsilon), (1, V), X$,

$$X^2 = (0, 1) + (0, \varepsilon) + 2(0, V) + (1, 1) + (1, \varepsilon) + 2(1, V) + 4X.$$

This complete tensor matches *R7_4_2* under $(0, 5, 4, 1, 2, 3, 6)$. The replay checks the group action, computes the central permutation characters and compares all 343 coefficients.

13.3 The twisted quaternion model from a Chern class

The remaining dimension-48 ring $R7_4_2019$ has a unitary group-theoretical model

$$\mathcal{C}(\Gamma, \Omega, S_3, 1), \quad \Gamma = Q_8 \rtimes S_3, \quad [\Omega] = c_2(V) - \inf c_2(R).$$

The action is the signed-axis action above. To construct the class, view unit quaternions inside $SU(2)$, and distinguish the scalar imaginary unit I from quaternion units i, j, k . Set

$$r = -\frac{1+i+j+k}{2}, \quad s = I \frac{i-j}{\sqrt{2}}.$$

Then $r^3 = s^2 = 1$ and $srs = r^{-1}$; conjugation implements the signed axis cycle and transposition. Thus r, s define a unitary representation $R : S_3 \rightarrow U(2)$ with determinant the sign character, and $V(q, p) = qR(p)$ is a representation of Γ . The ordinary second Chern classes give a well-defined class in $H^4(B\Gamma, \mathbb{Z}) \cong H^3(\Gamma, U(1))$ [3]. Its restriction to S_3 is zero.

Put $z = -1 \in Q_8$ and $K = \langle z \rangle \times S_3$. If $x = c_1(L)$ for the sign line of $\langle z \rangle$ and $y = c_1(\det R)$, the splitting principle gives

$$[\Omega]|_K = c_2(L \otimes R) - c_2(R) = x^2 + xy.$$

Writing elements of K as (a, p) and parity as $\epsilon(p)$, this restriction has the explicit normalized representative

$$\omega_K((a, p), (b, q), (c, r)) = (-1)^{abc + a\epsilon(q)\epsilon(r)}.$$

Indeed the real logarithm $a(bc + \epsilon(q)\epsilon(r))/2$ has integer coboundary $ab(cd + \epsilon(r)\epsilon(s))$, representing $x^2 + xy$. The replay checks all 12^4 such identities exactly. Choose a unitary global cocycle representing $[\Omega]$. Extend a correcting normalized two-cochain from K to Γ to make its restriction exactly ω_K . Then $\Omega|_{S_3} = 1$. The group-theoretical category is the category of $\mathbb{C}[S_3]$ -bimodules in Vec_Γ^Ω [2, Section 3 and Theorem 5.1]. The unitary cocycle and standard positive separable subgroup algebra make it unitary. This proves existence without constructing a table of 48^3 cocycle values.

The central double cosets have stabilizer S_3 . Over z , the multiplier is

$$\alpha_z(p, q) = \frac{\omega_K(z, p, q)\omega_K(p, q, z)}{\omega_K(p, z, q)} = (-1)^{\epsilon(p)\epsilon(q)}.$$

It has the one-dimensional projective representation $u(p) = I^{\epsilon(p)}$. Its lift t satisfies $t^2 = \varepsilon$: the central-degree tensorator is trivial and $u(p)^2$ is the sign character. Thus the six central simples are $1, \varepsilon, v, t, t\varepsilon, tv$, with the S_3 representation products and relation $t^2 = \varepsilon$. The noncentral double coset is free and gives one simple X of dimension six. Products $YX = XY = \text{FPdim}(Y)X$ follow from support and dimension. Each central product fibre of X^2 is a free transitive S_3 -set, so its projective monomial representation is projective regular, with irreducible multiplicities $1, 1, 2$ regardless of the cocycle phases. The noncentral fibres each have size four. Hence

$$X^2 = 1 + \varepsilon + 2v + t + t\varepsilon + 2tv + 4X.$$

In this order the model-to-census permutation is $(0, 6, 5, 1, 2, 3, 4)$. The supplement checks the representation relations, action, characteristic-class restriction, projective lift, and all 343 coefficients.

13.4 The unitary D_{12} model

The ring $R7_2_240$ is the even part of the D_{12} subfactor, equivalently the local bosonic modularization of $PSU(2)_{20}$. Existence and the split-object fusion rules follow from [41, Theorem 4.14] and [42, Theorem 2.2 and Proposition 2.4]. The resulting category is unitary modular. In the model basis $f_0, f_2, f_4, f_6, f_8, P, Q$, bulk products are obtained by folding the $SU(2)_{20}$ rules under $f_{2t} = f_{2(10-t)}$ and setting $f_{10} = P + Q$. The remaining rules are

$$Pf_{2j} = \sum_{k=5-j}^4 f_{2k} + \begin{cases} P & j \text{ even,} \\ Q & j \text{ odd,} \end{cases} \quad Qf_{2j} \text{ interchanges } P, Q,$$

$$P^2 = Q + f_2 + f_6, \quad Q^2 = P + f_2 + f_6, \quad PQ = f_0 + f_4 + f_8, \quad P^* = Q.$$

The model-to-census permutation is $(0, 5, 3, 6, 4, 1, 2)$. It matches all 343 ordered structure constants; the separate associativity check has 2401 equations. The category is supplied by the cited construction, not inferred from those finite ring checks.

13.5 An inversion orbifold model for $R7_4_2665$

Theorem 38. *The ring $R7_4_2665$ has a unitary categorification.*

Proof. Put $t = 2 + \sqrt{5}$. Start with Budinski's unitary near-group solution J_{16}^1 for $C_{16} + 16$, whose existence and exact certification are given in [43, Theorem 3.1.1, Table 3.1 and Section 3.3.2]. Its data include

$$\chi(g, h) = \exp(2\pi i 3gh/16), \quad a(g) = \exp(-\pi i 3g^2/16).$$

Let $\mathcal{C}$ denote this category and $H = 4C_{16}$. The central lifts of its invertibles make $A = \bigoplus_{h \in H} h$ a commutative separable algebra in $Z(\mathcal{C})$: $H = H^\perp$ and $\chi|_{H \times H} = 1$. The category $\mathcal{D} = \mathcal{C}_A$ is unitary. Since $a(4) = -1$, the orbifold rules in [44, Theorem 12.5] are

$$r^4 = 1, \quad \rho r = r^{-1} \rho, \quad \rho^2 = r^2 + (1 + r + r^2 + r^3) \rho. \quad (14)$$

This is the already established rank-eight model $R8_1_12$. Its canonical dual $\hat{H} = C_4$ action has a generator T fixing r and cycling the four dimension- t simples.

The Morita symmetry and its central permutation. The regular C_{16} algebra gives a self-Morita equivalence $\mathcal{M}$ of $\mathcal{C}$ by [44, Theorem 11.3]. In the fixed-point model, with inclusion $\kappa : M^\alpha \rightarrow M$,

$$\alpha'_g = \text{Ad } U(-g)|_{M^\alpha}, \quad \alpha_h \kappa = \kappa, \quad \alpha_h(U(g)) = \chi(h, g)U(g).$$

The module has one fixed simple $a_0 = \kappa$ of normalized dimension four and sixteen simples b_λ of dimension t in a free orbit. Its fixed simple has internal end the regular C_{16} algebra, whose structure is unique because $H^2(C_{16}, \mathbb{C}^\times) = 0$. The opposite module has the same property. The argument of [45, Lemma 4.7 and Remark 4.8] therefore makes $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{M}$ a regular bimodule with a tensor autoequivalence G_0 on the right; uniqueness of this algebra is the only part of that argument needed here.

The central permutation of $\mathcal{M}$ is inversion. Indeed, $\text{Ad } U(h)\rho = \rho\alpha_h$ gives

$$\text{Ad } \rho(U(g))(U(h)) = \chi(h, g)^{-1}U(h).$$

On $p_\lambda = 16^{-1} \sum_h \lambda(h)^{-1}U(h)$ in $\text{End}(\rho\kappa)$, left α_g and transported right α'_g shift the character by χ_g^{-1} and χ_g , respectively. Every invertible has a unique central lift by [44, Theorem 13.4] and

triviality of the universal grading. The free orbit identifies its image as $g \mapsto -g$. Thus the square fixes all central-invertible labels. The faithful action on these labels in [44, Corollary 13.5] implies $G_0 \simeq \text{Id}$. Inversion also preserves H and its regular central algebra A .

Descent to a coherent involution. We retain the algebra identification needed for central base change. An invertible bimodule whose center equivalence preserves A descends, after choosing that identification, to an invertible $\mathcal{C}_A\text{--}\mathcal{C}_A$ bimodule of left A -module objects. The identification balances the right action. This construction commutes with relative tensor products: both sides classify balanced functors with a compatible A -action, centrality moves A across the balancing variable, and separability splits the coequalizers. This comparison respects associators and inverse bimodules. Two algebra identifications differ by $\text{Aut}_{\text{alg}}(A) = \widehat{H}$, and therefore by a power of T after descent. The descended bimodule $\mathcal{L}$ satisfies

$$\mathcal{L} \boxtimes_{\mathcal{D}} \mathcal{L} \simeq \mathcal{D}_{T^j}$$

for some j modulo four.

The object a_0 has four A -module structures, each of normalized dimension one, and the free sixteen-element orbit gives four induced simples of dimension t . Left invertibles act freely on the former. The equation $\rho a_0 = \sum_{\lambda} b_{\lambda}$ shows that the four summands of $Q(\rho)$ take any chosen dimension-one module to the four dimension- t modules once each. Hence $\mathcal{L}$ is the regular left $\mathcal{D}$ -module, and its right action is an actual tensor autoequivalence F .

Write its left A -action on κ as scalars $\lambda_h : \alpha_h \kappa \rightarrow \kappa$. Left α_g multiplies them by $\chi(h, g)$; right α'_g , transported by $U(g)$, gives

$$U(g)\lambda_h\alpha_h(U(g)^*) = \chi(h, g)^{-1}\lambda_h.$$

This calculation on left A -modules is unaffected by the choice balancing the right algebra. Therefore $F(r) = r^{-1}$. Adjusting that choice by a character, and thus composing with T , gives

$$F(r) = r^{-1}, \quad F(\rho) = r^{-1}\rho. \quad (15)$$

At the tensor-functor level the regular-module identification retains an inner ambiguity: $F^2 \simeq \text{Inn}(r^{\ell})T^j$.

Central monodromy removes it. The crossing of a central invertible p with an A -module has defect on the summand h equal to $\chi(p, h)^2$. Checking A -linearity of that crossing gives a monoidal isomorphism between $\text{Inn}(Q(p))$ and the dual action of this character. The defect for $p = 1$ is the order-two character of H , whereas that for $p = 2$ is trivial. Consequently

$$T^2 \simeq \text{Inn}(r), \quad \text{Inn}(r^2) \simeq \text{Id}.$$

By (15), F^2 fixes simple labels. Its action on the noninvertibles gives $2\ell + j = 0 \pmod{4}$. Thus j is even and $F^2 \simeq \text{Inn}(r^{\ell+j/2}) \simeq \text{Id}$. The universal grading of $\mathcal{D}$ is trivial by (14), so $\text{Aut}_{\otimes}(\text{Id}_{\mathcal{D}}) = 1$. The two coherence composites for $F^3 \rightarrow F$ therefore agree. We obtain a coherent unitary C_2 action, not only an involution of the based ring.

The reflection dual and its pointed group. Form $\mathcal{E} = \mathcal{D} \rtimes_F C_2$, with odd invertible s . Its rules include

$$r^4 = s^2 = 1, \quad srs = r^{-1}, \quad \rho r = r^{-1}\rho, \quad \rho s = rs\rho,$$

together with (14). Its eight invertibles form D_8 , and the other eight simples are a free orbit of dimension t . The algebra $B = 1 + s$ exists from the coherent crossed product. Let $\mathcal{K} = {}_B\mathcal{E}_B$ and

$L = \{1, s\}$. On integral support, the double cosets L and r^2L each give two invertibles; LrL gives a unique dimension-two simple X . The two irrational double orbits are

$$U\rho, \quad r^2U\rho, \quad U = \{1, s, rs, r^{-1}\}.$$

Their stabilizers are trivial, so each gives one simple, denoted Y, Z , of dimension $2t$. Since $\rho^* = r^2\rho$, their duals interchange.

The group of four invertibles requires the associator, not just the D_8 multiplication table. In the crossed-product factor model of [44, Section 12.1],

$$\alpha_1^4 = \text{Ad } \lambda_4, \quad \alpha_1(\lambda_4) = \chi(1, 4)\lambda_4 = -i\lambda_4.$$

The cyclic kernel obstruction is thus a primitive order-four class in $H^3(C_4, \mathbb{C}^\times)$. The integral reflection dual is $\mathcal{K}_{\text{int}} = \mathcal{C}(D_8, L, \Omega, \psi)$. The group-theoretical exponent formula [47, Theorem 9.2] gives

$$\text{FSexp}(\mathcal{K}_{\text{int}}) = \bigsqcup_{C \leq D_8 \text{ cyclic}} |C| \text{ ord}(\Omega|_C).$$

The subgroup $\langle r \rangle$ forces divisibility by 16. On the other hand $\mathcal{K}_{\text{int}}$ is a Tambara–Yamagami category, since it has four invertibles and a unique dimension-two simple. Every such category on V_4 has Frobenius–Schur exponent four or eight by [48, Proposition 5.3]. The invertibles of $\mathcal{K}$ therefore form $C_4 = \langle g \rangle$. Here $1, g^2$ lie above L and fix Y, Z , while g, g^3 lie above r^2L and interchange them.

Complete fusion rules. The free induced bimodules are $X = BrB$, $Y = B\rho B$, $Z = Br^2\rho B$. The genuine relative-tensor identity

$$(BuB) \otimes_B (BvB) \simeq BuBvB$$

computes their products using the displayed rules of $\mathcal{E}$. It gives

$$\begin{aligned} X^2 &= 1 + g + g^2 + g^3, & XY &= YX = XZ = ZX = Y + Z, \\ Y^2 &= Z^2 = g + g^3 + X + 4Y + 4Z, & YZ &= ZY = 1 + g^2 + X + 4Y + 4Z. \end{aligned}$$

For example the integral support of Y^2 is $2r^2L + LrL$; Frobenius reciprocity gives g, g^3 once each, and the last part is X . Every irrational constituent occurs four times. The other three irrational products follow from the same identity. These rules, $gX = X$, and $gY = Z$ determine all coefficients. The independently generated model matches the full census tensor under the permutation

$$(1, g, g^2, g^3, X, Y, Z) \longmapsto (0, 1, 6, 2, 5, 3, 4).$$

All constructions preserve unitarity. This proves the theorem. $\square$

The exact replay checks all 343 source coefficients, all fusion axioms of the rank-seven and rank-sixteen tensors, the neutral $R8_1_12$ identification, and the eight relative-induction products used above. It does not infer the coherent action or the associator from those finite checks; those are supplied by the preceding proof. The same inversion-descent framework gives a second model with a different parent group and cyclic associator.

Theorem 39. *The ring $R7_4_35$ has a unitary categorification.*

Proof. Use Yu–Zhang’s unitary $C_4 \times C_4 + 16$ solution [49, Proposition 3.1 and Equation (13)], with

$$\chi((x, y), (u, v)) = i^{xu-yv}, \quad a(x, y) = \exp(2\pi i 3(x^2 - y^2)/8), \quad c = 1.$$

The subgroup $H = \langle (1, 1) \rangle$ is cyclic of order four, $H = H^\perp$, and $a|_H = 1$. Its quotient is generated by the class r of $(1, 0)$. Central de-equivariantization gives

$$r^4 = 1, \quad \rho r = r^{-1} \rho, \quad \rho^2 = 1 + (1 + r + r^2 + r^3) \rho. \quad (16)$$

We justify the parent Morita square despite $H^2(C_4 \times C_4, \mathbb{C}^\times) \neq 0$. In the fixed-point model of Theorem 38, the left internal-end algebra of κ is untwisted. On the right, choose $u_g = U(g) : \kappa \alpha'_g \rightarrow \kappa$. Right whiskering by α'_h precomposes the homomorphism and leaves the implementing intertwiner unchanged. Hence the right stabilizer cocycle is trivial as well:

$$u_h \circ (u_g \text{ right-whiskered by } \alpha'_h) = U(h)U(g) = U(g + h) = u_{g+h}.$$

Here U is the genuine representation in Izumi’s normalization. No application of α'_h to $U(g)$ is involved; $U(g)$ need not belong to its domain M^α . Thus the original and inverse left modules have the same untwisted regular internal-end algebra. Their square is regular up to a tensor autoequivalence. The central permutation is again inversion, and its square fixes all invertible labels. The faithful action in [44, Corollary 13.5] makes that autoequivalence trivial.

The central base-change and regularity arguments from Theorem 38 now apply. The descended symmetry can again be chosen with $F(r) = r^{-1}$ and $F(\rho) = r^{-1} \rho$. Indeed $\chi((1, 0), (j, j)) = i^j$, so the left and right character shifts are inverse and primitive. Their monodromy gives $T^2 \simeq \text{Inn}(r)$, while $(2, 0)$ gives a central lift of r^2 . These identities remove the same inner ambiguity in F^2 , and triviality of the universal grading from (16) supplies coherence. The unitary crossed product has eight invertibles D_8 and $\rho s = r s \rho$, now with invertible term 1 in ρ^2 .

The reflection algebra $B = 1 + s$ again has a dual with four invertibles, one dimension-two simple X , and two dimension-2t simples Y, Z . Here $\rho^* = \rho$, so both irrational double orbits are self-dual. The pointed cyclic associator is trivial: $(1, 0)$ is an actual order-four complement to H , and its extended automorphism has fourth power the identity, including on each $\lambda_{(j,j)}$ since $(i^j)^4 = 1$. Thus the associator Ω on the pointed D_8 part restricts trivially to its rotation C_4 . Every other cyclic subgroup has order two, with a third-cohomology class of order at most two. The formula in [47, Theorem 9.2] consequently gives

$$\text{FSexp}(\mathcal{K}_{\text{int}}) \mid 4.$$

The integral Tambara–Yamagami sector cannot have pointed group C_4 . For that group, [50, Theorem 3.4 and its specialization] gives

$$\nu_4(X) = \frac{1}{2} \sum_{x \in C_4} \chi(x, x)^{-1} = 1 + i \quad \text{or} \quad 1 - i,$$

because nondegeneracy forces $\chi(1, 1) = \pm i$. This differs from $\text{FPdim}(X) = 2$, independently of the sign of the TY parameter, contradicting the exponent bound. The pointed group is therefore $V_4 = \{1, g, h, gh\}$, with h fixing Y, Z and g interchanging them.

The same free-induction identity yields the complete rules

$$\begin{aligned} X^2 &= 1 + g + h + gh, & XY &= YX = XZ = ZX = Y + Z, \\ Y^2 &= Z^2 = 1 + h + X + 4Y + 4Z, & YZ &= ZY = g + gh + X + 4Y + 4Z. \end{aligned}$$

Together with $g^2 = h^2 = 1$, $gX = hX = X$, $gY = Z$, and $hY = Y$, these match all 343 census coefficients under

$$(1, g, h, gh, X, Y, Z) \mapsto (0, 1, 5, 2, 6, 3, 4).$$

The separate source-independent replay verifies both tensor ranks, all eight induction identities, and the fourth-indicator values. The categorical operations are unitary, proving the theorem. $\square$

The two constructions settle *R7_4_35* and *R7_4_2665* by one descent method. They do not decide the other two sources *R7_4_2053* and *R7_4_2603* in Proposition 29.

14 Extension to exact multiplicity five

14.1 Source, identifiers, and verification scope

The extension uses the supplied archive `rank7_multiplicity5_verified.zip`. Its expanded multiplication-table file has SHA-256

`a3fdcb75fbeb090913b1e4650e5124b9e0b1d016f1a71e0368a3f38e6e520c03`.

It contains 10006 rank-seven tensors through multiplicity five, including 5805 of exact multiplicity five. Fresh verification checks unit, duality, Frobenius reciprocity, associativity, nonnegative integral coefficients, and based-isomorphism uniqueness. All $7! = 5040$ permutations are unnecessary: the unit must be fixed, so $6! = 720$ permutations per tensor suffice. The exhaustive audit thus examines 7204320 unit-fixing permutations and finds no duplicates.

The 4201 previously published tensors occur exactly in the new input. The input does not put them in an initial contiguous block. We therefore retain every old identifier and decision and record an explicit transport between the previous and new physical line numbers. For the added band, *R7_5_j* denotes the j th row of exact multiplicity five in the supplied file, with $1 \leq j \leq 5805$. The new duality strata contain 4654, 1012, 138, and 1 rings with respectively zero, one, two, and three nonself-dual pairs; nine of the new rings are noncommutative.

The producer's run metadata reports complete enumeration of every rank-seven duality stratum with no time limit. The optimized enumeration source was supplied with the subsequent rank-eight multiplicity-three update. Its checksum agrees with both producer receipts, and the census repository now contains the source and independent implementation audits. The present update independently verifies the output tensors, uniqueness, and preservation of the earlier census; it does not rerun that enumeration or independently establish its exhaustive coverage. All classification counts for the extension are therefore statements about the supplied 5805 new classes, with completeness relative to all rings in the band conditional on the producer's enumeration claim.

14.2 A Frobenius-group realization

Proposition 40. *The new ring *R7_5_5678* has a unitary categorification $\text{Rep}(C_7 \rtimes C_6)$, where a generator of C_6 acts on C_7 by multiplication by 3.*

Proof. Write the group as $G = \langle a, b \mid a^7 = b^6 = 1, bab^{-1} = a^3 \rangle$. Its six linear characters are $\chi_j(a^x b^y) = \zeta_6^{jy}$, $0 \leq j < 6$. The action of C_6 is transitive on the six nontrivial characters of $\langle a \rangle$. Inducing any such character gives an irreducible character ρ of degree six, with values 6 at the identity, -1 at nonidentity elements of $\langle a \rangle$, and 0 elsewhere. These characters are orthonormal,

and their squared degrees sum to $6 + 36 = 42 = |G|$, so they are the complete irreducible character list. Pointwise multiplication gives

$$\chi_i \chi_j = \chi_{i+j \bmod 6}, \quad \chi_i \rho = \rho, \quad \rho^2 = \sum_{j=0}^5 \chi_j + 5\rho.$$

In the model order $(\chi_0, \dots, \chi_5, \rho)$, the permutation $(0, 1, 4, 6, 3, 2, 5)$ gives the corresponding census labels. The accompanying checker verifies equality of all 343 structure constants under this map. Finite-group representation categories admit a unitary structure, proving the claim. This is a known construction; its occurrence in the extended census is not a claim of a new tensor category. $\square$

14.3 The noncyclic maximal near-group obstruction

The only noncommutative ring remaining after the first arithmetic passes is $R7_5_4812$. Its invertible labels are $G = \{0, 1, 2, 3, 5, 6\}$ and its remaining simple is $X = b_4$, with

$$X^2 = \sum_{g \in G} g + 5X.$$

Every invertible fixes X on both sides. The group G is nonabelian: $b_1 b_3 = b_6$ whereas $b_3 b_1 = b_5$. Nikshych–Ostrik’s maximal near-group theorem [13, Appendix A, Theorem A.14(ii)] states that a near-group category with parameter $n = |G| - 1$ must have cyclic G . It therefore excludes this tensor for arbitrary complex fusion categories. The appendix supplies the general near-group reduction; pivotality is not an extra hypothesis here. The exact verifier checks the invertible group, the entire near-group rule, and the displayed noncommuting products.

14.4 Integral formal-codegree refinement

Proposition 41. *Let R be a commutative fusion ring with a positive integral dimension character (d_i) , and put $D = \sum_i d_i^2$. If R admits a complex categorification, then D/f is a positive integer for every formal codegree f of R . In particular, every formal codegree is an integer dividing D .*

Proof. The positive character is the Frobenius–Perron character. A hypothetical category has integral Frobenius–Perron dimension D and hence is pseudo-unitary by [28, Proposition 8.24]. Give it its canonical spherical structure. Each one-dimensional character of R corresponds, by central induction, to a simple summand of the induced unit having dimension D/f ; see [13, Theorem 2.13]. The forgetful functor preserves these dimensions, and its image is a nonnegative integral sum of the simples of the original category. Since all d_i are integers, D/f is a positive integer. Thus f is rational; as a formal codegree it is an algebraic integer, so f is an integer dividing D . $\square$

The checker first verifies the positive dimension character against every product and then factors the exact Casimir characteristic polynomial. It yields the following six obstructions; only one failed factor per ring is needed.

Census ID	D	Obstructing codegree or factor
$R7_5_1390$	112	$t^2 - 40t + 80$
$R7_5_1404$	240	$t^2 - 40t + 80$
$R7_5_2454$	1428	$t^2 - 48t + 144$
$R7_5_4695$	220	12
$R7_5_5187$	96	64
$R7_5_5759$	96	64

The quadratic factors are irreducible over $\mathbb{Q}$, whereas the listed integer codegrees do not divide D . These tests strengthen the basic d -number test for this integral subfamily; they do not assert that arbitrary fusion categories have integral dimensions.

14.5 A cyclic-stabilizer refinement

The prime-order criterion in Proposition 8 admits the following extension to arbitrary cyclic order.

Proposition 42. *Let $H \cong C_n$ be a subgroup of invertible objects in a complex fusion category. Suppose that a simple X is fixed on the left by H , a simple Y is fixed on the right, and a simple Z is fixed on both sides. If $N = N_{XY}^Z > 0$, then*

$$\frac{\gcd(n, N)}{n} \text{FPdim}(Z) \quad \text{is an algebraic integer.}$$

Proof. As in Proposition 8, the fixed simple trivializes the associator class on H , and $H^2(C_n, \mathbb{C}^\times) = 0$ supplies module structures. Thus $A = \bigoplus_{h \in H} h$ is a connected separable algebra and $X \otimes Y$ is an A -bimodule. The Z -isotypic summand is stable under both actions, because H fixes Z on both sides. It is therefore an A -bimodule with underlying object NZ . Its Frobenius–Perron dimension in the Morita dual is $N \text{FPdim}(Z)/n$, which is an algebraic integer. Write $g = \gcd(n, N) = uN + vn$ with $u, v \in \mathbb{Z}$. Then

$$\frac{g}{n} \text{FPdim}(Z) = u \frac{N}{n} \text{FPdim}(Z) + v \text{FPdim}(Z)$$

is also an algebraic integer. $\square$

This extends the prime-order argument without a pivotality assumption. An application records a cyclic subgroup, its left and right actions, the coefficient N , and an exact nonintegrality witness for the scaled Frobenius–Perron dimension. The proof is independent of any finite search or numerical approximation. The extended checker finds 54 witnesses among the 178 tensors surviving the basic passes. Of these, 49 already satisfy the prime-order test. The five additional order-four applications are:

Census ID	g	$X = Y = Z$	N	Minimal polynomial of $\text{FPdim}(Z)$
<i>R7_5_4702</i>	b_1	b_4	1	$t^2 - 6t - 4$
<i>R7_5_4704</i>	b_1	b_4	3	$t^2 - 8t - 4$
<i>R7_5_4706</i>	b_1	b_4	5	$t^2 - 10t - 4$
<i>R7_5_4757</i>	b_1	b_4	3	$t - 10$
<i>R7_5_5676</i>	b_1	b_3	5	$t^2 - 10t - 4$

Each N is odd, so the necessary quantity is $\text{FPdim}(Z)/4$. For a quadratic $t^2 - at - 4$, its minimal polynomial after division by four is $t^2 - (a/4)t - 1/4$, which is not integral. In the linear case the quotient is $5/2$. The exact replay also verifies the subgroup actions and the minimal polynomial of the Perron eigenvalue, using exact root isolation. The application counts in this subsection may overlap earlier tests; Table 2 charges each excluded ring only once.

14.6 A cubic Fourier-module obstruction

Proposition 43. *The fusion ring *R7_5_5007* has no complex fusion-category categorification.*

Proof. Write $g = b_1$, so $g^2 = b_2$, and write the four remaining, self-dual basis objects as X_3, X_4, X_5, X_6 . Each is fixed by g .

The pointed subcategory generated by g has trivial associator class: the action on any one of these fixed simple objects supplies a rank-one module category. Thus $A = 1 \oplus g \oplus g^2$ is a separable algebra. The category of right A -modules has one simple over the free orbit $\{1, g, g^2\}$ and three simples over each X_i . The dual group $\widehat{C}_3$ acts by twisting the module structure, cyclically permuting each triple and fixing the free-orbit simple. This action commutes with the left action of the original category. For a primitive cube root ζ , its ζ -eigenspace therefore yields a four-dimensional representation $b_i \mapsto T_i$ of the fusion ring. In a Fourier basis,

$$(T_i)_{ab} = n_0 + n_1\zeta + n_2\zeta^2, \quad n_j \in \mathbb{Z}_{\geq 0}, \quad n_0 + n_1 + n_2 = N_{i,a}^b, \quad a, b \in \{3, 4, 5, 6\}.$$

Frobenius reciprocity for the module action gives $T_{i^*} = T_i^*$. In particular the four matrices T_i for $i = 3, 4, 5, 6$ are Hermitian. The matrix T_g is diagonal, with cube roots of unity on the diagonal. Since $gX_i = X_i$, the identity $T_g T_i = T_i$ implies that a nontrivial diagonal entry of T_g forces the corresponding row of every T_i ($i = 3, 4, 5, 6$) to vanish. But a sum of three nonnegative multiples of $1, \zeta, \zeta^2$ vanishes only when their total is divisible by three. The entries $N_{3,3}^4 = 1$, $N_{3,4}^3 = 1$, $N_{3,5}^3 = 2$, and $N_{3,6}^3 = 2$ consequently force $T_g = I$.

Set $T = T_3$. The following identity holds in the fusion ring, as checked by multiplying the given integer matrices:

$$(X_3 - 9)(X_3 + 1)(X_3^3 - X_3^2 - 4X_3 - 1) = 6 - 3g - 3g^2.$$

Thus $(T - 9I)(T + I)(T^3 - T^2 - 4T - I) = 0$. The matrix $B = (N_{3,a}^b)_{a,b=3}^6$ is

$$B = \begin{pmatrix} 3 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 3 \\ 2 & 3 & 3 & 3 \end{pmatrix}.$$

Its entries dominate the absolute values of the entries of T . The leading principal minors of $9I - B$ are 6, 41, 268, 47, so $\rho(B) < 9$ and 9 is not an eigenvalue of T . The characteristic polynomial of a Hermitian matrix over $\mathbb{Q}(\zeta)$ has rational coefficients; here they are algebraic integers and hence integers. Since $x^3 - x^2 - 4x - 1$ is irreducible over $\mathbb{Q}$, the only possibilities are $(x + 1)^4$ and $(x + 1)(x^3 - x^2 - 4x - 1)$. Reduction modulo $(1 - \zeta)$ shows $\text{tr } T \equiv \text{tr } B = 9 \equiv 0 \pmod{3}$, which excludes the first possibility. Consequently

$$\text{tr } T = 0, \quad \text{tr } T^2 = 10, \quad \text{tr } T^3 = 15.$$

A real diagonal entry with total multiplicity N has the form $N - 3b$ with $0 \leq b \leq \lfloor N/2 \rfloor$. Together with the first two moments, this forces diagonal $(0, -1, 1, 0)$. For an off-diagonal entry with multiplicity 1 or 2, its squared norm is at least one; for multiplicity three its squared norm is at least zero. There are four of the former entries and two of the latter. Since the diagonal squared norm is two, the second-moment bound is saturated: the four entries have norm one, and the other two vanish. The only nonzero triangle is on labels 3, 4, 5. The diagonal contributions and the edge contributions to $\text{tr } T^3$ cancel, leaving six times the real part of a product of three norm-one entries. Its absolute value is at most six, contradicting $\text{tr } T^3 = 15$. The certificate checker also exhausts all 81 possible matrices after these bounds, obtaining cubic traces only -3 and 6 . $\square$

14.7 Subring descent and involution modules

A full fusion-closed, dual-stable subset of the basis generates a fusion subcategory in any categorification. Thus the general rank-six certificates from [19] give the following five ambient exclusions. Each replay checks the restricted tensor and its nested exclusion certificate, rather than relying on an ID or a recorded status alone.

Rank-seven ring	Rank-six subring	Ambient labels
$R7_5_1033$	$R6_4_304$	0, 1, 2, 3, 5, 6
$R7_5_1283$	$R6_3_194$	0, 1, 2, 3, 5, 6
$R7_5_2019$	$R6_3_244$	0, 1, 2, 3, 5, 6
$R7_5_5232$	$R6_2_51$	0, 1, 2, 4, 5, 6
$R7_5_5681$	$R6_4_19$	0, 1, 2, 3, 4, 6

The anti-invariant representation of Proposition 17 also applies to the extension. Exact spectral, parity, and matrix bounds exclude

$R7_5_1396$, $R7_5_1781$, $R7_5_2181$, $R7_5_2247$, $R7_5_2454$, $R7_5_3078$, $R7_5_3102$.

The checker allows more than two fixed simples; the initial cheaper involution pass was limited to smaller blocks. These are further applications of the same general module-category argument.

The one-spectrum criterion of [6, Theorem 6.8] was also checked on the full new band. It supplies 66 exact witnesses, all on tensors already excluded by the basic passes, so it contributes no additional first-stage exclusions. The supplementary evidence includes its independent replay and positive-model controls.

14.8 A unified quadratic and sphericalization continuation

The continuation of 20 October applies three related mechanisms: the full induced-unit compression bound, exhaustive sphericalization followed by dimension-field reduction, and simultaneous first and second central twist moments. Their roles are different. Compression can exclude an arbitrary categorification directly; a spherical obstruction requires a separate argument covering every signed quotient; and a pseudo-unitary center obstruction additionally requires the spherical dimension character to belong to the Frobenius–Perron Galois orbit.

Proposition 44. *The rings $R7_5_4653$, $R7_5_5029$ and $R7_5_5040$ admit no complex fusion categorification.*

Proof. For a real quadratic field L of fundamental discriminant q , collect the irreducible factors $(x^2 - t_j x + n_j)^{m_j}$ of the regular Casimir characteristic polynomial whose roots generate L . Define

$$Q_L(R) = q \sum_j m_j \frac{t_j^2 - 4n_j}{n_j^2}.$$

Corollary 14 gives the necessary condition $Q_L(R) \leq 4$ for arbitrary complex categorification. In particular, it does not require the original category to admit a pivotal structure. The exact factors and rational values are

Ring	Quadratic Casimir factor	q	$Q_L(R)$
$R7_5_4653$	$x^2 - 3960x + 7260$	165	49
$R7_5_5029$	$x^2 - 1584x + 6336$	88	49/9
$R7_5_5040$	$x^2 - 1584x + 6336$	88	49/9

Each exceeds four. Recall why the general bound differs from the spherical bound $[K : L]Q_L(R) \leq 1$: in sphericalization the unsigned sector has Casimir $2B$, so its contribution is $Q_L(R)/4$. The other sector contributes a nonnegative squared norm. The obstruction therefore holds independently of any proposed choice of signed quotient. $\square$

Proposition 45. *The ring $R7_5_4569$ admits no complex fusion categorification.*

Proof. Its Casimir characteristic polynomial is

$$(x - 14)(x - 3)(x - 7)^3(x^2 - 2940x + 17640).$$

For $L = \mathbb{Q}(\sqrt{15})$, of fundamental discriminant 60, this gives $Q_L(R) = 81/49 > 1$. Corollary 13 excludes a spherical categorification, but this inequality alone would not exclude an arbitrary fusion category.

Apply the complete oriented sphericalization enumeration to the original tensor. All its simples are self-dual. Choose the lift with positive second indicator for each simple, with the actual tensor unit as unit. The difference tensor T has the same parity as N , satisfies $-N_{ij}^k \leq T_{ij}^k \leq N_{ij}^k$, and obeys the oriented cyclic identities. The odd-square-channel condition from [6, Theorem 3.9] imposes $T \equiv N \pmod{4}$ on the relevant channels. These constraints apply in sphericalization and do not assume a pivotal structure on the source category.

Exhausting associativity on these finite domains gives exactly one solution, namely $T = N$. The replay visits 2285932869 search nodes; completion is regenerated, rather than inferred from a candidate list. The multiplicity of an opposite lift is $(N - T)/2$, hence is zero in every product. The selected lifts form a spherical tensor subcategory, and forgetting is a tensor equivalence to the original category. This contradicts its spherical obstruction. $\square$

Proposition 46. *Every complex fusion categorification of any of*

$$R7_5_443, \quad R7_5_1214, \quad R7_5_3895, \quad R7_5_5039$$

admits a spherical structure and has a pseudo-unitary Galois conjugate. This is a structural reduction, not an existence assertion.

Proof. Complete oriented enumeration gives respectively 1, 1, 2, 4 signed quotients. For the first two rings the sole quotient is the original tensor. Both quotients of $R7_5_3895$, and two quotients of $R7_5_5039$, are gauge equivalent to the original tensor. Switching the lift choices by that gauge gives a tensor section of sphericalization, as in Proposition 45.

For either remaining quotient of $R7_5_5039$, the unsigned and signed Casimirs contain respectively

$$x^2 - 1440x + 6400, \quad x^2 - 616x + 1760.$$

Their roots generate $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{11})$. The two sectors of the sphericalization have twice these Casimir roots. The algebras are commutative, so these are twice the corresponding formal codegrees. Every formal codegree of a spherical category belongs to its dimension field: a constituent of the induced unit has dimension D/f , and forgetting expresses this dimension as an integer combination of source simple dimensions. Thus the sphericalization's dimension field must contain the degree-four compositum of these two quadratic fields. But every character component of either signed quotient has degree at most two. Its spherical dimension character belongs to that signed sector, giving a contradiction. This excludes the nongauge quotients.

Finally the exact character decomposition of each original ring shows that every character outside the Frobenius–Perron orbit either vanishes on a simple object or takes unequal values on

dual objects. Neither is a spherical dimension character. Galois conjugation therefore makes all simple dimensions equal to their Frobenius–Perron values, proving the assertion. $\square$

14.9 The conductor-sixty two-moment obstruction

Theorem 47. *The rings $R7_5_443$ and $R7_5_1214$ admit no complex fusion categorification.*

By Proposition 46, it suffices to exclude a pseudo-unitary categorification with its canonical spherical structure. The following argument extends Section 9 to conductor sixty. A relative-trace refinement avoids any assumption that paired individual twists coincide.

Put $s = \sqrt{15}$ and $\eta = 4 + s$. The source dimension vectors are

$$\begin{array}{c|c} 443 & (1, 1, 4 + s, 5 + s, 5 + s, 4 + s, 6 + 2s) \\ 1214 & (1, 1, 4 + s, 4 + s, 5 + s, 5 + s, 6 + 2s). \end{array}$$

Both global dimensions are $D = 60\eta$, with $\eta\bar{\eta} = 1$. The quadratic Casimir factor is $x^2 - 480x + 3600$, so $Q_{60} = 1$. The remaining formal codegrees are respectively $(12, 5, 4, 6, 6)$ and $(5, 12, 12, 4, 4)$.

There is a cyclotomic automorphism that flips s and fixes every other source character. To check the latter assertion, use the separating matrix $\sum_i w_i N_i$, where $w = (1, 3, 11, 37, 101, 307, 997)$. Its characteristic polynomials are

$$\begin{aligned} 443 : & (x - 683)(x - 422)(x - 230)(x - 184)(x + 298) \\ & \cdot (x^2 - 15896x - 26866796), \\ 1214 : & (x - 953)(x - 24)(x + 28) \\ & \cdot (x^2 - 16436x - 22501976)(x^2 + 712x - 572). \end{aligned}$$

All roots are distinct. The only other nonrational character field is $\mathbb{Q}(\sqrt{3})$ in the second ring, which is disjoint from $\mathbb{Q}(\sqrt{15})$. Formal-codegree rationality alone would not suffice for this choice of automorphism.

Write $d_i = a_i + b_i s$, $a(v) = \sum_i a_i v_i$ and $b(v) = \sum_i b_i v_i$. The full character transport formula (4) gives the integral involution

$$T(v) = v + (b_i)_i (a(v) - 4b(v)).$$

Every central restriction row has partner $T(v)$ or $-T(v)$, with one common sign making all entries nonnegative. A mixed-sign image is impossible. The Galois image J of the central unit has restriction row $(1, 0, 1, 1, 1, 1, 2)$ and dimension $\eta^2 = 31 + 8s$.

For $w = e_1 - e_J$, the known induced-unit coordinates give

$$\text{proj}_{I(1)} Sw = (D^{-1} - \bar{D}^{-1})w = -\frac{w}{\sqrt{60}}.$$

This saturates the compression bound. The equality and primitive-mode argument of Section 9 implies

$$z/D \notin \mathbb{Q} \implies \theta^{60} = 1, \quad \chi_{60}(k) = -1 \text{ for primitive } \theta = e^{2\pi i k/60}.$$

The units modulo sixty do not all square to one. Consequently we do *not* conclude that individual paired twists are equal. Instead let t be normalized relative trace to $\mathbb{Q}(\sqrt{15})$. Modular Galois symmetry replaces θ by θ^{u^2} , while every square automorphism fixes this quadratic field. Hence $t(\theta^{u^2}) = t(\theta)$ and $t(\theta^{2u^2}) = t(\theta^2)$. Paired row types have equal multiplicities at each pair of relative-trace values, which is exactly the equality used below.

Lemma 48. *For every root of unity, $(t(\theta), t(\theta^2))$ lies in the convex hull of the following nine vectors, where (a, b, c, d) denotes $(a + bs, c + ds)$:*

$$(1, 0, 1, 0), (-1, 0, 1, 0), (1/2, 0, -1/2, 0), (-1/2, 0, -1/2, 0), (0, 0, -1, 0), \\ (0, 1/8, -1/8, 0), (0, -1/8, -1/8, 0), (0, 0, 0, 1/8), (0, 0, 0, -1/8).$$

For the nonrational normalized-dimension rows, the first five vertices and $(0, -1/8, -1/8, 0)$ suffice.

Proof. If the order n is not divisible by sixty, the first trace is the Ramanujan value $\mu(n)/\varphi(n)$. For $60 \mid n$, its rational part vanishes; its irrational part is nonzero precisely when $n = 60m$, where m is squarefree and coprime to thirty. The corresponding pair is

$$\left(\frac{\pm s\mu(m)\chi_{60}(m)}{8\varphi(m)}, -\frac{\mu(m)}{8\varphi(m)} \right).$$

For $m = 1$ these are the primitive-sixty vertices. Otherwise $w = 1/\varphi(m) \leq 1/6$; place weight w on the matching primitive vertex and absorb the rational second-coordinate correction in the vertical segment $(0, [-1, 1])$ contained in the rational five-vertex hull. If only the second moment has an irrational part, $n = 120m$ with the same restrictions, and the pair lies between the two primitive-120 vertices. All remaining pairs are rational. Odd squarefree orders give (u, u) , twice odd squarefree orders give $(-u, u)$ with $-1/2 \leq u \leq 1$, and orders divisible by four give $(0, u)$ with $-1 \leq u \leq 1$. These lie in the same five-vertex hull. The saturated twist constraint leaves only the negative-character primitive vertex in addition to the rational hull. This covers all root orders; no finite cutoff on possible twists is imposed. $\square$

Let B be the central induction Gram matrix in (2). The exact necessary integrality of D/z for a central simple of dimension $z = a + bs$ gives

$$a^2 - 15b^2 \mid 60a, \quad a^2 - 15b^2 \mid 60b.$$

The six nonunit rows in $I(1)$ have dimensions D/f , include the forced row for J , sum to $B_{0,*} - e_0$, and leave a positive semidefinite residual Gram matrix. Exhaustive bounded enumeration gives 110 frames for $R7_5_443$ and 157 for $R7_5_1214$. Every bound comes from B . Before frame-specific constraints there are respectively 3670 and 3741 possible residual rows. The exact test $vv^T \leq R$ includes singular residual matrices and is also applied to each Galois partner.

The odd-isotypic pivotal forms on XYX^* and X^3 force all seven second indicators to $+1$. Combine the 21 Gram equations with the 12 rational-coordinate equations for each of

$$\text{Tr}(\theta \mid I(X_i)) = 0, \quad \text{Tr}(\theta^2 \mid I(X_i)) = D\nu_2(X_i), \quad i \neq 0.$$

Multiply all columns by eight and pair Galois rows at a common hull vertex. The resulting system has integer columns. A necessary integral Gram-span condition modulo a small prime rejects respectively 13 and 20 frames. The remaining 97 and 137 frames are separated by respectively 9 and 13 integer covectors y satisfying

$$y \cdot c \geq 0 \quad \text{for every allowed column } c, \quad y \cdot b < 0.$$

These inequalities rule out nonnegative real solutions, and therefore every possible center. The exact replay regenerates all domains and checks the inequalities without invoking an optimizer. Together with Proposition 46, this proves Theorem 47.

14.10 Further complete signed and induction proof trees

The same distinction between spherical obstructions and general categorification is essential in six additional decisions. For $R7_5_4530$ and $R7_5_4531$, every candidate spherical dimension character either vanishes on a simple or lies in one cubic orbit. In that orbit a constituent of the induced unit has dimension D/f . An exact linear functional, integer-valued on the integer span of the simple dimensions, takes the value $4/3$ on D/f . Forgetting the constituent would express D/f in that integer span, a contradiction. This dimension-order argument uses spherical dimensions without a positivity assumption.

Complete oriented enumeration gives respectively four and one signed quotients. The sole quotient for $R7_5_4531$ is its original tensor. For the three nongauge quotients of $R7_5_4530$, the character covers consist of a zero simple dimension or a quartic orbit failing formal-codegree divisibility at the source codegree nine. The gauge branches transfer sphericity by a tensor section. Thus both conclusions have general complex scope.

The four rings $R7_5_1964$, $R7_5_5343$, $R7_5_5623$ and $R7_5_5778$ share one induction mechanism. Each original spherical character cover leaves only the Frobenius–Perron Galois orbit after zero-dimension, unequal-dual-dimension and norm obstructions. That orbit reduces to a pseudo-unitary induction problem. In each ring its complete 5025 low induction frames all fail the ordinary induction moment inequality. Although the certificate is handled by the character-aware verifier, these subproofs need no character labelling, Galois-pruning or rotation refinement.

The complete signed quotient covers are summarized below. The last four columns count character-component obstructions across nongauge quotients: Z is a zero simple dimension, D is unequal dual dimensions, N is a norm obstruction and F is a dimension-field degree obstruction. They are proof-tree branch counts, not numbers of distinct fusion rings.

Ring	Signed quotients	Gauges	Z	D	N	F
$R7_5_1964$	52	4	160	0	72	0
$R7_5_5343$	12	2	22	10	10	2
$R7_5_5623$	40	8	96	0	48	0
$R7_5_5778$	16	4	24	12	12	4

The gauge branches transfer sphericity by a tensor section and invoke the original spherical proof. Every remaining quotient is covered by the displayed character obstructions. This exhausts the sphericalizations of a hypothetical category and proves a general complex exclusion for each of the four rings.

The source tensors, complete proof trees, exact normal and optimized replay receipts, and separate scope audits for this continuation are stored under `evidence/rank7_m5_20261020/`. The aggregate M5 replay binds accepted certificates to the canonical source tensors. Successful finite verification establishes the stated arithmetic and enumeration; the categorical arguments remain subject to independent mathematical verification.

14.11 An integral Morita descent at multiplicity five

The next argument couples the complete right action of a Morita dual, then descends to a fusion subcategory of dimension 148. It applies to arbitrary fusion categories over $\mathbb{C}$.

Lemma 49. *Let $p > 3$ be prime. If an integral fusion category has dimension type $(1^4, 4^{(p-1)/4})$, its group of invertible objects is cyclic of order four. In particular this dimension type with invertible group $C_2 \times C_2$ cannot occur.*

Proof. Its dimension is $4p$. Jordan–Larson’s classification [24, Theorem 1.1] implies that it is group-theoretical: in that theorem put $q = 2$, and note that neither $p = 2$ nor $p \mid q + 1$ is possible. Write it as $\mathcal{C}(G, \omega, H, \psi)$, where $|G| = 4p$. The simples supported on a double coset HgH have dimensions

$$[H : H \cap gHg^{-1}] \dim(\pi),$$

where π runs through irreducible projective representations of $H \cap gHg^{-1}$ for the prescribed cocycle. The simples supported on H form a fusion subcategory equivalent to $\text{Rep}(H)$; this statement incorporates the cochain ψ and is independent of its choice. We use the standard double-coset description [2, Theorem 5.1].

The Sylow p -subgroup P is normal, since its number divides four and is one modulo p . By Schur–Zassenhaus, $G = P \rtimes K$ with $P \cong C_p$ and $|K| = 4$. If $|H| = p$, the identity double coset already supplies $p > 4$ invertibles. If $|H| = 2p$, then H is cyclic or dihedral: the former supplies $2p$ invertibles and the latter supplies dimension-two simples. Both contradict the given dimension type. If $H = G$, the category is equivalent to $\text{Rep}(G)$. Since K is abelian and P is cyclic, the ordinary irreducible dimensions are the orbit sizes of K on $\hat{P}$. For completeness, each character of P extends to its stabilizer $P \rtimes K_\chi$ by taking value one on K_χ , so no projective representation factor changes these sizes. A dimension-four simple therefore requires a faithful action of C_4 on P . Then $[G, G] = P$, and the invertible group $G/[G, G]$ is C_4 .

It remains to consider H of order at most four; conjugating H places it in K . For $|H| \leq 2$, the double-coset formula cannot produce dimension four. Suppose $|H| = 4$. If $H \cong C_2 \times C_2$, its action on P has a kernel Z of order at least two, because $\text{Aut}(P)$ is cyclic. This kernel is central in G , hence is contained in every $H \cap gHg^{-1}$. Thus the index is at most two. An intersection of order two has only one-dimensional projective irreducibles; an intersection of order four has projective irreducibles of dimension at most two. Again no simple has dimension four. If $H \cong C_4$ but its action on P is not faithful, the same kernel argument applies, and every intersection is cyclic, so all projective irreducibles are one-dimensional. Finally suppose that $H = C_4$ acts faithfully. It acts freely on $P \setminus \{1\}$, so $N_G(H) = H$. A dimension-one simple must have double-coset index one; hence every invertible lies in the identity-double-coset subcategory $\text{Rep}(H)$. Its four invertibles form $\hat{H} = C_4$, as required. This argument allows arbitrary ω and ψ . $\square$

Proposition 50. *The fusion ring $R7_5_4982$ has no categorification over $\mathbb{C}$.*

Proof. Write its basis as $1, g, g^2, X, Y, Z, W$, where $g^3 = 1$ and g fixes the four noninvertibles. These are self-dual, have dimensions 3, 12, 12, 12, and, with $S = 1 + g + g^2$, satisfy

$$\begin{aligned} X^2 &= S + 2X, & XV &= 3V \quad (V = Y, Z, W), \\ Y^2 &= S + 3X + 2Y + 4Z + 5W, & Z^2 &= S + 3X + 5Y + 2Z + 4W, \\ W^2 &= S + 3X + 4Y + 5Z + 2W, \\ YZ &= 4Y + 5Z + 3W, & YW &= 5Y + 3Z + 4W, & ZW &= 3Y + 4Z + 5W. \end{aligned}$$

Suppose that a categorification $\mathcal{C}$ exists. The action of the pointed subcategory on the single simple X trivializes its associator class, so $A = 1 \oplus g \oplus g^2$ is a connected separable algebra. There are thirteen simple right A -modules: A itself, and three lifts of each of X, Y, Z, W . Their normalized dimension vector is

$$d = (1, 1, 1, 1, 4, 4, 4, 4, 4, 4, 4, 4, 4).$$

The algebra automorphism twisting by a primitive cube root ζ cyclically permutes each set of three lifts and fixes A . Use labels 0 for A , 1, 2, 3 for the X lifts, and consecutive triples starting at 4, 7, 10 for the other lifts.

Exhausting the module actions. On a nontrivial Fourier sector of the twisting action, a source simple has a 4×4 matrix over $\mathbb{Z}[\zeta]$. If the total multiplicity between two underlying fixed simples is n , each entry is of the form

$$a + b\zeta + c\zeta^2, \quad a, b, c \in \mathbb{Z}_{\geq 0}, \quad a + b + c = n. \quad (17)$$

Duality makes the four noninvertible matrices Hermitian. The matrix of g is diagonal with cube-root entries. If a diagonal entry were not one, the relation $gV = V$ would force the corresponding row of every noninvertible matrix to vanish. This is impossible: for the X coordinate use $N_{XX}^X = 2$, and for the other three coordinates use the diagonal entries 2, 5, 4 of multiplication by Y . None is divisible by three, whereas a zero entry in (17) requires that n be divisible by three. Thus $T_g = I$. The fixed-simple block of multiplication by X is $\text{diag}(2, 3, 3, 3)$. Its off-diagonal entries vanish, and $T_X^2 = 3I + 2T_X$. The real domains in (17) therefore force

$$T_X = \text{diag}(-1, 3, 3, 3).$$

Commutation splits $T_Y = 0 \oplus T$, where T is Hermitian of size three and its entries satisfy (17) with majorant

$$B = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 5 & 3 \\ 5 & 3 & 4 \end{pmatrix}.$$

The raw fusion tensor gives the annihilator

$$T(T - 12I)(T^3 + T^2 - 12T + 11I) = 0.$$

The leading principal minors of $12I - B$ are 10, 54, 47, so B has spectral radius less than twelve. The entrywise bound $|T_{ab}| \leq B_{ab}$ excludes eigenvalue twelve. The characteristic polynomial of T has rational integral coefficients: it is real, has coefficients in $\mathbb{Q}(\zeta)$, and all its eigenvalues are algebraic integers. The cubic $t^3 + t^2 - 12t + 11$ is irreducible. Consequently either the entire three-dimensional spectrum is zero, or its characteristic polynomial is this cubic. The zero case contradicts $\text{tr}(T) \equiv \text{tr}(B) = 11 \pmod{3}$. The necessary moments are therefore

$$\text{tr}(T) = -1, \quad \text{tr}(T^2) = 25, \quad \text{tr}(T^3) = -70.$$

Exact enumeration of the finite entry domains (17) gives 639 matrices for the first two moments and eighteen for all three. On this sector the remaining action matrices are forced by

$$T_Z = 7I - 3T - T^2, \quad T_W = -8I + 2T + T^2;$$

the identities are verified in the cubic spectral summand of the original regular representation. Each of the eighteen choices reconstructs nonnegative integral 13×13 module matrices M_i satisfying every source fusion relation. Explicit basis permutations fixing label zero identify all eighteen module actions. In particular it suffices to use the first one. It satisfies

$$M_g = M_{g^2} = I_{13}, \quad M_X = (J_4 - I_4) \oplus 3I_9.$$

Exhausting the complete Morita right action. Let $\mathcal{D} = {}_A\mathcal{C}_A$, and let B_U denote the right module action of a simple U of $\mathcal{D}$. Associativity in the Morita bicategory gives the integral Gram identity

$$G_{(a,b),(c,e)} = \sum_i (M_i)_{c,a} (M_i)_{e,b} = \sum_U (B_U)_{a,b} (B_U)_{c,e}. \quad (18)$$

Indeed both sides compute the same multiplicity by associating $m_a(m_c^* m_e)$ in the two possible ways. Equivalently, G is the Gram matrix of the vectorized right actions. They lie in $\text{im}_{\mathbb{Q}} G$, commute with the M_i , are permuted by transposition, and satisfy the entrywise bounds

$$0 \leq (B_U)_{a,b} \leq \left\lfloor \sqrt{G_{(a,b),(a,b)}} \right\rfloor, \quad \text{vec}(B_U) \text{vec}(B_U)^t \leq G. \quad (19)$$

Here $\text{rank}(G) = 37$. The restriction of G to pairs $(a, 0)$ is $3I_{13}$. Every first column of B_U is therefore a unit vector, and each unit vector occurs three times. No first column is zero: label zero generates the indecomposable module category and a commuting nonzero right action cannot annihilate it. If the column is e_a , then $\text{FPdim}(U) = d_a$. Hence $\mathcal{D}$ has dimension type $(1^{12}, 4^{27})$. Transposition gives the same statement for first rows.

For a noninvertible action, write its first column and row as e_a and e_b^t , with $a, b \geq 4$. Since $B_U M_X e_0 = M_X B_U e_0 = 3e_a$, its first four by four block is zero, by nonnegativity. The cross-Gram block

$$(G_{(a,0),(0,b)})_{a,b=4}^{12} = \text{diag}(J_3, J_3, J_3) \quad (20)$$

forces a, b to lie in the same triple and forces exactly one noninvertible simple for every such ordered pair. The permutation centralizer of the M_i has order 72; its stabilizer of label zero has order 18 and is transitive on each of these three triples. Independently multiplying an action on the left and right by elements of this stabilizer preserves (18)–(19), and moves its first column and row to a common anchor 4, 7, or 10. These multiplications are used only to enumerate necessary matrices, without asserting that the permutations are tensor functors.

For each anchor, the affine subspace in $\text{im}_{\mathbb{Q}} G$ with that first row, first column, and zero small block has dimension twelve. Selecting twelve actual matrix entries as coordinates gives a common denominator 31. Integrality imposes two independent congruences modulo 31; their solved coordinates have bounds at most four, so each residue has at most one representative. Each of the three remaining integer boxes has exactly 2,048,000 points. Checking (19) retains exactly one matrix per anchor. Stabilizer translates give exactly 27 distinct matrices, one per allowed ordered pair in (20), so all occur once.

Subtract their outer products from G . Every invertible right action is a permutation commuting with the M_i , hence lies in the above group of order 72. Its image is a subgroup H of order dividing twelve; a possible kernel would make every element of H occur with the same multiplicity $12/|H|$. Exhaustive subgroup closure gives respectively 1, 9, 13, 7, 13, 10 subgroups of orders 1, 2, 3, 4, 6, 12. Exactly two satisfy the remaining Gram identity, and both have order twelve. Thus the invertible right actions are faithful in either case. This step does not assume that all 39 action matrices are linearly independent; their span has dimension only 37.

Recovering the forced fusion subcategory. For each of the two complete frames, products involving an invertible are read directly from matrix multiplication. For two noninvertibles U, V , the invertible coefficient of h is forced by Frobenius reciprocity:

$$[UV : h] = [U^* h : V].$$

Subtract these twelve coefficients from $B_U B_V$. Its first column gives, for each of the nine large module coordinates, the sum of the remaining three coefficients with that first column. Their total is at most four, by the dimension equation $16 = \sum_h [UV : h] + 4 \sum_W [UV : W]$. Enumerating these nonnegative compositions and checking all matrix entries gives exactly one solution for each of the $27^2 = 729$ products, in both frames. The resulting two rank-39 tensors satisfy the based-ring axioms, including associativity and both Frobenius reciprocity identities.

In the recorded ordering of each frame, the simple with index 14 generates the same closed subset

$$\{0, 1, 4, 9, 14, 15, 18, 23, 25, 27, 30, 33, 38\}.$$

Its dimension type is $(1^4, 4^9)$ and its dimension is 148. The four invertibles have indices 0, 1, 4, 9, and all are self-inverse, so their group is $C_2 \times C_2$. A based subset closed under products and duals gives a fusion subcategory of a hypothetical categorification. Applying Lemma 49 with $p = 37$ gives the contradiction.

Every finite enumeration in this proof is regenerated from the original seven-dimensional tensor by `evidence/rank7_m5_20261020/modules/verify_r4982.py`. The exhaustive Fourier data, normalized bounded domains, two right-action frames, fusion tensors, and closed-subring tensors are retained with the final certificate `r4982_exclusion_certificate.json`. The verifier uses exact integer or rational arithmetic throughout; it invokes neither a numerical feasibility test nor a pivotality assumption. $\square$

14.12 Necessary structure of the final quadratic pair

After the preceding exclusions, the rings $R7_5_3895$ and $R7_5_5039$ require additional analysis. Proposition 46 reduces any categorification to a pseudo-unitary Galois conjugate. The following restrictions apply to that positive representative. They do not by themselves give a categorification or a general exclusion. The final proposition excludes braided categorifications of both rings.

Full character transport. Write $d_i = a_i + b_i\sqrt{5}$. In source order the vectors are

	(a_i)	(b_i)
3895	(1, 5, 5, 10, 8, 8, 9)	(0, 2, 2, 4, 4, 4, 4)
5039	(1, 5, 5, 8, 8, 10, 9)	(0, 2, 2, 4, 4, 4, 4).

Both global dimensions are $D = 720 + 320\sqrt{5} = 80(9 + 4\sqrt{5})$. Let P swap labels 4, 5 for the first ring and labels 3, 4 for the second. The signed normalized Galois transport of a column restriction vector is the integral involution

$$T = P + \mathbf{b} \left(\mathbf{a} - \frac{9}{4}\mathbf{b} \right)^t. \quad (21)$$

The permutation term records the action on the non-Frobenius–Perron characters; omitting it gives an incorrect transport. The exact checker constructs the Galois automorphism on the separating fusion matrix by Chinese remaindering over its character factors. If A is this algebra automorphism and B is the Casimir, the i th column of T is the transpose of row zero of $BA(B)^{-1}A(N_i)$. Evaluating in all source characters verifies

$$CT = \text{diag}(f)P_\sigma \text{diag}(1/f)C.$$

This checks the complete transport without numerical root choices or an assumed labelling of repeated formal codegrees. In particular $Te_0 = (1, 2, 2, 4, 4, 4, 4)^t$, the forced restriction row of the Galois image of the central unit. Complete row transport, character reciprocity and duality initially leave two low induction frames for each source ring. The refinements below remove one frame in each case. For a positive restriction row, its actual transported row is the nonnegative choice among Tv and $-Tv$. Orbit counts must retain this sign; the linear map alone need not preserve nonnegativity.

A second saturated quadratic character. Besides the Frobenius–Perron pair, each source has a pair of formal codegree-five characters exchanged by $\sqrt{5} \mapsto -\sqrt{5}$. In the orthonormal basis of the two corresponding character-difference vectors, the full known induced-unit compression is

$$B_5 = \begin{pmatrix} -\sqrt{5}/10 & 0 \\ 0 & \delta\sqrt{5}/5 \end{pmatrix}, \quad \frac{1}{5}I - B_5^*B_5 = \begin{pmatrix} 3/20 & 0 \\ 0 & 0 \end{pmatrix}, \quad \delta = \pm 1.$$

Thus the codegree-five difference saturates the conductor-five norm bound even though the Frobenius–Perron difference does not. The sign δ is determined by each exact character assignment.

The equality case gives a criterion that is independent of these two rings and of the prime five.

Proposition 51 (Saturated quadratic compression fixes primitive phases). *Let $p \equiv 1 \pmod{4}$ be prime. In an anomaly-one unitary congruence representation of $\mathrm{SL}_2(\mathbb{Z})$, let w be fixed by the upper unipotents and have diagonal-torus character the primitive quadratic character χ_p of conductor p . Let L be a subspace of the twist-one space containing w . Suppose*

$$\mathrm{proj}_L(Sw) = \frac{\delta}{\sqrt{p}}w, \quad \delta \in \{1, -1\}.$$

Then the cyclic representation generated by w factors through $\mathrm{SL}_2(\mathbb{F}_p)$, and the full twist-one projection of Sw equals the displayed known projection. Every nontrivial twist in the support of Sw is a primitive p th root ζ_p^k with $\chi_p(k) = \delta$.

Proof. We first determine the equality space at every modulus p^a . The upper-unipotent subgroup U is the stabilizer of $(1, 0)$ in the transitive action on primitive column vectors (r, s) modulo p^a . Frobenius reciprocity embeds every irreducible with a nonzero U -fixed vector isometrically in the corresponding permutation representation. In its U -fixed, χ_p -torus-character subspace, the functions supported on s a unit form the line generated by

$$F(r, s) = \chi_p(s)[p \nmid s].$$

Write Σ for its orthogonal complement, supported on the small cell $p \mid s$, where r is a unit. Let P_U be unipotent averaging and $A = P_U S P_U$ the Weyl compression. Since $\chi_p(-1) = 1$, this compression is self-adjoint. Direct averaging gives

$$AF = G, \quad G(r, s) = \chi_p(-r)[p \mid s].$$

Indeed, $\chi_p(r + ts)$ is constant if $p \mid s$ and averages to zero otherwise. The Weyl translate of a function in Σ vanishes on the small cell after averaging, so $A\Sigma \subseteq \mathbb{C}F$. Moreover $\|G\|^2/\|F\|^2 = 1/p$. Self-adjointness now gives $AG = F/p$ and makes A vanish on $\Sigma \cap G^\perp$. Consequently $\|A\| = 1/\sqrt{p}$, and its entire maximal-singular subspace is $\mathrm{span}(F, G)$. Both functions pull back from level p , as does the invariant subrepresentation they generate. Equality therefore forces level p on the cyclic subrepresentation of the equality vector.

At another prime the local torus character is trivial and the compression bound is one. Equality says that both the vector and its Weyl translate are upper-unipotent-fixed; hence the vector is fixed by upper and lower unipotents. These generate the local SL_2 , so this factor is trivial. Chinese remaindering and orthogonal irreducible decomposition give the global norm bound $1/\sqrt{p}$ and the same equality conclusion: the top eigenspace of a tensor product of positive operators A^*A is the tensor product of their top eigenspaces. Finally, the hypothesized projection onto L already has squared norm $\|w\|^2/p$. Since L lies in the full twist-one space, Pythagoras forces all additional

twist-one components to vanish. This is an equality refinement of the local compression calculation in [34, Lemma 4.1].

Put $h = \|w\|^2$. The twist-one squared weight of Sw is h/p ; let x_+, x_- be its remaining squared weights on the two quadratic classes of primitive p th roots. Thus $x_+ + x_- = h(p-1)/p$. The diagonal torus acts transitively within each class, so the squared weights there are equal. The average phases in the two classes are $(-1 \pm \sqrt{p})/(p-1)$, by the quadratic Gauss sum. Also $S^{-1}w = Sw$, because S^2 acts by $\chi_p(-1) = 1$. Using $Tw = w$ and $STS = T^{-1}ST^{-1}$ gives

$$\langle Sw, TSw \rangle = \langle w, Sw \rangle = \frac{\delta h}{\sqrt{p}}.$$

Consequently

$$\frac{h}{p} + x_+ \frac{-1 + \sqrt{p}}{p-1} + x_- \frac{-1 - \sqrt{p}}{p-1} = \frac{\delta h}{\sqrt{p}}, \quad x_+ - x_- = \frac{\delta h(p-1)}{p}.$$

Nonnegativity implies $x_- = 0$, proving the phase assertion. $\square$

For the two present rings apply the proposition to the unnormalized codegree-five difference w , of squared norm two. Its known twist-one projection has squared norm $2/5$. Thus every additional center row with unequal coordinates 4, 5 in the first source, or 3, 4 in the second, has primitive fifth-root twist of residue class δ . In fact $(Sw)_Y = (v_p - v_q)/\sqrt{5}$, up to the sign of w , for the indicated pair (p, q) . Unitarity gives the small integral budget

$$\sum_{Y \text{ additional}} (v_p - v_q)^2 = 8.$$

Each permitted primitive eigenvalue contributes four. Hence every additional row satisfies $|v_p - v_q| \leq 2$; a transported orbit with difference two exhausts this budget.

There is one further cut for $R7_5_5039$. Its characters

$$(1, -1 \pm \sqrt{-2}, -1 \mp \sqrt{-2}, 0, 0, 0, 1)$$

have formal codegree eight. Their difference has odd conductor-eight torus character. If v is the restriction row of one of the two induced-unit constituents, the compression is $\sqrt{-2}(v_1 - v_2)/4$, of absolute value at most $1/\sqrt{8}$. Thus $|v_1 - v_2| \leq 1$. One of the two frames violates this inequality; the other, with three identical codegree-eight rows, survives it. This uses the local bound for all torus characters, including the odd quadratic character, and does not follow merely from the real quadratic Casimir criterion.

A conductor-eight partial-projection criterion. The odd character above also gives a uniform refinement that records small two-primary twist orders, including twists with arbitrary odd parts. The distinction between these two projectors is essential.

Proposition 52 (Odd conductor-eight projection ladder). *Let χ_{-8} be the primitive quadratic character modulo eight, with values $(1, 1, -1, -1)$ at $(1, 3, 5, 7)$. Let w be a unipotent-fixed vector of this torus character in a unitary congruence representation. For $r = 0, 1, 2, 3$ let Q_r project onto invariants of the two-primary unipotent subgroup generated by T^{2^r} , acting as the identity on the odd-primary factors. If $b = \|Q_0 Sw\|^2$, then*

$$\|Q_r Sw\|^2 = 2^r b, \quad b \leq \frac{\|w\|^2}{8}.$$

Thus the squared masses of Sw with two-primary twist order exactly 1, 2, 4, 8 are respectively $b, b, 2b, 4b$. The same upper bounds hold if Q_r is replaced by the full projector onto T^{2^r} -invariants. Equality in a two-primary bound forces that local cyclic factor to descend to level eight. Equality in the full bound also makes all odd-primary cyclic factors trivial.

Proof. At modulus $M = 2^a$, $a \geq 3$, use the primitive-vector universal representation as in Proposition 51. Its unipotent-fixed torus-character space is the orthogonal sum of

$$F(u, v) = \chi_{-8}(v)[v \text{ odd}]$$

and the small-cell functions $f(u, v) = \chi_{-8}(u)g(uv)$, supported on v even and u odd. Unipotent invariance implies

$$g(z) = \chi_{-8}(1 + tz)g(z + tz^2).$$

The map $z \mapsto z + tz^2$ permutes the even residues and the congruence classes below: its difference at z, z' is $(z - z')(1 + t(z + z'))$, with odd second factor. Taking $t = 2$ on $z \equiv 2 \pmod{4}$ and $t = 1$ on $z \equiv 4 \pmod{8}$ gives

$$\sum_{z \equiv 2 \pmod{4}} g(z) = \sum_{z \equiv 4 \pmod{8}} g(z) = 0,$$

since the character multiplier is minus one in each case. Use the Weyl convention $(Sf)(u, v) = f(-v, u)$ and write $R_r = P_{(T^{2^r})}S$ locally. Direct averaging gives

$$R_r F(u, v) = \chi_{-8}(u)[v \equiv 0 \pmod{2^{3-r}}],$$

and, on v odd and $u \equiv 0 \pmod{2^r}$,

$$R_r f(u, v) = \chi_{-8}(-v) \frac{2^r}{M} \sum_{z \equiv 0 \pmod{8}} g(z),$$

with value zero elsewhere. The two cancellation identities justify retaining only the residues divisible by eight in the latter sum. The first output has squared norm $2^{r-3}\|F\|^2$. The second has $M^2/2^{r+1}$ possible nonzero positions, and therefore

$$\|R_r f\|^2 = 2^{r-1} \left| \sum_{z \equiv 0 \pmod{8}} g(z) \right|^2 \leq M 2^{r-4} \sum_{z \text{ even}} |g(z)|^2 = 2^{r-3} \|f\|^2.$$

For $r = 0, 1, 2$, the outputs of the two cells have opposite parity of v , so are orthogonal. Their exact norm expressions give $R_r^* R_r = 2^r R_0^* R_0$. To include $r = 3$, use $t = 2$ separately on $z \equiv 2, 6 \pmod{8}$; each class is preserved and each sum vanishes. Only the zero class modulo eight survives. The small-cell formula above now has squared norm $4|\sum_{8|z} g(z)|^2$. The big-cell image is $\chi_{-8}(u)[u \text{ odd}]$ with squared norm $\|F\|^2$. These images are disjoint by parity of u , and both squared norms are eight times their $r = 0$ values. Thus $R_3^* R_3 = 8R_0^* R_0$ as well. Equality on the small cell requires g to be constant on the multiples of eight and zero elsewhere. Thus the complete extremal space is spanned by F and $\chi_{-8}(u)[8 \mid v]$, both pulled back from level eight.

Frobenius reciprocity and orthogonal decomposition transfer these identities to arbitrary local unitary representations. For Q_r , the odd-primary Weyl operators are unitary, preserving both the exact ratios and the bound. For the full projector they are instead unipotent compressions of norm at most one. Equality there forces upper- and lower-unipotent invariance at each odd prime. The asserted factorization and triviality conclusions follow as in Proposition 51. $\square$

For $R7_5_5039$, the character-difference vector w_8 has squared norm two and $(Sw_8)_Y = \sqrt{-2}(v_1 - v_2)/4$. Hence

$$\sum_Y (v_1 - v_2)^2 = 16.$$

In squared-difference units, the exact two-primary masses at orders 1, 2, 4, 8 are $c, c, 2c, 4c$ with $0 \leq c \leq 2$. Once the exponent prime support $\{2, 5\}$ is imposed below, the combined nonzero phase groups of root orders $\{1, 5\}$, $\{2, 10\}$ and $\{4, 20\}$ consequently have budgets at most 2, 2, 4. They are inequalities: a root whose odd part is divisible by 25 has zero first and second relative trace and can carry additional mass in the same two-primary class. The independent audit and finite convention checks are in `evidence/rank7_m5_20261020/modules/CHI_MINUS8_PARTIAL_PROJECTION_AUDIT.md`.

Exact phase values and a forced third rotation. By the Cauchy theorem for spherical categories [25, Theorem 3.9], the Frobenius–Schur exponent has only the prime divisors 2, 5, since $D/80$ is an algebraic unit. Thus every central twist order is $2^a 5^b$. For normalized relative trace t to $\mathbb{Q}(\sqrt{5})$, the pair $(t(\theta), t(\theta^2))$ has exactly the following ten possible values. In coefficient notation $(r, s, t, u) = (r + s\sqrt{5}, t + u\sqrt{5})$ they are

$$\begin{aligned} (1, 0, 1, 0), \quad (-1, 0, 1, 0), \quad (0, 0, -1, 0), \quad (0, 0, 0, 0), \\ P_{\pm} = (-1/4, \pm 1/4, -1/4, \mp 1/4), \\ Q_{\pm} = (1/4, \pm 1/4, -1/4, \pm 1/4), \quad R_{\pm} = (0, 0, 1/4, \pm 1/4). \end{aligned}$$

Nonzero pairs come from orders 1, 2, 4, 5, 10, 20; larger allowed prime powers give zero in both coordinates by the Ramanujan or induced quadratic Gauss-sum formula. These are exact trace values, not only vertices of a containing convex hull. Counts at each value on an orbit of restriction rows are therefore nonnegative integers.

Since three is prime to the exponent, the third relative twist moment is the field conjugate of the first. Put $h = D/\bar{D}$. The complete row transport implies that each dimension-weighted orbit sum W satisfies $W = hT\bar{W}$: the row and its dimension acquire the same transport sign, which cancels in their product. The full first induction moment is $L_1 = De_0$, including the induced-unit constituents. Hence

$$L_3 = hT\bar{L}_1 = DT e_0, \quad (\nu_3(X_i))_i = (1, 2, 2, 4, 4, 4, 4).$$

The trace identity is [47, Theorem 4.1]. Each entry equals $\dim \operatorname{Hom}(1, X_i^3)$, so every third rotation operator is the identity. This is a necessary categorical consequence and presently yields no contradiction.

A source-fixed conductor-eight dichotomy. For $R7_5_5039$, call a center simple active when $k_Y = v_{Y,1} - v_{Y,2} \neq 0$. Source duality changes k_Y to $-k_Y$, while center duality preserves the twist. Each active object therefore has a distinct dual with the same squared weight. The mass c above is consequently an even integer. There are exactly two alternatives:

	exact two-primary masses	possible active twist orders
$c = 2$	2, 2, 4, 8 at orders 1, 2, 4, 8	1, 2, 4, 5, 8, 10, 20, 40
$c = 0$	0 at all orders at most 8	16, 32, 64, 80, 160, 320.

Here the finite order lists use Galois actions fixing the complete source field, as follows; they do not assume that the rational transport matrix has a positive sign on every row.

All source characters, codegrees and dimensions lie in $L = \mathbb{Q}(\sqrt{5}, \sqrt{-2})$. A cyclotomic automorphism fixing L fixes a center restriction row under the categorical Galois permutation. Indeed, its sign is positive from the fixed positive entry $S_{0,Y} = \dim(Y)/D$, and the entries $S_{A_E,Y} = \chi_E(v_Y)/f_E$ then determine v_Y through the complete invertible source character table. The twist is transported by the square of the automorphism. For an object of twist order $2^a 5^b$, choose $u = 3 \pmod{8}$ at the two-primary factor and the identity at the fifth-primary factor. If $a \geq 4$, then $v_2(u^2 - 1) = 3$ and this squared action has order 2^{a-3} . Independently, $u = 6 \pmod{25}$ at the fifth-primary factor, with the identity at the two-primary factor, has squared action of order 5^{b-1} when $b \geq 2$. Both choices fix L , and Chinese remaindering makes them independent. Thus the row-fixed orbit has size divisible by

$$m(a, b) = 2^{\max(a-3, 0)} 5^{\max(b-1, 0)}.$$

Its distinct source-dual row gives a disjoint orbit, so an active object costs at least $2m(a, b)k_Y^2$ in its exact two-primary mass. If $c = 2$, the four masses exhaust the total sixteen. Each is at most eight, excluding $b \geq 2$, which would cost at least ten. If $c = 0$, then $a \geq 4$; $b \geq 2$ would cost at least twenty, and $a \geq 7$ would cost at least thirty-two. These contradictions prove the displayed lists. Both alternatives must be included in any subsequent exhaustive argument.

The full source phase domain. There is a stronger finite domain than the ten pairs over $\mathbb{Q}(\sqrt{5})$. Put $F = \mathbb{Q}(\sqrt{5}, \sqrt{2}, i)$ and $\xi = \zeta_8$. For normalized relative trace to F , the complete first/second moment domain consists of the following 37 distinct pairs:

$$\begin{aligned} &(\xi^j c_1, \xi^{2j} c_2) \quad (0 \leq j < 8), \quad (0, \xi^j c_1) \quad (j = 1, 3, 5, 7), \quad (0, 0), \\ &(c_1, c_2) \in \left\{ (1, 1), \left(\frac{-1 + \delta\sqrt{5}}{4}, \frac{-1 - \delta\sqrt{5}}{4} \right) : \delta = \pm 1 \right\}. \end{aligned}$$

The first 24 pairs have root orders dividing 40; the next 12 have orders 16 or 80. Larger allowed prime powers have zero trace in both coordinates: the relevant cyclotomic kernels fixing F negate the first or second power at two-primary order at least 32, and average to zero at fifth-primary order at least 25. Exact averaging in $\mathbb{Q}[x]/(\Phi_{80}(x))$ verifies the finite list and its distinctness. Every squared cyclotomic automorphism fixes F , so the relative phases are constant on transported row orbits.

The same row-fixed actions prove additional integer restrictions. Each of the twelve order-sixteen phase counts at each fixed row is even. The genuine zero-phase multiplicity at a fixed row belongs to $4\mathbb{Z}_{\geq 0} + 5\mathbb{Z}_{\geq 0}$: partition by twist order and use the row-fixed cycles of length divisible by four or five. For the active rows of $R7_5_5039$, the dichotomy sharpens these conditions. In the branch $c = 2$ only the first 24 phases are allowed, with exact two-primary masses 2, 2, 4, 8. In the branch $c = 0$ only the twelve order-sixteen phases and the zero phase are allowed; the latter count at each active row is divisible by four, since its possible orders are 32, 64, 160, 320. The full all-order argument and exact field calculation are recorded in `models/sqrt5/FULL_SOURCE_PHASES_PROOF.md` under the continuation evidence directory. The source-fixed dichotomy is independently audited in `modules/CHI_MINUS8_SIGN_SAFE_DICHOTOMY.md`.

The distinct-row frame is excluded. The two possible low frames for $R7_5_3895$ differ only in the restriction rows of its two codegree-sixteen constituents. In the first they are

$$(1, 0, 2, 0, 1, 1, 2), \quad (1, 2, 0, 0, 1, 1, 2);$$

in the second both are $(1, 1, 1, 0, 1, 1, 2)$. The first frame is now excluded by a complete exact integer certificate. Here is its categorical and finite-domain content.

The phase multiplicities are nonnegative integers. When the actual positive quadratic Galois transport fixes a row or takes it to its source dual, the six nonrational phase types over $\mathbb{Q}(\sqrt{5})$ have even multiplicity: compose the Galois permutation with duality if needed; its odd iterates change the fifth-primary twist and cannot fix an object of fifth-primary order five. The relevant columns are therefore doubled. In addition to the restriction Gram matrix, first and second indicator moments, and induced-unit *STS* identities, impose all 56 unordered triples among the six nonunit induced-unit objects. Their symmetric Verlinde coefficients are

$$\frac{D}{f_E f_H f_J} \sum_Y \frac{\chi_E(v_Y) \chi_H(v_Y) \chi_J(v_Y)}{\dim(Y)} \in \mathbb{Z}_{\geq 0}.$$

This is N_{EH}^{J*} ; the absence of a complex conjugate is appropriate for the symmetric triple formulation. Its irrational coefficient vanishes and its rational coefficient is an integer, encoded by a nonnegative integer variable.

Conductor-five saturation fixes the phase of every row with $v_4 \neq v_5$, and its squared-difference budget is eight. After integral spectral filtering there are 58 possible active row orbits: 25 have weight two, 27 have weight eight, and six have weight four with even multiplicity. Hence there are exactly

$$\binom{28}{4} + 27 + 6 = 20508$$

active allocations. Exact residual Gram bounds reduce these to 126 for the distinct-row frame. Integer-lattice containment reduces them to 32 per positive-indicator character labelling. There are four such labellings. Exact Farkas covectors exclude 28 allocations in each, leaving 16 combined branches. Complete integer split trees close all sixteen with 775 terminal integer Farkas inequalities. At each node the split $x \leq k$ or $x \geq k + 1$ covers every integer value; at each terminal node the covector is nonnegative on every allowed column and negative on the residual right side. Branch pruning uses explicit entry bounds and negative quadratic-form witnesses for the residual Gram matrix. The negative-indicator cases are excluded by a further exact covector.

The complete entry point `evidence/rank7_m5_20261020/integral_gram/verify_frame6.py` regenerates all 15 initial low frames and 163766 high rows from the canonical source, both surviving orbit inventories, all eight positive-indicator systems across the two frames, and every active allocation. It then replays the negative-indicator, lattice, separation, pruning and integer-tree certificates. Normal and optimized Python both pass without an optimizer. Its conclusion is precisely that the distinct-row frame (index six) is impossible; this certificate leaves the repeated-row frame (index eight) untreated. Thus every hypothetical categorification must, after the established positive Galois reduction, use the repeated codegree-sixteen restriction row.

The exact transport, low-block compression, and equality computations are recorded in `evidence/rank7_m5_20261020/models/sqrt5/` and the independent saturation audit in the adjacent `modules/` directory. Earlier integral phase and low-block Verlinde witnesses certify their recorded truncated systems only; subsequent higher-word constraints reject those particular witnesses. They are not complete modular data or models. These truncated computations imply neither coherent associators nor a general exclusion.

Proposition 53 (The quadratic pair has no braided categorification). *Neither $R7_5_3895$ nor $R7_5_5039$ is the Grothendieck ring of a braided fusion category over an algebraically closed*

field of characteristic zero. Braiding is essential to this proposition; a general exclusion requires a separate argument.

Proof. Proposition 46 permits a pseudo-unitary Galois conjugate, and a braiding is preserved under this operation. In both source tensors each nonunit basis element generates the whole ring. Consequently the Müger center is either Vec or the whole category. The second possibility is symmetric and hence integral [27, Corollary 9.9.11], contrary to the simple dimension $5 + 2\sqrt{5}$. The canonical positive spherical structure therefore makes the category modular [27, Propositions 9.5.1 and 8.20.12].

For $R7_5_5039$, modularity requires the formal codegree $D/(5 + 2\sqrt{5})^2 = 16$ [33, Section 3]. Its exact Casimir polynomial is

$$(x - 8)^3(x - 5)^2(x^2 - 1440x + 6400),$$

which does not vanish at 16.

For $R7_5_3895$, matching every simple dimension d_j with the formal codegree D/d_j^2 gives exactly four character assignments p . Each forces a symmetric real orthogonal matrix

$$S_{ij} = \frac{\chi_{p(j)}(X_i)d_j}{20 + 8\sqrt{5}}.$$

Its entries lie in $K = \mathbb{Q}(\sqrt{5})$, and the nontrivial field automorphism has character permutation $\pi = (0\ 6)(4\ 5)$ in the zero-based source labelling.

We first bound all possible twist orders; no trial cutoff is used. The categorical Cauchy theorem [25, Theorem 3.9] permits only the primes 2, 5, since $D = 80(9 + 4\sqrt{5})$ and $9 + 4\sqrt{5}$ is a unit. For the actual twists, Galois symmetry gives

$$\frac{\theta_{\pi_\sigma(i)}}{\theta_{\pi_\sigma(0)}} = \sigma^2(\theta_i) \tag{22}$$

by [26, Proposition 4.7, equation (4.12)]. The denominator is retained; the modular category need not have trivial anomaly. An automorphism fixing K has $\pi_\sigma = 1$ and therefore square-fixes every twist. Taking exponent 3 on the two-primary part and 1 on the fifth-primary part gives $v_2(3^2 - 1) = 3$. Taking exponent 6 on the fifth-primary part and 1 on the two-primary part gives $v_5(6^2 - 1) = 1$. Both choices fix K , so every twist has order dividing 40.

The exponent 17 modulo 40 acts nontrivially on K and squares to 9. Writing $\theta_i = \zeta_{40}^{k_i}$, equation (22) becomes $k_{\pi(i)} = k_6 + 9k_i \pmod{40}$, with $k_0 = 0$. Its complete solution set is

$$\begin{aligned} k_6 &= 8c, & 0 \leq c < 5, & & k_1, k_2, k_3 &\equiv -c \pmod{5}, \\ 0 \leq k_4 &< 40, & & & k_5 &\equiv 8c + 9k_4 \pmod{40}. \end{aligned}$$

There are $5 \cdot 8^3 \cdot 40 = 102400$ candidates per matrix. Exact reduction modulo Φ_{40} imposes every balancing identity

$$\theta_i \theta_j \sqrt{D} S_{ij} = \sum_k N_{ij}^k d_k \theta_k.$$

Only four assignments remain: with $\zeta = \zeta_5$, their twists are

$$(1, -1, -1, 1, \zeta^2, \zeta^3, 1), \quad (1, -1, -1, -1, \zeta, \zeta^4, 1),$$

together with their conjugates. The modular indicator formula [47, Theorem 7.5] gives respectively

$$\begin{aligned}(\nu_2(X_k))_{k=0}^6 &= (1, 5, 5, 10, 9, 9, 9), \\ (\nu_2(X_k))_{k=0}^6 &= (1, 1, 1, 2, 1, 1, 1).\end{aligned}$$

Every source simple is self-dual, so each second indicator must be 1 or -1 . Index 3 contradicts this in all four cases.

The verifier `verify_braided_frontier.py`, in the continuation's `criteria/` directory, binds the canonical tensors and replays the signed sphericalization and positive Galois reduction, fusion closures, character assignments, complete twist domain and exact indicators. Both normal and optimized executions pass. Fibonacci and semion indicator controls give $(1, 1)$ and $(1, -1)$, so negative indicators are explicitly allowed. This certificate excludes braidings and leaves the general ledger unchanged. $\square$

14.13 Coupled cyclic rotations and fusion integrality

The rotation constraints below couple indicators of a simple object with indicators of its tensor powers. They are uniform necessary conditions; the subsequent finite-domain argument must separately justify its source dimensions, phase inventory and exhaustive coverage.

Proposition 54 (Coupled cyclic multiplicities). *Let X be an object of a pivotal fusion category over $\mathbb{C}$, and put $h_n(X) = \dim \text{Hom}(\mathbf{1}, X^{\otimes n})$. There exist nonnegative integers $u_0, \dots, u_3$ and $v_0, \dots, v_7$ satisfying*

$$\begin{aligned}\sum_{j=0}^3 u_j &= h_4(X), & \sum_{j=0}^3 i^j u_j &= \nu_4(X), & \sum_{j=0}^3 (-1)^j u_j &= \nu_2(X^{\otimes 2}), \\ \sum_{j=0}^7 v_j &= h_8(X), & \sum_{j=0}^7 \zeta_8^j v_j &= \nu_8(X), \\ \sum_{j=0}^7 i^j v_j &= \nu_4(X^{\otimes 2}), & \sum_{j=0}^7 (-1)^j v_j &= \nu_2(X^{\otimes 4}).\end{aligned}$$

For a simple X_i , the same indicators occurring in these equations satisfy

$$\nu_4(X_i^{\otimes 2}) = \sum_k N_{ii}^k \nu_4(X_k), \quad \nu_2(X_i^{\otimes r}) = \sum_k [X_i^{\otimes r} : X_k] \nu_2(X_k) \quad (r = 2, 4).$$

Proof. Let E_n be pivotal rotation on $\text{Hom}(\mathbf{1}, X^{\otimes n})$. The block-composition identity [46, equation (5.4)] identifies E_n^r , for $r \mid n$, with rotation of the r -fold blocks. Moreover $E_n^n = 1$ by [46, Theorem 5.1]. Thus u_j and v_j can be taken as the eigenspace dimensions for eigenvalues i^j and ζ_8^j . Taking traces of the indicated powers gives the first identities. Additivity of indicators gives the last two. $\square$

For explicit use, write $\nu_4(X) = a + ib$, $m = h_4(X)$ and $r = \nu_2(X^{\otimes 2})$. The four required multiplicities are

$$\frac{1}{4}(m + r + 2a, \quad m - r + 2b, \quad m + r - 2a, \quad m - r - 2b).$$

For the eighth rotation put $m = h_8(X)$, $q = \nu_2(X^{\otimes 4})$, $\nu_4(X^{\otimes 2}) = s + it$, and write $\nu_8(X) = \sum_{j=0}^3 \eta_j \zeta_8^j$. Set

$$A_0 = \frac{m + q + 2s}{4}, \quad A_2 = \frac{m + q - 2s}{4}, \quad A_1 = \frac{m - q + 2t}{4}, \quad A_3 = \frac{m - q - 2t}{4}.$$

Then $v_j = (A_j + \eta_j)/2$ and $v_{j+4} = (A_j - \eta_j)/2$. Requiring these eight values to be nonnegative integers gives the exact cyclic constraint. In the basis $1, \sqrt{2}, i, i\sqrt{2}$, coefficients of $\sqrt{2}$ and $i\sqrt{2}$ can be half-integral; imposing integral coordinates in that basis would add an invalid restriction.

Proposition 55 (Extraction from center restrictions). *Assume now that the category is spherical. Let F be the forgetful functor from its center, let D be its global dimension, and write $d_i = \dim(X_i)$, $v_{Y_i} = [F(Y) : X_i]$ and $z_Y = \sum_i v_{Y_i} d_i$ for a central simple Y . Choose the center twist convention of [47, Theorem 4.1]. Then*

$$D\nu_n(X_i) = \sum_Y v_{Y_i} z_Y \theta_Y^n.$$

If K contains the source dimensions and ζ_8 , normalized relative trace to K therefore gives the complete fourth and eighth indicators:

$$D\nu_n(X_i) = \sum_Y v_{Y_i} z_Y t_K(\theta_Y^n), \quad n = 4, 8.$$

Proof. The first identity is the center-indicator formula. Its relative trace commutes with the coefficients $v_{Y_i} z_Y \in K$. Proposition 54 places $\nu_n(X_i)$ in $\mathbb{Q}(\zeta_n) \subset K$ for $n = 4, 8$, so the left side is fixed by that trace. The opposite center convention replaces every twist by its inverse, as in [47, Remark 4.3]; changing conventions uniformly permutes the cyclic eigenvalue labels. $\square$

The restriction Gram identity, fusion tensor and complete relative-phase inventory now give rational linear equations in nonnegative integral restriction and cyclic multiplicities. Clearing denominators produces an integer system. A modular row functional, an exact rational separator, or an exhaustive finite branch certificate may prove its infeasibility. This conclusion requires verification of the whole finite domain and every dimension/indicator branch. A general complex exclusion additionally uses the independently established reduction from arbitrary categorifications to the spherical branches under consideration.

Proposition 56 (Integral fusion-norm inequality). *Let A, B be central simple objects and put $n_Y = [A \otimes B : Y]$, $a = [F(A) \otimes F(B)]$ and $q = \dim \text{End}(A \otimes B)$. Partition the central simples into a specified low set $\mathcal{L}$ and its complement $\mathcal{H}$. For a rational source-coordinate vector x define*

$$g^*(t) = \max_{k \in \mathbb{Z}_{\geq 0}} (tk - k^2).$$

For arbitrary integers $b_G \geq 0$, $G \in \mathcal{L}$, one has

$$\begin{aligned} q \geq & x \cdot a + \sum_{G \in \mathcal{L}} (2b_G + 1 - x \cdot v_G) n_G - \sum_{G \in \mathcal{L}} b_G (b_G + 1) \\ & - \sum_{Y \in \mathcal{H}} g^*(x \cdot v_Y). \end{aligned}$$

Proof. Semisimplicity and restriction give $q = \sum_Y n_Y^2$ and $a = \sum_Y n_Y v_Y$. For $Y \in \mathcal{H}$, the definition of g^* gives $n_Y^2 \geq (x \cdot v_Y) n_Y - g^*(x \cdot v_Y)$. For $G \in \mathcal{L}$, the product of the consecutive integers $n_G - b_G$ and $n_G - b_G - 1$ is nonnegative, so $n_G^2 \geq (2b_G + 1) n_G - b_G (b_G + 1)$. Summing proves the inequality. The maximum defining g^* is computed exactly by checking $\max(0, \lfloor t/2 \rfloor)$ and the next integer. $\square$

For A, B in a fixed labelled low frame, a and the low restriction rows are fixed. The low coefficients n_G and q are respectively triple and quartic Verlinde moments; the final sum is linear in high-row multiplicities. Thus fixed rational choices of x and integral choices of b_G produce valid linear inequalities for the complete necessary system. A failed trial witness may suggest those choices, but the proof of the inequality does not depend on that witness or on an optimizer.

Proposition 57 (All-order power-of-two phase inventory). *Put $K = \mathbb{Q}(\sqrt{5}, \zeta_8)$ and let $k \geq 1$. For roots of unity θ whose orders have no prime factors other than 2 and 5, the vectors*

$$(t_K(\theta), t_K(\theta^2), \dots, t_K(\theta^{2^k}))$$

take exactly $25 + 12k$ values. All nonzero values are realized at orders dividing $40 \cdot 2^k$. The zero vector corresponds precisely to exact orders $2^a 5^b$ with $a \geq k + 4$ or $b \geq 2$.

Proof. The two-primary trace vanishes when $a > j + 3$ at the 2^j -th power; otherwise the powered root lies in $\mathbb{Q}(\zeta_8)$ and is unchanged by the trace. For $b \geq 2$ every indicated five-primary trace vanishes. For $b = 0$ the five-primary factor is 1, whereas for $b = 1$ its successive traces alternate between $(-1 + \delta\sqrt{5})/4$ and $(-1 - \delta\sqrt{5})/4$, with $\delta = \pm 1$. The cyclotomic Chinese remainder decomposition makes the two factors independent. If the first nonzero coordinate has index $h = 0$, there are eight eighth-root phases and three five-primary choices, giving 24 vectors. If $1 \leq h \leq k$, the first nonzero eighth-root phase is primitive, giving four choices and hence 12 vectors for each h . These vectors are distinct, and adjoining zero proves the count. $\square$

For $k = 3$ this gives 61 records for moments 1, 2, 4, 8: 60 visible records represented at orders dividing 320, and one genuine zero record covering arbitrarily large orders. Thus 320 is a modulus for the visible trace values, rather than an asserted bound on all twists. The prime restriction used in this application comes from the separately proved categorical Cauchy argument, not from cyclic rotation alone.

Packet divisibility uses a field containing both source-character fields, $L = \mathbb{Q}(\sqrt{5}, \sqrt{-2})$, which is smaller than the trace field K . It equals the source-character field for $R7_5_5039$; that of $R7_5_3895$ is $\mathbb{Q}(\sqrt{5})$. Thus the L -fixing subgroup is a valid, possibly smaller subgroup for either ring. The square action of automorphisms fixing L has a primitive-root orbit of size $2^{\max(a-3,0)} 5^{\max(b-1,0)}$ at order $2^a 5^b$: its two-primary image is the principal subgroup congruent to 1 modulo 8, and its five-primary image is congruent to 1 modulo 5. Consequently the zero record has per-row multiplicity in $16\mathbb{Z}_{\geq 0} + 5\mathbb{Z}_{\geq 0}$ for $k = 3$. As in the earlier transport argument, restriction rows are fixed only after the actual positive source-field Galois action and its signs have been justified. No unsigned raw transport assertion is used here.

14.14 Excluding the conductor-eight quadratic case

Proposition 58. *The based ring $R7_5_5039$ has no categorification over $\mathbb{C}$.*

Proof. The source-bound signed reduction covers all four oriented signed quotients. The reduction establishes sphericity and a dimension character in the Frobenius–Perron Galois orbit. After Galois conjugation, any categorification would therefore give one of the pseudo-unitary center systems considered below. The incompatible signed branches are excluded by their quadratic character fields. The independent domain enumeration reconstructs all four induced-unit frames and all 142 510 possible high restriction rows. Character reciprocity leaves two eligible frames. One has twist-one conductor-eight difference-square weight 8, exceeding its upper bound 2. The other

has twelve label assignments, comprising two classes of identical character-to-row assignments. Its second-indicator vector is uniquely $(1, 0, 0, 1, 1, 1, 1)$.

The sign-safe conductor-eight argument gives $c \in \{0, 2\}$ and exact difference-square masses $c, c, 2c, 4c$ at two-primary twist orders $1, 2, 4, 8$. Use the complete 61-record inventory of Proposition 57 and its source-fixed packet divisibilities, together with the fourth and eighth rotation equations. For either label class with $c = 2$, an integral row functional modulo 256 annihilates every one of the 1700 columns but takes value 128 on the right-hand side. This excludes both complete $c = 2$ systems.

For $c = 0$, complete aggregate and inactive-lattice enumeration reduces 615 phase allocations to four. The five integral possibilities for the fourth rotation of source object 1 reduce to the unique multiplicity vector $(3, 1, 0, 0)$. The conductor-five allocation has 1004 raw possibilities, exactly 50 of which satisfy the residual source Gram constraints. Exact rational separators exclude 184 of the resulting 200 combinations in label class 0, and 192 of 200 in label class 1.

For the remaining 24 combinations, the $\sqrt{10}$ and $i\sqrt{10}$ eighth-indicator equations and exact Gram capacities force one order-320 packet of multiplicity 16 in the designated restriction orbit (orbit 50 in the certificate). There are two packet choices for each active phase, giving 48 final branches. Other phase contributions in that orbit remain allowed. The verifier reconstructs the complete packet domains and all capacity inequalities from the original induction Gram matrix; it checks each capacity by an integer scan and rounds only by the gcd of the packet units.

Every final branch has two exact separating leaves. If u is the multiplicity of $-i$ in the fourth rotation for source object 5, the leaves respectively exclude $u \leq 18$ and $u \geq 19$. They cover every integer u , giving 96 exact leaves in total. These separators use the Gram, first/second moment, modular STS , prime-budget, integral fusion-norm, fourth-rotation and capacity constraints. Their eighth-rotation coefficients are zero: the eighth indicators enter earlier through the forced packet domain. No numerical infeasibility result is used as a certificate. All possible categorical branches therefore give a contradiction. $\square$

The read-only verifier `verify_r5039_cyclic_chain.py` in `evidence/rank7_m5_20261020/models/sqrt5/` binds the original tensor, reconstructs every domain and system, and checks all conditional branches. Normal and optimized replays passed; an independent optimized replay also passed with every SciPy import disabled. The proof uses the integer inequality in Proposition 56, rather than treating failure of a trial necessary-data witness as a categorical exclusion.

14.15 The complete obstruction for R7_5_3895

The second quadratic ring uses the same all-order phase inventory and coupled rotations, together with an additional integrality check on the fusion-norm slack. Its Frobenius–Perron dimensions, in source order, are

$$(1, 5 + 2\sqrt{5}, 5 + 2\sqrt{5}, 10 + 4\sqrt{5}, 8 + 4\sqrt{5}, 8 + 4\sqrt{5}, 9 + 4\sqrt{5}), \quad D = 720 + 320\sqrt{5}.$$

The formal-codegree polynomial is

$$(t - 4)(t - 16)^2(t - 5)^2(t^2 - 1440t + 6400).$$

Proposition 59 (Finite all-word integrality). *Fix commuting simple classes $A_1, \dots, A_t$ in a pseudo-unitary modular category, and a finite spectral domain of s candidate points containing their complete possible simultaneous eigenvalue tuples. Suppose each point lies in $\mathcal{O}_K^t$ for a number field K ; write $\lambda_i = (\lambda_i(Y))_Y \in \mathcal{O}_K^s$ for the evaluation vector of the i th generator. Then the*

unit-coefficient functional is integral on every integer polynomial in the A_i if and only if it is integral on an additive basis of

$$B = \mathbb{Z}[1, \lambda_1, \dots, \lambda_t] \subseteq \mathcal{O}_K^s.$$

Such a basis is obtained by exact additive-lattice closure under the generators, with a certified stabilization test. There is no assumed maximum word degree.

Proof. For a pseudo-unitary modular category $\mathcal{M}$, the unit weights in simultaneous modular diagonalization are $d_Y^2 / \text{FPdim}(\mathcal{M})$. If $\mathcal{M} = Z(\mathcal{C})$ and $D = \text{FPdim}(\mathcal{C})$, simultaneous modular diagonalization gives

$$\tau(P) = [\mathbf{1} : P(A_1, \dots, A_t)] = \sum_Y \frac{z_Y^2}{D^2} P(\lambda_1(Y), \dots, \lambda_t(Y)).$$

This is an integer for every integer polynomial P , and a nonnegative integer for each monomial. An arbitrary integer combination of monomials need not have nonnegative unit coefficient. The additive group $\mathcal{O}_K^s$ is finitely generated, hence Noetherian. Starting with 1 and the eigenvalue tuples, repeatedly adjoin products of an additive lattice basis with each generator. This increasing chain stabilizes; equality of successive exact lattice bases certifies closure. The stable lattice is exactly B . Additivity of τ proves the stated equivalence. Extra candidate points are permitted by assigning them multiplicity zero, but omitting an actual spectral point would invalidate the implication. $\square$

For the present source, $K = \mathbb{Q}(\sqrt{5})$, $\mathcal{O}_K = \mathbb{Z}[\varphi]$ with $\varphi = (1 + \sqrt{5})/2$, and $D = 80(5 + 8\varphi)$, where $5 + 8\varphi$ is a unit. Thus $W_Y = 6400z_Y^2/D^2 \in \mathcal{O}_K$. Weighted Galois invariance of the complete actual row orbits is checked explicitly. Consequently

$$\tau(P) = \frac{1}{12800} \sum_Y \text{Tr}_{K/\mathbb{Q}}(W_Y P(\lambda(Y))).$$

For arbitrary $h \in \mathcal{O}_K^s$, define $\rho(h) = \sum_Y \text{Tr}_{K/\mathbb{Q}}(W_Y h_Y) / 12800$. Then $\rho(12800h)$ is integral, so checking integrality on B is equivalent to checking $B + 12800\mathcal{O}_K^s$. It suffices to compute this latter lattice modulo $12800\mathcal{O}_K^s$. Closure now takes place in a finite abelian group, and exact row Hermite forms certify stabilization. The resulting congruences are linear in the orbit multiplicities and use signed integer slacks. They do not incorrectly impose nonnegativity on an arbitrary lattice-basis polynomial.

Lemma 60 (Integral norm slack for the two remaining systems). *In each of the two remaining character labellings for R7_5_3895, the fixed instance of Proposition 56 has a nonnegative integer slack in every categorification.*

Proof. The multiplier is $x = (0, \frac{1}{2}, \frac{1}{2}, 0, 1, 1, 0)$. All low chord weights and the constant term are integral. The low product coefficients and the quartic squared norm are integers because they are fusion multiplicities and a sum of their squares. For a high restriction row v , put $n = v_1 + v_2 + 2v_4 + 2v_5$. The integer maximum defining g^* gives

$$2g^*(x \cdot v) = 2g^*(n/2) = \lfloor n^2/8 \rfloor.$$

There are half-integral orbit costs, so their integrality cannot be asserted separately. Instead, the mod-two sum of the eight all-word congruences with stored zero-based row indices

$$346, 347, 359, 362, 373, 382, 385, 390$$

has, on every phase column, coefficient twice its high-orbit cost. It is zero on auxiliary columns and has even right-hand side. These congruences follow from Proposition 59; their moduli are respectively 6400, 320, 80, 80, 8, 8, 8, 8. Thus the total high-row contribution is integral. These coefficient identities are independently reconstructed on all 3925 columns from the canonical tensor; the norm row, indexed 345, is not used. The entire gap in the norm inequality is therefore a nonnegative integer, as required for its unscaled integer slack. $\square$

Proposition 61 (General exclusion of $R7_5_3895$). *No complex fusion category has Grothendieck ring $R7_5_3895$.*

Proof. The complete signed reduction in Section 14.12 has two oriented signed quotients, both gauge equivalent to the unsigned ring. The corresponding chosen lifts form a tensor subcategory of the sphericalization equivalent to the original category, so sphericity descends. Every possible spherical dimension character belongs to the Frobenius–Perron Galois orbit: the other characters either vanish on a simple object or disagree on duals. After Galois conjugation, it is enough to exclude a pseudo-unitary categorification. All subsequent modular constraints belong to the center of this rank-seven category.

Complete source enumeration gives 15 low induction frames and 163766 candidate high rows. Character reciprocity leaves precisely frames 6 and 8. The row transport uses the actual nonnegative row obtained by correcting the sign of the linear Galois transport. All eight negative-second-indicator cases, four in each frame, have exact covector contradictions. For the four positive labellings of frame 6, the conductor-five budget has 20508 raw allocations, 126 after Gram compatibility, and 32 per labelling after the inactive lattice check. Of these 128 branches, 112 have rational Farkas certificates and the remaining 16 have complete integer trees with 775 leaves.

In frame 8, the same raw budget gives 166 Gram-compatible allocations and 32 for each positive labelling after the lattice check. Another 112 exact Farkas certificates reduce these alternatives to allocation indices $\{0, 19, 160, 165\}$. Equality of the character attached to every restriction row gives two labelling classes, $\{0, 4\}$ and $\{2, 6\}$. Both representatives 0 and 2 are retained; their opposite conductor-five residue choices are not identified. The saturated quadratic-character criterion (Proposition 51) applies at every congruence level and gives the active budget $\sum_{Y \text{ high}} (v_{Y4} - v_{Y5})^2 = 8$.

For each label representative, the complete 61-record phase inventory of Proposition 57 and the coupled rotations of Proposition 54 produce a system with 3925 nonnegative integer variables and 665 equations. Its necessary conditions include the restriction Gram matrix, center traces, low modular relations, simultaneous fusion-eigenvalue integrality, conductor-five restrictions, all 56 nonunit low Verlinde triples, and the all-word unit-coefficient congruences. The latter close an integral order on the complete candidate spectral domain; no maximum word-degree assumption is imposed. The fixed fusion-norm inequality has an integer slack by Lemma 60. The final verifier recomputes every low-Verlinde triple on the filtered domain of 145 high orbits with caching disabled and compares the entire payload before rebuilding these systems.

After fixing a conductor-five allocation, exact entrywise positivity and positive semidefiniteness of the residual Gram matrix remove impossible packets and bound every retained orbit multiplicity. A single packet already violating either condition has multiplicity zero; all capacities are computed by rational arithmetic. For the first nonunit source object, the fourth-rotation multiplicities in eigenvalue order $(1, -1, i, -i)$ have the form

$$(r, 9 - r, s, 2 - s), \quad 0 \leq r \leq 9, \quad 0 \leq s \leq 2.$$

There are 30 choices for each of the eight label/allocation pairs. Exact Farkas covectors exclude 225 of these 240 choices. The remaining fifteen have $s = 1$; their complete integer-chain leaf counts are as follows. A dash denotes a choice already excluded in the preceding 225-case cover.

Label case	Allocation	Leaves for $r = 5$	Leaves for $r = 6$
0	0	114	179
0	19	111	211
0	160	39	232
0	165	39	146
2	0	—	145
2	19	108	121
2	160	90	236
2	165	147	208

Together these chains have 2111 forced integer splits and fifteen terminal contradictions, hence 2126 exact leaves.

Each split is the exhaustive alternative $x_j \leq k$ or $x_j \geq k + 1$. After retained bounds are represented by nonnegative integer slacks, an exact covector is nonnegative on every matrix column and strictly negative on the target. It excludes one side; the final covector excludes the last retained side. The independent chain verifier reconstructs the source Gram matrix, row identities, fixed packets, fourth-rotation coordinates, packet removals and capacities, then checks every leaf. Numerical solver outcomes are not premises. All remaining positive-dimension cases are covered, so no general categorification exists. $\square$

The strict read-only entry point is `evidence/rank7_m5_20261020/models/sqrt5/verify_r3895_cyclic_chain.py`. It binds the original tensor and every proof input, reconstructs the complete finite domains and checks all branches. Partial probe logs, missing terminal covectors and unfrozen dependencies do not pass this whole-chain gate. The accompanying `R3895_CYCLIC_CHAIN.md` records the complete proof inventory.

14.16 Recorded outcome

Among the 5805 new tensors, 5804 are excluded, 1 has a verified unitary model, and 0 remain undecided. Every supplied tensor in the band has a recorded general exclusion or unitary model. A feasible necessary system is not a categorical construction, and a time-limited or incomplete search is not a nonexistence certificate. No new tensor category is established by this continuation. The model in Proposition 40 is the only new realized based-ring type added to the catalog in this continuation.

15 Reproducibility and research status

The verification package contains the pinned census tensors, exclusion certificates, model tensors and exact replay programs. From the repository root, restore the packaged inputs and run the current publication checks:

```
python3 tools/restore_data.py
python3 tools/verify_current_publication.py --historical
python3 -O tools/verify_current_publication.py --historical
```

The dispatcher checks the current publication identities and classification records, then runs the component replays in isolated trees with their required input conventions. The extension's complete exact checks can also be run directly:

```
python3 supplements/rank7/software/verify_m5.py --jobs 4
python3 -O supplements/rank7/software/verify_m5.py --jobs 4.
```

They replay every new negative certificate, reconstruct the positive character tensor, verify the transport of all supplied tensors, and check that the earlier 4201 decision records are unchanged. The `--metadata-only` option checks source and record bindings without replaying mathematical certificates. The extension archive is `supplements/rank7/release/rank7-certified-through-m5.zip`; it retains the earlier multiplicity-four release as a historical package. The earlier component checks include

```
tools/verify_upper_residual_20261002b.py
tools/verify_upper_residual_20261002c.py.
```

These check the prime-descent and signed certificates, the D_{12} model, source identities and ledger consistency. Individual proof checks remain available under

```
evidence/theoretical_20261016/rank7/
```

and are named

```
verify_signed_pair.py,
verify_C3_signed_general.py,
verify_cyclic_module.py,
verify_R695_general.py,
verify_three_fixed.py,
verify_three_fixed_gram.py,
verify_psd_gram.py,
verify_C3_A4_reduction.py,
verify_A4_quadratic.py,
verify_scoped_dimension_block.py.
```

Each accepts `--repo /path/to/repository`; the last makes no general decision by itself. The Gram verifiers use the exact helpers in `gram_support/`; their replay does not invoke an optimizer. Run each normally and with `-O`. Further proof verifiers are in

```
evidence/unified_20261016/rank7/.
```

They are

```
verify_R1022_inverse_morita.py,
verify_three_case_galois_pu.py,
verify_four_case_dihedral.py,
verify_R2591_twisted_v4.py,
verify_R2665_inversion_model.py,
verify_R35_inversion_model.py,
verify_r601_center_lattice.py.
```

The first checks the inverse Morita exclusion. The next three check Proposition 26 and Propositions 29–30 without changing a classification status. The next two check the complete model tensors and induction identities of Theorems 38–39. The last regenerates all 336 induced-unit blocks and checks the exact finite-field certificates in Theorem 37. All accept the same `--repo`

argument. The conductor-equality proof and its independent audit are in `evidence/unified_20261016/compression/`. The earlier complete base-release rank-seven loop was executed successfully over all 4201 source records. Its historical component replay chain and classification checks also passed in normal and optimized Python. The full-loop and component receipts document their exact inputs and scope in the repository verification record. These calculations verify finite identities, source integrity and certificate applications; the categorical reductions and cited theorems remain mathematical inputs. A checksum or a successful computation does not replace a proof of those inputs. Discovery programs are not needed for certificate replay. For the multiplicity-five extension, install `supplements/rank7/requirements-m5.txt` and run `supplements/rank7/software/verify_m5.py`, normally and with `-O`. Python 3.12 and a C++17 compiler are used; the complete Morita calculation additionally requires NetworkX. Its assertion-based components execute in an isolated unoptimized child in either aggregate mode. The receipts in `supplements/rank7/output/m5/` state the ledger hash and complete coverage of each run; a receipt for an earlier ledger does not certify the current revision.

AI disclosure. This manuscript and its supporting proofs, code and data were generated with AI assistance, including GPT-6 Astra. They have not yet been independently verified by human experts or formalized in Lean. The stated results are provisional pending that verification.

A Independent signed verification of two quadratic cases

A.1 A quadratic-field reduction for two signed quotients

Proposition 62. *For a hypothetical categorification of $R7_4_1212$ or $R7_4_1213$, choose positive-second-indicator lifts in its sphericalization and let d_i be their dimensions. Then*

$$K = \mathbb{Q}(d_0, \dots, d_6) = \mathbb{Q}(\sqrt{5}), \quad \delta = \sum_i d_i^2 \in \{180, 360, 270 \pm 90\sqrt{5}, 450 \pm 90\sqrt{5}\}.$$

Proof. The two source Casimir polynomials are respectively

$$\begin{aligned} (z - 90)(z - 20)^2(z - 4)(z - 2)(z^2 - 900z + 6480), \\ (z - 90)(z - 12)(z - 10)(z - 6)(z - 2)(z^2 - 900z + 6480). \end{aligned}$$

Their traces are $B = 1036, 1020$, and the least common multiple of the rational formal codegrees is 180. As in Proposition 6, the signed multiplication matrices are commuting symmetric integer matrices. The formal-codegree field and divisibility theorems for the sphericalization [13, Theorem 2.13 and Corollaries 2.14–2.15] therefore give $\mathbb{Q}(\sqrt{5}) \subseteq K$ and $\beta = \delta/180 \in \mathcal{O}_K$. Every embedding of K is real, and β is totally positive.

The nontrivial invertible g fixes all five noninvertibles in the source. The square-channel congruence gives $T_{ii}^g = 1$, and Frobenius reciprocity gives $gX_i = X_i$ in the signed quotient. Thus $(1, -1, 0, \dots, 0)$ is a character of codegree two, distinct from every conjugate of the dimension character. If $e = [K : \mathbb{Q}]$, then e is even, $e \leq 6$, and

$$180 \operatorname{Tr}_{K/\mathbb{Q}}(\beta) + 2 \leq B.$$

Total positivity and the arithmetic–geometric mean inequality give $\operatorname{Tr}_{K/\mathbb{Q}}(\beta) \geq e$. Hence $e \neq 6$.

Suppose $e = 4$ and put $r = [\mathbb{Q}(\beta) : \mathbb{Q}] \mid 4$. If $r = 4$, its minimal polynomial is $z^4 - az^3 + bz^2 - cz + d$, with positive integer coefficients and

$$a \leq 5, \quad b \leq \lfloor 3a^2/8 \rfloor, \quad c \leq \lfloor a^3/16 \rfloor, \quad d \leq \lfloor a^4/256 \rfloor.$$

These bounds follow from the trace inequality and Maclaurin's inequalities. Factoring every polynomial in this finite box and counting its positive roots exactly gives no irreducible totally positive quartic. The complete check is included in the verifier. If $r = 2$, a totally positive irreducible quadratic algebraic integer has trace at least three, so its trace from K is at least six, contradicting the same bound. Thus $r = 1$, and $4\delta + 2 \leq B$ forces $\delta = 180$.

There would then be four signed Casimir eigenvalues 180, one of value two, and two further positive eigenvalues u, v . The reciprocal-codegree identity gives $90(u + v) = 43uv$. Their residual quadratic has integer coefficients, so it is

$$z^2 - 43kz + 90k \quad (1 \leq k \leq 7).$$

Each of these seven quadratics is irreducible, while $(43k)^2/(90k) = 1849k/90$ is not integral. Their roots are therefore not d -numbers, contrary to the formal-codegree theorem [33]; doubling a codegree does not change the d -number property. This excludes $e = 4$.

Consequently $e = 2$ and $K = \mathbb{Q}(\sqrt{5})$. Write $\beta = (a + b\sqrt{5})/2$, with $a, b \in \mathbb{Z}$ of the same parity. Total positivity and the trace bound give $1 \leq a \leq 5$ and $a^2 > 5b^2$. The six possibilities are $(a, b) = (2, 0), (3, \pm 1), (4, 0), (5, \pm 1)$, proving the assertion. $\square$

The exact arithmetic underlying this reduction is preserved in

`evidence/upper_residual_20261002c/scoped_research/QUADRATIC_REMAINDER.md`

at source commit `1592905dc573227ad03659a6370a9459d40db26c`. The present verifier independently replays its complete quartic box, seven quadratic d -number failures, and source Casimir checks.

A.2 Two quadratic signed cases

Proposition 63. *The fusion rings $R7_4_1212$ and $R7_4_1213$ have no categorification over $\mathbb{C}$.*

Proof. We use the sphericalization conditions and Proposition 62. Suppose that a categorification exists, and choose the lifts of positive second Frobenius–Schur indicator in its sphericalization. Write T for the signed quotient by the central sign object. Its structure constants satisfy

$$|T_{ij}^k| \leq N_{ij}^k, \quad T_{ij}^k \equiv N_{ij}^k \pmod{2}, \quad T_{ii}^j \equiv N_{ii}^j \pmod{4}.$$

All basis elements are self-dual, so trace cyclicity and dual reflection make T_{ij}^k fully symmetric. The nontrivial invertible element g acts positively on all five noninvertible basis elements. The spherical dimension character $d = (1, 1, d_2, \dots, d_6)$ generates $K = \mathbb{Q}(\sqrt{5})$, all its coordinates are nonzero, and its codegree $\delta = \sum_i d_i^2$ belongs to

$$\{180, 360, 270 \pm 90\sqrt{5}, 450 \pm 90\sqrt{5}\}.$$

Every signed formal codegree lies in K , by applying the formal-codegree field theorem to the sphericalization and dividing its codegrees by two. It is also an algebraic integer, since it is an eigenvalue of the integer signed Casimir matrix $\sum_i M_i^2$. The characters d and $\bar{d}$ are distinct, including when δ is rational. The character $(1, -1, 0, \dots, 0)$ has codegree 2.

Consider multiplication by X_2 on the $g = 1$ summand, in the basis $(1 + g, X_2, \dots, X_6)$. Its matrix M is self-adjoint for $G = \text{diag}(2, 1, 1, 1, 1, 1)$. Set

$$p(t) = \det(tI - M), \quad q(t) = \det(tI - M_{\{1, \dots, 5\}, \{1, \dots, 5\}}).$$

Here the first row of M is $(0, 1, 0, 0, 0, 0)$ and its first column is $(0, 2, 0, 0, 0, 0)^t$. For an eigenvalue λ , let w_λ be the first diagonal entry of its G -orthogonal spectral projection. The resolvent identity gives

$$\frac{q(t)}{p(t)} = \sum_{\lambda} \frac{w_{\lambda}}{t - \lambda}.$$

Each common character of codegree h contributes $2/h$ to this weight. Indeed its eigenvector has coordinates $(1, \chi(X_2), \dots, \chi(X_6))$, whose squared G -norm is h . Consequently, if d_2 is simple, then $2p'(d_2) - \delta q(d_2) = 0$.

There are exactly 97 200 and 259 200 possible matrices M , respectively, under the displayed coefficient conditions. We enumerate only these matrices, without imposing the remaining associativity equations. For each matrix we test for common roots of p with

$$2p' - 180q, \quad 2p' - 360q, \quad (2p' - 270q)^2 - 40500q^2, \quad (2p' - 450q)^2 - 40500q^2.$$

The complete exact calculation, supplied by `verify_signed_pair.py` and `one_matrix.cpp`, is summarized below. The modular tests use the primes 1009 and 1013; all surviving gcds are subsequently computed over $\mathbb{Q}$. Since p is monic, a characteristic-zero common factor cannot disappear upon reduction modulo either prime.

ring	matrices	first prime	both primes	simple roots surviving
R7_4_1212	97 200	996	372	0
R7_4_1213	259 200	1 888	288	0

More precisely, every common root remaining over $\mathbb{Q}$ lies in a repeated factor of p . All repeated factors are linear and have multiplicity two or three. Thus d_2 is rational, and the distinct characters d and $\bar{d}$ occupy its eigenspace. Their combined weight is

$$b = \frac{2}{\delta} + \frac{2}{\bar{\delta}} = \begin{cases} 1/45, & \delta = 180, \\ 1/30, & \delta = 270 \pm 90\sqrt{5}, \\ 1/90, & \delta = 360 \text{ or } 450 \pm 90\sqrt{5}. \end{cases}$$

For multiplicity two we must have $w_{d_2} = b$. For multiplicity three, the third character has codegree $h = 2/(w_{d_2} - b)$, which must be a positive integer. Exact rational projections exclude every multiplicity-three branch and every multiplicity-two branch except four matrices for R7_4_1213, all with $d_2 = -3$ and $\delta = 180$.

Each of these four matrices has

$$\begin{aligned} p(t) &= (t - 2)(t + 3)^2(t^3 - 3t^2 - 5t + 3), \\ q(t) &= (t + 3)(t^4 - 2t^3 - 12t^2 + 12t + 13). \end{aligned}$$

The simple eigenvalue 2 therefore has codegree $2p'(2)/q(2) = 10$. The cubic factor is irreducible. Its three eigenvalues are simple, so each corresponding character and codegree is defined over its cubic field. The codegree also lies in K and hence is rational. Galois conjugacy makes the three codegrees equal to one positive integer h . The reciprocal-codegree identity then reads

$$\frac{1}{2} + \frac{2}{180} + \frac{1}{10} + \frac{3}{h} = 1,$$

forcing $h = 54/7$, a contradiction.

The supplied verifier rereads the source census, constructs the full matrix domains, reruns the enumeration, and checks every polynomial gcd and rational projection. It uses no floating-point computation and passes unchanged with Python optimization enabled. $\square$

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