

Rank twelve, multiplicity one: complex categorification

Fusion-ring census research notes

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Abstract

We classify complex categorifiability of the 632 supplied rank-twelve, multiplicity-one fusion rings: 480 are excluded and 152 admit models. The models include 151 unitary realizations and one additional complex/pseudo-unitary realization whose unitarity is not proved. The exact ledger binds every decision to the original full multiplication tensor, categorical proof and reproducible certificate. The complete proof archive retains the earlier family reductions and constructions; this manuscript gives a unified reading guide and the final theoretical arguments. A new equality-case refinement of the dimension-block conductor bound excludes the final two commutative rings. An algebraic near-group quotient realizes the last noncommutative ring and yields a uniform family. Enumeration completeness is producer-reported; exact tensor validity and based-isomorphism uniqueness were independently checked computationally, without repeating the full search. The proofs and computations are AI-generated and await independent human verification and formalization.

Classification and proof guide

The inputs are the 632 physical lines of the supplied rank-twelve multiplicity-one tensor file, with permanent labels `R12_1_1`, ..., `R12_1_632`. A model means a fusion category over $\mathbb{C}$ together with an exact based-ring identification; it does not classify associators. An exclusion rules out arbitrary complex fusion categories, with any pivotal, spherical or pseudo-unitary reduction justified separately.

The complete accepted ledger, source tensors, code, proofs and separate assistant audits are in the [rank-twelve proof archive](#). Its top-level ledger is authoritative. Historical ledgers retain their original, explicitly dated scope; their former open counts are not current. Every earlier settled decision is retained. The first stage supplies 393 exclusions and 127 unitary models. The three subsequent family continuations add respectively (32, 14), (13, 7) and (40, 3) exclusions and unitary models. The final arguments below add two exclusions and one complex/pseudo-unitary model. Thus

$$393 + 32 + 13 + 40 + 2 = 480, \quad 127 + 14 + 7 + 3 + 1 = 152.$$

No status follows merely from failure to find a model, a numerical optimization, or a conditional modular datum.

The archive's four earlier reports give the full proofs and accepted replays for lower-rank subring inheritance, cyclic isotypic integrality, formal-codegree and dimension-order obstructions, paired stabilizers, pointed Morita extensions, cohomology and relative-algebra reconstructions. The model certificates use standard quantum-group and group-theoretical categories, pointed and TY extensions, coherent equivariantizations, central zests and actual Morita duals. Each construction checks its categorical hypotheses before identifying all fusion coefficients. The accepted replay instructions and frozen source manifests accompany each report. The current verifier binds the entire preceding packet byte for byte; its optional baseline replay executes the full earlier audit. The final three arguments are proved in detail below.

1 Decisions and common structure

Ring	Decision	Reason
R12_1_561	Excluded over $\mathbb{C}$	Central C_3 and conductor saturation
R12_1_628	Excluded over $\mathbb{C}$	The same argument
R12_1_623	Modeled over $\mathbb{C}$	Cyclic order-36 near-group quotient

All three have six invertible objects forming $G = C_6 = \langle g \rangle$ and six noninvertibles forming a free G -orbit. Put $z = g^3$ and

$$d = 3 + \sqrt{10}, \quad d^2 = 1 + 6d, \quad D = 6(1 + d^2) = 120 + 36\sqrt{10}.$$

Their presentations are

	ρg	ρ^2
R12_1_561	$g\rho$	$1 + \sum_{h \in G} h\rho$
R12_1_628	$g\rho$	$z + \sum_{h \in G} h\rho$
R12_1_623	$g^{-1}\rho$	$z + \sum_{h \in G} h\rho$

The universal grading of each ring is trivial: the products XX^* generate the entire based ring. The complete tensors, rather than only these abbreviated presentations, are bound in the accompanying packet. The 629 previously settled decisions are retained unchanged. Together with the three decisions below, they give the full classification of the supplied census.

2 The central C_3 premise for the two commutative rings

We recall the categorical premise from the preceding continuation, with its exact verification included again. A hypothetical complex categorification can be replaced by a pseudo-unitary Galois conjugate using the two-dimension theorem [1, Lemma 3.1]. Let $H = \langle g^2 \rangle \cong C_3$. Conjugation by the actual invertible objects gives a coherent H -action, even when the pointed associator is nontrivial: its scalar occurs with its inverse on the dual factor. Form the genuine equivariantization $E = \mathcal{C}^H$.

Every source simple has three equivariant lifts, since its stabilizer is cyclic. The pointed group A of E is a central extension of C_6 by $K = \hat{H}$, hence is C_{18} or $C_6 \times C_3$. The noninvertible action is free. Choosing a lift r of ρ gives

$$r^2 = t' + \sum_{s \in S} sr,$$

where S selects one element in every fiber $A \rightarrow C_6$. Associativity forces the lifted right action to be trivial; duality imposes $S^{-1} = t'^{-1}S$. There are exactly 72 possibilities across the two rings and the two pointed groups. For each character λ of A , write $s_\lambda = \sum_{s \in S} \lambda(s)$ and $\epsilon = \lambda(t')$. The two associated formal codegrees satisfy

$$f_+ f_- = 18^2(4 + s_\lambda^2/\epsilon).$$

Central induction requires $(3D)/f_\pm$ to be algebraic integers. Exact cyclotomic norms eliminate 66 sections: a denominator has a prime factor outside $\{2, 3, 5\}$, while the numerator is a unit at that prime. The six surviving sections are subgroup complements B of K in $C_6 \times C_3$, three for each ring. The replay enumerates this entire small domain; it is not a sampled search.

For a surviving section, E_{ad} has simples $B \cup Br$. The forgetful functor $E_{\text{ad}} \rightarrow \mathcal{C}$ sends these bijectively to distinct simple objects, and is therefore a tensor equivalence. Equivariant structures for the inner action provide central lifts of every $h \in H$; equivalently, use the reverse central embedding from the center of the pointed H -subcategory into the center of its relative center. Select the unique lift lying in E_{ad} . Half-braidings on a fixed invertible are unique because $U(E_{\text{ad}}) = 1$, so

the selected lifts close under tensor product as central objects. Transporting them gives a fully faithful braided C_3 in $Z(\mathcal{C})$ whose forgetful image is the original H .

This argument proves an actual central lift, not just commutativity of the fusion ring. Its braiding is either Tannakian or one of the two nondegenerate forms $q(a) = \zeta_3^{ra^2}$, $r = 1, 2$. The next section excludes all three choices.

3 One conductor argument excludes both rings

3.1 The equality refinement

For a spherical fusion category let N be its Frobenius–Schur exponent. Its center has a canonical unitary modular representation

$$\varrho : \mathrm{SL}_2(\mathbb{Z}/N) \longrightarrow U(V), \quad \varrho(s) = S, \quad \varrho(t) = T,$$

where S is normalized. This unitarity concerns the matrices of the modular representation; it does not assume that the original tensor category is unitarizable. The dimension-block bound [2] associates to a dimension-field character χ a twist-fixed vector w_χ , with diagonal-torus character χ , and bounds its S -matrix coefficient by $\mathfrak{f}(\chi)^{-1/2} \|w_\chi\|^2$.

Lemma 1 (Auxiliary-prime rigidity at equality). *Suppose this bound is attained. For every prime $p \mid N$ with $p \nmid \mathfrak{f}(\chi)$, the vector w_χ is fixed by the entire local factor $\mathrm{SL}_2(\mathbb{Z}/p^{v_p(N)})$.*

Proof. Use the Chinese remainder decomposition of the modular representation. For each prime $\ell \mid N$, let U_ℓ be the upper unipotent group, P_ℓ its fixed-space projection, and $B_\ell = P_\ell \varrho(s_\ell) P_\ell$. The local estimate in [2, Lemma 4.1] bounds the product of the compressions at primes other than p by $c = \mathfrak{f}(\chi)^{-1/2}$ on the relevant character space; $\|B_p\| \leq 1$. The operators commute. Equality gives

$$c \|w\|^2 = |\langle w_{\bar{\chi}}, B_{\mathrm{rest}} B_p w \rangle| \leq c \|w\| \|B_p w\| \leq c \|w\|^2.$$

Here $\|w_{\bar{\chi}}\| = \|w\|$. Thus $\|P_p \varrho(s_p) w\| = \|w\|$, so $\varrho(s_p) w$ is U_p -fixed. The vector w is fixed by U_p and by its opposite $s_p^{-1} U_p s_p$. These elementary unipotent subgroups generate $\mathrm{SL}_2(\mathbb{Z}/p^{v_p(N)})$. This works on direct sums of tensor product constituents as well; w need not be a pure tensor in the Chinese remainder decomposition. $\square$

In our two rings the simple-dimension and global-dimension fields are both $\mathbb{Q}(\sqrt{10})$. Its nontrivial character has conductor 40. Writing $D' = 120 - 36\sqrt{10}$ gives

$$DD' = 1440, \quad D/D' = d^2, \quad \left(\frac{1}{D} - \frac{1}{D'} \right)^2 = \frac{1}{40}.$$

Let A_σ be the simple summand of $I(1)$ associated to the conjugate dimension character. Then $\dim A_\sigma = d^2$, $\theta_{A_\sigma} = 1$, and

$$w = e_1 - e_{A_\sigma}, \quad |\langle w, Sw \rangle| = \frac{\|w\|^2}{\sqrt{40}}.$$

The lemma therefore applies at the prime 3 whenever $3 \mid N$.

3.2 Tannakian and metric alternatives

If the central C_3 is Tannakian, the actual regular algebra gives tensor de-equivariantization of dimension $D/3$. Its four simple dimensions are $1, 1, d, d$, because the C_3 -action on source simples is

free. A pseudo-unitary Galois conjugate of this quotient would violate the same dimension-block bound:

$$\left(\frac{3}{D} - \frac{3}{D'}\right)^2 = \frac{9}{40} > \frac{1}{40}.$$

No rank-four classification is needed.

Otherwise the central C_3 is a nondegenerate pointed category $\mathcal{B}$. Apply modular decomposition *inside the braided center*:

$$Z(\mathcal{C}) \simeq \mathcal{B} \boxtimes \mathcal{D}.$$

The induction-character identity gives $S_{A_\sigma, h}^{\text{un}} = \dim(A_\sigma)$ for $h \in C_3$, since both dimension characters send its underlying invertible to 1. As h is invertible, its monodromy with A_σ is therefore 1. Thus $A_\sigma \in \mathcal{D}$ and

$$w = e_0 \otimes (e_1 - e_Y)$$

is a nonzero pure tensor for this modular decomposition.

The pointed twists force $3 \mid N$. Put $n = N/3^{v_3(N)}$, so $3 \nmid n$. The element $L = st^n s^{-1}$ lies entirely in the local 3-factor, because t^n is the identity at every other prime power. The equality lemma forces $\varrho(L)w = w$. Normalized modular matrices factor exactly, so

$$\varrho(L) = (S_{\mathcal{B}} T_{\mathcal{B}}^n S_{\mathcal{B}}^{-1}) \otimes (S_{\mathcal{D}} T_{\mathcal{D}}^n S_{\mathcal{D}}^{-1}).$$

For $q(a) = \zeta_3^{ra^2}$, set $u = \zeta_3^{rn} \neq 1$. The Fourier matrix of $\mathcal{B}$ gives

$$S_{\mathcal{B}} T_{\mathcal{B}}^n S_{\mathcal{B}}^{-1} e_0 = \frac{1}{3}(1 + 2u, 1 - u, 1 - u).$$

This is not proportional to e_0 . The second tensor factor is invertible and sends $e_1 - e_Y$ to a nonzero vector. Hence $\varrho(L)w \neq w$, a contradiction.

Theorem 2. *Neither R12_1_561 nor R12_1_628 is the Grothendieck ring of a complex fusion category.*

The proof uses no bound on the unknown exponent N , no modular representation phase choice for either factor, no ambient braiding on $\mathcal{C}$, and no categorical unitarity assumption. The additional equality argument is developed here from the proof of [2].

4 A complex model for R12_1_623

4.1 An existing algebraic precursor

Let $G_0 = C_{36}$, additively. The algebraic construction [3, Theorems 2.3 and 2.10] supplies an actual spherical near-group category $\mathcal{N}$ with simple objects $G_0 \cup \{X\}$ and

$$gX = Xg = X, \quad X^2 = \sum_{g \in G_0} g + 36X, \quad \text{FPdim}(X) = 18 + 6\sqrt{10}.$$

The construction first has negative categorical dimension for X ; its positive spherical Galois conjugate is also available. Its quadratic data are

$$\chi(g, k) = e^{2\pi i g k / 36}, \quad a(g) = e^{\pi i g (36 - g) / 36}.$$

In the conventions of [5], the parameters needed below are $b = \chi^{-1}$ and $a_I = a^{-1}$, by [3, Lemma 2.14]. We use the spherical existence theorem, not the conjecture about unitarity at order 36.

4.2 The actual central algebra

Take $H = 6\mathbb{Z}/36\mathbb{Z}$, of order 6. Then

$$b|_{H \times H} = 1, \quad H^\perp = H, \quad a_I(6j) = (-1)^j.$$

The central-invertible calculation of [4, Section 4.3], also written explicitly in the proof of [5, Theorem 13.4], applies algebraically to the explicit mixed associators of this precursor. For clarity, this is not an assertion that every spherical near-group category belongs to a unitary classification. In the reconstructed mixed-vertex basis v_t , the three group insertions have the form

$$Z(h)v_t = \chi(h, t)v_t, \quad X(h)v_t = v_{t+h}, \quad Y(h)v_t = \frac{a(t)}{a(t-h)}v_{t-h}.$$

Their product satisfies

$$X(h)Y(h)Z(h) = a(h)^{-1}I.$$

The other identities needed for the invertible half-braiding are bicharacter multiplication and $a(h)^2\chi(h, h) = 1$. For $h \in H$, the candidate mixed crossing uses $a(h)$; $a(h)^2 = 1$ checks the duality channel and the displayed product checks the trivalent channel. All maps are inverse mixed vertices or bilinear dual maps in the algebraic reconstruction [3, Section 4.7]. There is no positivity condition and no all- X associator block in these hexagons. The detailed convention audit is retained in the packet.

The central lifts restrict to a Tannakian H : their mutual braiding is $b|_{H \times H} = 1$. They close coherently because $U(\mathcal{N}) = 1$ makes half-braidings on invertibles unique. The pointed associator is trivial, since the fixed simple X is a rank-one module category for the pointed subcategory. Consequently

$$A = \bigoplus_{h \in H} h$$

with regular group multiplication is an actual connected commutative separable algebra in $Z(\mathcal{N})$. Its module category $\mathcal{M} = \mathcal{N}_A$ is a fusion category.

4.3 Fusion rules without a unitary reconstruction assumption

The quotient has six invertibles u_j indexed by $G_0/H = C_6$. The induced X splits into six distinct simple summands using the primitive Fourier idempotents of $\mathbb{C}[H]$. The perfect pairing $G_0/H \rightarrow \widehat{H}$ makes their pointed action free. Left and right multiplication shift the character with opposite signs, giving

$$\sigma u_j = u_{-j}\sigma.$$

The pointed-channel calculation in the isotropic orbifold [5, Theorem 12.5] is the algebraic character average

$$\frac{1}{6} \sum_{h \in H} a_I(h)b(h, g) = \begin{cases} 1 & g \equiv 3 \pmod{6}, \\ 0 & \text{otherwise.} \end{cases}$$

This uses only the mixed vertices and their inverse dual maps. Thus the unique pointed term of σ^2 is $z = u_3$, and $\sigma^* = z\sigma$.

The remaining coefficients require no transported operator-algebra calculation. Write

$$\sigma^2 = z + \sum_{j=0}^5 m_j u_j \sigma.$$

Associativity with the invertibles forces $m_{j+2} = m_j$. The noninvertible dimension is d , so $\sum_j m_j = 6$. The inversion rule gives $(u_j \sigma)^* = u_{j+3} \sigma$, while $(\sigma^2)^* = \sigma^2$. Duality therefore forces $m_{j+3} = m_j$.

Shifts by 2 and 3 generate C_6 , so all six coefficients equal 1. We obtain exactly

$$\sigma g = g^{-1} \sigma, \quad \sigma^2 = z + \sum_{g \in C_6} g \sigma.$$

The coordinate-to-census bijection, listing $1, g, \dots, g^5, \sigma, g\sigma, \dots, g^5\sigma$, is

$$[0, 1, 10, 11, 9, 2, 3, 8, 5, 4, 7, 6].$$

Two independent implementations check all 1728 structure constants, duality, and the exact quadratic-form specialization.

Theorem 3. *The ring R12_1_623 has a complex categorification and a pseudo-unitary categorification. The present argument does not establish unitarity.*

The strict isotropic statement above is essential: a version requiring $a_I|_H = 1$ would miss the shift z and give the wrong ring.

4.4 A uniform family

The same construction works for every $k \equiv 2 \pmod{4}$. Start from the cyclic near-group category $C_{k^2} + k^2$ and take $H = k\mathbb{Z}/k^2\mathbb{Z}$. The pairing is trivial on H and $H = H^\perp$, while $a_I(kj) = (-1)^j$. The quotient therefore has k invertibles, k noninvertibles, inversion mixed action, and pointed square term $g^{k/2}$. Associativity makes the noninvertible coefficients constant on parity classes; duality shifts the index by the odd integer $k/2$ and equates those classes. Their sum is k , hence every coefficient is 1. Thus the construction supplies, uniformly over $\mathbb{C}$,

$$\rho g = g^{-1} \rho, \quad \rho^2 = g^{k/2} + \sum_{j=0}^{k-1} g^j \rho, \quad k \equiv 2 \pmod{4}.$$

The case $k = 6$ is R12_1_623. This corollary inherits the same distinction between complex existence and unproved unitarity.

5 Evidence, totals, and limits

The accompanying archive contains the updated 632-ring ledger, the three complete input tensors, the new proofs, separate proof reviews, and portable exact checks. It preserves the entire preceding proof packet. The current checks replay the forced-central- C_3 premise, the rational and cyclotomic equality calculations, the algebraic quotient data, and both independent full-tensor identifications. There is no numerical associator search or long classification run.

Stage	Excluded	Modeled	Unresolved
Previous ledger	478	151	3
This continuation	2	1	−3
Final ledger	480	152	0

The new source inputs are the pinned preprints [2, 3]. The exact checks certify their application and the census bindings; they are not replacements for the categorical proofs or an independent reconstruction of the analytic precursor theorem. The model is “new” only as an explicit realization added to this project, with no claim of novelty up to tensor equivalence or exoticness. The companion census repository records enumeration separately from these categorification decisions.

References

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- [3] H. He, Z. Liu, F. Lu, S. Palcoux and Y. Ren, *Triangular prism equations for near-group categories: algebraic solutions and a unitary conjecture*, [arXiv:2610.11652v1](https://arxiv.org/abs/2610.11652v1), 8 October 2026. Theorems 2.3, 2.10, 5.4; Lemmas 2.14, 4.14; Corollary 6.6.
- [4] D. E. Evans and T. Gannon, *Near-group fusion categories and their doubles*, [arXiv:1208.1500](https://arxiv.org/abs/1208.1500), Section 4.3, equations (4.48)–(4.50).
- [5] M. Izumi, *A Cuntz algebra approach to the classification of near-group categories*, [arXiv:1512.04288v2](https://arxiv.org/abs/1512.04288v2), Section 12.1, Theorem 12.5, and the proof of Theorem 13.4.