

A UNITARY CATEGORIFICATION OF A RANK-TEN MULTIPLICITY-FREE FUSION RING

ABSTRACT. We identify the census fusion ring `R10_1_189` with the Grothendieck ring of a fusion subcategory arising from the known conformal embedding $SU(4)_6 \subset SU(10)_1$. A short proof uses two exterior-Pieri products, conformal energies, and ground-state branching. It gives an actual unitary model and corrects the entry's previously unresolved status. We also describe a rank-fourteen unitary modular category with a connected étale algebra whose module category is this rank-ten model. An exact based-ring certificate specifies the identification with the census tensor. These are identifications within established categorical constructions; no claim of a new tensor category is made.

1. THE RING AND ITS CATEGORICAL MODEL

Let R be the based ring with basis

$$1, g, g^2, g^3, g^4, x, gx, g^2x, g^3x, g^4x$$

and relations

$$(1) \quad g^5 = 1, \quad gx = xg, \quad x^* = x, \quad x^2 = 1 + (g + g^2 + g^3 + g^4)x.$$

The involution sends g to g^{-1} . Put $d = 2 + \sqrt{5}$; then

$$(2) \quad d^2 = 1 + 4d, \quad \text{FPdim}(R) = 5(1 + d^2) = 50 + 20\sqrt{5}.$$

Appendix A identifies R exactly with `R10_1_189`.

Write $\mathcal{C} = \mathcal{C}(\mathfrak{sl}_4, 6)$ for the unitary WZW modular category. Its simple objects are denoted $\mathbf{abc}$, where (a, b, c) are Dynkin labels, $a, b, c \geq 0$ and $a + b + c \leq 6$. The conformal embedding induced by the symmetric-square representation of $\mathfrak{sl}_4$ is

$$(3) \quad SU(4)_6 \subset SU(10)_1.$$

This is a realized conformal inclusion, with vacuum branching object

$$(4) \quad B = 000 \oplus 060 \oplus 202 \oplus 222;$$

see [1, p. 29 of the arXiv version] and [2, §§1.4.3–1.4.5]. In particular, B carries an actual connected commutative separable algebra structure. Its category of local modules is braided equivalent to $\mathcal{C}(\mathfrak{sl}_{10}, 1)$. These assertions concern the conformal inclusion and its unitary extension, rather than the existence of an algebra inferred from a numerical modular invariant.

Let $\mathcal{D} = \mathcal{C}_B$ be the unitary fusion category of right B -modules, with relative tensor product. Write $F : \mathcal{C} \rightarrow \mathcal{D}$ for free induction and U for forgetting the module structure. The functor F is tensor and central [3, §§3.3–3.4, equation (17)]. Denote the invertible simple local modules by τ_j , $j \in \mathbb{Z}/10$, with $\tau_i \tau_j = \tau_{i+j}$. We suppress the symbol $\otimes_B$ for products in $\mathcal{D}$.

Theorem 1.1. *Put $\beta = F(010)$, $g = \tau_2$ and $x = \tau_5\beta$. The ten objects*

$$g^j, \quad g^j x \quad (0 \leq j < 5)$$

are pairwise nonisomorphic simple objects of $\mathcal{D}$ and are closed under tensor products and duality. Their full replete semisimple subcategory $\mathcal{K}$ is a unitary fusion category satisfying $K_0(\mathcal{K}) \cong R$ as based rings. Thus `R10_1_189` is unitarily categorifiable.

The proof below needs only a small amount of ground-state branching; it does not require a completeness assertion about an affine branching table. The additional construction in Section 3 identifies $\mathcal{K}$ with the neutral component of $\mathcal{C}_B$ for the natural $\mathbb{Z}/4$ grading.

2. A MINIMAL BRANCHING ARGUMENT

Let $v = 010$, the exterior square of the defining four-dimensional representation. The exterior-Pieri rule gives

$$(5) \quad v^2 = 000 + 101 + 020,$$

$$(6) \quad v^3 = 3010 + 2111 + 030 + 200 + 002.$$

All dominant summands encountered have level at most three, so these classical products are also the level-six fusion products.

Lemma 2.1. *For $j \in \mathbb{Z}/10$,*

$$(7) \quad \dim \operatorname{Hom}_{\mathcal{C}}(v^2, U\tau_j) = \delta_{j,0},$$

$$(8) \quad \dim \operatorname{Hom}_{\mathcal{C}}(v^3, U\tau_j) = \begin{cases} 1, & j \in \{1, 3, 7, 9\}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Conformality of (3) means equality of the two Sugawara conformal vectors. The lowest conformal energy of the level-one $\mathfrak{sl}_{10}$ module indexed by j is

$$h_j = \frac{j(10-j)}{20}, \quad 0 \leq j \leq 9.$$

An affine $\mathfrak{sl}_4$ highest-weight module of highest weight λ occurring in it must satisfy

$$(9) \quad h_\lambda = h_j + n, \quad n \in \mathbb{Z}_{\geq 0}.$$

If $n = 0$, its multiplicity is the ordinary finite-dimensional restriction multiplicity in the ground-state representation $\Lambda^j \mathbb{C}^{10}$. With long roots of squared length two, $h_\lambda = (\lambda, \lambda + 2\rho)/20$. The needed values are

λ	000	101	020	010	111	200, 002	030
h_λ	0	2/5	3/5	1/4	3/4	9/20	21/20.

The residues of the h_j are 0, 9/20, 4/5, 1/20, 1/5, 1/4. Consequently 101 and 020 cannot occur in any $U\tau_j$, while 000 occurs only in $U\tau_0$, with multiplicity one. This proves (7).

For (6), the residue of h_{111} excludes that weight. The only possible index for 010 is $j = 5$, but $h_5 = 5/4 > h_{010} = 1/4$, contradicting (9). The weights 200, 002 can occur only for $j = 1, 9$, at ground energy. The defining ten-dimensional representation restricts as $\operatorname{Sym}^2 V$, $V = \mathbb{C}^4$, or its dual according to orientation. Thus precisely one of these two weights occurs, once, at each of $j = 1, 9$.

Finally, 030 can occur only for $j = 3, 7$, again at ground energy. The ordinary Schur-functor identity

$$(10) \quad \Lambda^3(\operatorname{Sym}^2 V) = \mathbb{S}_{(4,1,1)} V \oplus \mathbb{S}_{(3,3)} V$$

has highest weights 301 and 030. The latter is self-dual and occurs once; dualizing gives the same multiplicity for $j = 7$. This proves (8). Identity (10) is also checked as a full weight-character identity in the accompanying certificate. $\square$

Proof of Theorem 1.1. The object v is self-dual, so β is self-dual. Rigidity, the tensor property of F , and its adjunction to U give

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{D}}(\beta, \tau_j \beta) &= \dim \operatorname{Hom}_{\mathcal{D}}(\beta^2, \tau_j) = \dim \operatorname{Hom}_{\mathcal{C}}(v^2, U\tau_j), \\ \dim \operatorname{Hom}_{\mathcal{D}}(\beta^2, \tau_j \beta) &= \dim \operatorname{Hom}_{\mathcal{D}}(\beta^3, \tau_j) = \dim \operatorname{Hom}_{\mathcal{C}}(v^3, U\tau_j). \end{aligned}$$

Lemma 2.1 proves that β is simple and its ten translates $\tau_j\beta$ are pairwise nonisomorphic. It also gives one copy of each of $\tau_1\beta, \tau_3\beta, \tau_7\beta, \tau_9\beta$ in β^2 . Self-duality gives one copy of the unit.

For $[r] = \sin(r\pi/10)/\sin(\pi/10)$ the quantum Weyl formula gives

$$\text{FPdim}(\beta) = \text{FPdim}(v) = \frac{[4][3]}{[2]} = 2 + \sqrt{5} = d.$$

The tensor functor F preserves Frobenius–Perron dimension. The known constituents of β^2 have total dimension $1 + 4d = d^2$; positivity therefore proves the exact decomposition

$$(11) \quad \beta^2 = \mathbf{1} + \tau_1\beta + \tau_3\beta + \tau_7\beta + \tau_9\beta.$$

Centrality of F supplies isomorphisms $\beta\tau_j \cong \tau_j\beta$. Hence $x = \tau_5\beta$ is self-dual, $g = \tau_2$ has order five, and (11) becomes (1). The five invertibles and the five indicated translates of β are distinct, their dimensions being 1 and $d > 1$. Relations (1) show closure under products and duals. The full subcategory they span is consequently a unitary fusion subcategory of $\mathcal{D}$, as asserted. $\square$

3. AN ACCOMPANYING RANK-FOURTEEN MODULAR CATEGORY

The same conformal inclusion supplies a concrete modular category and an algebra which recover $\mathcal{K}$. This strengthens the existence statement beyond consistency of candidate modular data.

Proposition 3.1. *Let $J = 600$, let $L = \mathbf{1} \oplus J^2$, and put $\mathcal{Q} = \mathcal{C}_L^0$. The image of J in $\mathcal{Q}$ generates a semion subcategory $\mathcal{S}$. Its Müger centralizer $\mathcal{E} = \mathcal{S}' \subset \mathcal{Q}$ is a unitary modular category of rank fourteen and dimension $100d^2$. There is a connected étale algebra*

$$A = \mathbf{1} \oplus U_0 \in \mathcal{E}, \quad \text{FPdim}(U_0) = d^2,$$

obtained from the inclusion $L \subset B$, and a tensor equivalence

$$\mathcal{E}_A \simeq \mathcal{K} \simeq (\mathcal{C}_B)_0.$$

Here the subscript 0 denotes the neutral component of the $\mathbb{Z}/4$ grading by $a + 2b + 3c$ modulo four.

Proof. The simple current J has order four, with $J^2 = 060$, $\theta_J = i$ and $\theta_{J^2} = 1$. We first justify the algebra inclusion $L \subset B$. The regular right B -module is simple because B is connected. Tensoring it by the invertible J^2 again gives a simple right module. Multiplication induces a nonzero right-module map $J^2 \otimes B \rightarrow B$ and hence an isomorphism. Its restriction to the summand $J^2 \otimes J^2 = \mathbf{1}$ is nonzero and lands in the unique unit summand of B . Thus multiplication restricts to $\mathbf{1} \oplus J^2$. The restricted commutative algebra is the separable order-two group algebra object; in particular L is connected étale.

The image of J in $\mathcal{C}_L^0$ is invertible of order two and twist i . It generates a nondegenerate semion subcategory. Since $\mathcal{Q}$ is unitary modular, its centralizer $\mathcal{E}$ is unitary modular as well. To enumerate its simples, define the charge of $\mathbf{abc}$ by $t = a + 2b + 3c \pmod{4}$. All four summands of B have charge zero, so this grading descends to $\mathcal{C}_B$. Simple-current monodromy with J is $\exp(\pm 2\pi i t/4)$, where the sign depends on the braiding convention. Locality for L requires even charge; centralizing the image of J requires charge zero.

On weights of charge zero the action of J^2 is

$$(a, b, c) \mapsto (c, 6 - a - b - c, a).$$

There are twenty-two such weights. The only fixed weights are 121 and 303. A nonfixed orbit gives one simple local L -module. Each fixed weight gives two: its stabilizer is cyclic of order two, and its twisted group algebra has two one-dimensional irreducible representations. The resulting rank is $(22 - 2)/2 + 2 \cdot 2 = 14$.

For completeness, Table 1 gives all dimensions and twists. They follow from the quantum Weyl formula

$$\dim(\mathbf{abc}) = \frac{[a+1][b+1][c+1][a+b+2][b+c+2][a+b+c+3]}{[2]^2[3]}$$

and $\theta_\lambda = \exp(2\pi i h_\lambda)$. For a nonfixed orbit the module dimension is the dimension of a representative; at a fixed weight each of the two resolved modules has half that dimension. Their squared dimensions sum to $100d^2$.

TABLE 1. Simple objects of $\mathcal{E}$, with $\zeta = \exp(2\pi i/5)$. A row with two representatives describes two distinct orbits; a fixed-weight row describes its two resolved modules.

Orbit representative(s)	Number	Dimension	Twist
000	1	1	1
004, 024	2	$1 + d$	ζ
303 (fixed)	2	$1 + d$	ζ^{-1}
012, 032	2	$1 + 3d$	ζ^{-1}
121 (fixed)	2	$1 + 3d$	ζ
020	1	$2d$	ζ^{-2}
101	1	$2d$	ζ^2
105	1	d	1
113	1	$5d$	-1
202	1	d^2	1

The algebra B descends along $L \subset B$ to a connected étale algebra A in $\mathcal{Q}$ by [3, Proposition 3.16]. Its underlying object is $\mathbf{1} \oplus U_0$, where U_0 is the module of the orbit $\{202, 222\}$. Both summands have charge zero, so A lies in $\mathcal{E}$ and has dimension $1 + d^2$.

We detail the module-in-stages identification to fix its scope. Giving an A -module in $\mathcal{C}_L^0$ amounts to giving a B -module whose restriction to L is local: the action maps are the same after balancing over L , and the relative tensor products agree. A B -module supported on charge-zero objects restricts to a local L -module and centralizes the residual semion. Conversely, any A -module in $\mathcal{E}$ has underlying $\mathcal{C}$ -object of charge zero. Thus this tensor equivalence identifies $\mathcal{E}_A$ with precisely $(\mathcal{C}_B)_0$.

Under the symmetric-square embedding the central element iI_4 maps to $-I_{10}$. It therefore acts on the entire affine module τ_j by $(-1)^j$: its ground state has this character and the current modes preserve it. Thus $U\tau_j$ has charge $2j \pmod{4}$. Consequently τ_2 has charge zero, whereas τ_5 and β both have charge two. It follows that $\mathcal{K} \subset (\mathcal{C}_B)_0 \simeq \mathcal{E}_A$. Finally,

$$\text{FPdim}(\mathcal{E}_A) = \frac{100d^2}{1 + d^2} = 5(1 + d^2) = \text{FPdim}(\mathcal{K}),$$

using $(1 + d^2)^2 = 20d^2$. The full fusion-subcategory inclusion is therefore an equivalence. $\square$

Remark 3.2. The proposition constructs both $\mathcal{E}$ and its algebra A inside known unitary categories. Table 1 records their dimensions and twists; it is not used to assert uniqueness of a modular category or to identify a full S -matrix from those invariants alone. The rank-ten existence proof in Theorem 1.1 is independent of this optional modular construction.

APPENDIX A. EXACT IDENTIFICATION WITH THE CENSUS TENSOR

Let $(e_0, \dots, e_9)$ be the original, zero-indexed census basis of R10_1_189. The based-ring map sends the ordered basis in Section 1 to

$$(12) \quad (e_0, e_1, e_8, e_7, e_2, e_9, e_4, e_6, e_5, e_3).$$

In particular, $g \mapsto e_1$ and $x \mapsto e_9$. If $p = (0, 1, 8, 7, 2, 9, 4, 6, 5, 3)$, the accompanying verifier checks

$$N_{p(i), p(j)}^{\text{census } p(k)} = N_{i,j}^{Rk} \quad (0 \leq i, j, k < 10)$$

for all 1000 structure constants. The right-hand tensor is generated directly from (1). The source tensor is bound by SHA-256 of its compact JSON serialization, namely

c44e21d1844ded91dba30cd6c691180afa4d4f1be1c4f27806bbc12c8c8292f4.

This binds the certificate to the tensor itself rather than to a potentially changing display label.

The standard-library Python files accompanying this note have the following roles:

- `verify_conformal_model.py` checks the two Pieri products, the needed intersections with the published branching table, dimension exhaustion, and the exact census map (12); its receipt is `conformal_model_certificate.json`.
- `verify_minimal_branching.py` checks the conformal energies over the rationals and verifies (10) as an equality of four-variable weight characters by enumerating the exterior-cube and semistandard-tableau characters. Its receipt is `minimal_branching_certificate.json`.
- `verify_r189_rank14_wzw.py` checks the charge-zero orbits, fixed-point count, dimensions and twists in Table 1, using exact arithmetic in $\mathbb{Q}[t]/(t^4 - 5t^2 + 5)$ with $t = 2\cos(\pi/10)$. Its receipt is `r189_rank14_wzw_receipt.json`.

The finite checks supplement the categorical arguments. Existence of the associativity constraints and of the algebra multiplications comes from the realized conformal inclusion and the module constructions above.

REFERENCES

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