Seven unresolved fusion rings: audit and theoretical continuation

The three previous exclusions are retained; seven problems remain open

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Outcome

The fresh review retains the general-complex exclusions of **R9_2_1607, R9_2_1975, and R9_2_1976**. It found no gap requiring their withdrawal. The inherited spherical/Galois and Morita reductions remain explicit dependencies of these conclusions.

This continuation gives **no additional complete census-ring exclusion and no new census-ring model**. All seven requested rings remain unresolved. Each was investigated separately; the new results below include categorical restrictions and exclusions of construction routes, which must not be counted as exclusions of the rings themselves.

Ring	Principal additional result	Census status
R6_4_155	Nonpointed algebra family reduced to two cases; one algebra is forced but returns to a known module category.	Unresolved
R6_8_1660	One center subcase removed; a pairing obstruction also tests proposed Morita realizations.	Unresolved
R8_3_928	Its nonpointed modular centralizer must be simple, with only one nontrivial connected étale algebra.	Unresolved
R8_3_941	Its rank-22 fermion centralizer must be nonsplit and have no proper nonpointed fusion subcategory.	Unresolved
R8_3_1154	All six third indicators are forced to be 2; all six trivalent rotations are identity.	Unresolved
R8_3_1768	A pairing obstruction rules out the higher-Haagerup Morita construction family.	Unresolved
R9_2_137	A nonpointed algebra is forced, but returns to an existing Morita route; both TY signs pass the strengthened exact test.	Unresolved

The source snapshot is the classification repository at `6a426acf6465dd674351be92dcca41b72f371521`. This is offline research: no online repository, website, manuscript release, or classification ledger was changed. The computations in this pass were short exact replays or bounded certificate-discovery tests, not long associator searches.

1. Fresh audit of the three exclusions

For **R9_2_1975**, the review checked the opposite-bimodule and normalizer conventions, cancellation for arbitrary complex cocycles, the strict even-V_4 subgroup used in the projective parity argument, and

the normalized character formula for the distinguished Galois central object. The required restriction remains

$$F(B) = 1 + X_0 + X_1 + X_2 + X_3.$$

In the rotation branch its multiplicity at X_0 is odd, whereas the forced anticommuting stabilizer operators require it to be even. Reflection shifts are separately impossible. The audit preserves the eighth-root coboundary witness: a fourth-root-only test would be invalid. The inherited extreme-branch argument and complete signed gate were also checked as stated dependencies. The fresh gate replay retained two branches and certified completeness.

For **R9_2_1607 and R9_2_1976**, the review specifically checked that pivotal symmetry of the Frobenius pairing, rather than merely braided commutativity, is available. The selfdual multiplicity-one Galois summand of the canonical algebra has a symmetric nondegenerate pairing. Transporting that pairing forces all four large second indicators in the zero-shift Morita category to be $+1$. The dimension-weighted second-indicator total is Morita invariant. Comparison of irrational coefficients then forces the same sum to be four in the original category. The original rings have, respectively, only two and zero selfdual large simples. This is the contradiction. The irrational-coefficient comparison avoids a dependence on TY sign conventions. Both complete signed gates were replayed successfully.

These arguments do not assume that arbitrary tensor functors preserve numerical indicators. Their pairing transport and Morita-invariant total are different, explicitly justified steps. Full fresh reviews are `audit_parity/AUDIT_R1975.md` and `audit_fs2/AUDIT.md`.

2. A useful common obstruction: pairings under Morita forgetful functors

Let $\mathcal{C}$ be pseudo-unitary, and let A be a connected separable algebra. With its positive module trace, A can be equipped with the compatible symmetric separable Frobenius structure. The functor

$$U : {}_A\mathcal{C}_A \longrightarrow \mathcal{C}$$

is pivotal Frobenius monoidal. It preserves a chosen nondegenerate pairing and its pivotal symmetry sign. In particular, if a selfdual simple bimodule Q has second indicator ε, then every selfdual simple constituent of $U(Q)$ having the opposite indicator must occur with even multiplicity.

Indeed, on such an isotypic block the multiplicity space must carry a nondegenerate alternating form. Its dimension is even. This supplies a short necessary test for proposed Morita restrictions. It does not assert equality of the numerical indicators of Q and $U(Q)$.

The positive pivotal normalization matters. The bimodule dimension is $\dim_{\mathcal{C}}(U(Q))/\dim_{\mathcal{C}}(A)$, so the induced structure is the canonical positive one. The relevant primary inputs are Schaumann, *Traces on module categories over fusion categories*, Proposition 5.8 and Theorem 6.6, and Mulevičius, *Condensation inversion and Witt equivalence via generalised orbifolds*, Example 4.18.

3. R6_4_155: forced algebras and a Morita loop

Write the invertibles as $1, h, g, hg$, and the other simples as X, Y, with $X^* = Y$, $hX = X$, $gX = Y$, and

$$XY = 1 + h + 4(X + Y), \qquad X^2 = Y^2 = g + hg + 4(X + Y).$$

Put $d = 4 + 3\sqrt{2}$. Consider a connected separable algebra with underlying object $A_k = 1 + kX + kY$. Four free pointed modules are simple and distinct. Subtracting them from the induction Gram matrix leaves nonnegative integral vectors a, b with

$$\|a\|^2 = \|b\|^2 = 1 + 8k - 4k^2, \qquad a \cdot b = 8k - 4k^2.$$

Thus $k \leq 2$. At $k = 1$ the only viable residual modules have underlying objects $2X + Y$ and $X + 2Y$. The other Gram possibilities would contain a module with underlying object X, whose internal end would have dimension $2 + 4\sqrt{2}/3$, not an algebraic integer. At $k = 2$ the two residual modules have underlying objects X and Y. Each surviving case has a unique necessary based module of rank six.

There is also a conditional existence statement. The canonical central subalgebra corresponding to the pointed subcategory has underlying central object $1+B$, where B is the Galois-unit object. Its forgetful algebra splits through its augmentation as

$$F(1 + B) \cong 1 \times A_2, \qquad [A_2] = 1 + 2X + 2Y.$$

So A_2 is forced in any hypothetical categorification. However the residual module underlying X has internal end $1 + h$. This returns to the already available pointed-algebra module category and supplies neither a construction of the original category nor a contradiction. The potentially different algebra A_1 is not forced or constructed.

An initially tempting trace contradiction was rejected during review: the trace on the four pointed free modules must be included. The total is eight, not four. The accepted proof and replay are `audit_fs2/R155_ALGEBRA_REDUCTION.md` and its companion verifier.

4. R6_8_1660: a narrower valid center restriction

The two remaining possibilities for the negative charge-conjugation representation meeting the four spin phases were

$$(16, 2) \oplus (20, 2), \qquad (44, 2).$$

The second possibility cannot occur in low frame 0. In that frame all high restriction columns have equal X and gX multiplicities. The phase-12 Gram identities force exactly one column there. It is selfdual, leaving no negative charge-conjugation line at that phase, whereas $(44, 2)$ requires one.

A proposed extension to all of frame 1 failed independent checking. The total asymmetric-column budget is eight, not four; the earlier four counted only one member of each pair. The remaining budget can support phase-36 columns that change an apparent half-integral Verlinde entry to the integer zero. Additional negative irreducibles away from the spin phases also cannot be omitted. Thus the entire $(44, 2)$ branch is not excluded, and both overall alternatives remain.

The pairing test in Section 2 does exclude two specific proposed Morita module realizations, labelled `III_mixed0` and `IV_mixed1`. A selfdual simple of their respective duals would restrict to $\sigma + T$ and $X + gX + \sigma + T$. Here σ has indicator -1 and every other displayed simple has indicator $+1$. Their odd multiplicities cannot carry a pairing with a single symmetry sign. The conditional route to R941 passes this test and is retained.

There is also a further rigidity condition: fusion by one distinguished Galois-unit simple acts on the negative charge-conjugation space as a signed permutation. Its eigenvalues there are ± 1; in the normalized dual-difference basis it is a symmetric integral matrix whose square is identity. Hence each row has precisely one entry ± 1. This has not yet yielded a contradiction.

No forced algebra or categorical realization resolving R1660 was found. The independent reviews are `audit_fs2/R1660_SPIN_AUDIT.md` and `audit_fs2/R1660_BIMODULE_PARITY_AUDIT.md`.

5. R8_3_928 and R8_3_941: simple centralizers

Let D be their common global dimension. Their only proper nontrivial fusion subcategory is pointed of rank two. Consequently any central fusion subcategory containing a nonintegral object forgets dominantly onto the whole source category and has dimension at least D. Equality would make the source braided, impossible for these noncommutative rings.

For **R928**, write its center as $S \boxtimes \mathcal{B}$, where S is Semion and $\dim \mathcal{B} = D^2/2$. The unit is the only integral simple in $\mathcal{B}$. A proper nontrivial subcategory and its centralizer would both dominate the source. Their dimensions would therefore have product at least D^2, contradicting the centralizer formula $D^2/2$. Thus $\mathcal{B}$ is **simple**, not just indecomposable as a tensor product. Its possible ranks are 17, 20, and 23.

The earlier all-module rigidity theorem implies a further statement: the entire center has exactly three connected étale algebra classes, forming $1 \subset A \subset I(1)$, with $\dim A = D/2$. The factor $\mathcal{B}$ contains exactly one nontrivial such algebra, namely A, and

$$\mathcal{B}_A \simeq \mathcal{C}, \qquad \mathcal{B}_A^0 \simeq S^{\mathrm{rev}}.$$

For **R941**, the fermion centralizer $\mathcal{E}$ has rank 22, dimension $D^2/2$, and integral part exactly sVec. It cannot split as $\mathcal{M} \boxtimes \mathrm{sVec}$: that would make the center have rank 33 or 44, whereas the complete retained list gives 34, 37, 40, or 43. The same dominant-image and centralizer argument then proves that its only fusion subcategories are Vec, sVec, and $\mathcal{E}$.

These results sharply specify the missing modular or slightly degenerate categories, but do not prove them nonexistent. The current *Anyon Atlas* was checked; its higher-rank numerical search is not certified exhaustive and cannot justify exclusion by absence. The full proofs and exact rank checks are in `r8_928_941/REPORT.md`.

6. R8_3_1154: every trivalent rotation is identity

The preceding actual equivariantization reduction gives a hypothetical rank-ten category with a central sign object u and lifts x, ux of X_4, satisfying $x^* = ux$. For odd n, the two character subspaces of $\mathrm{Hom}(1, X_4^n)$ are the rotations on $\mathrm{Hom}(1, x^n)$ and $\mathrm{Hom}(1, (ux)^n)$. Spherical duality gives equal indicators of x and x^*. Therefore

$$\nu_n(X_4) = 2\nu_n(x) \quad (n \text{ odd}).$$

The same holds for X_5. Their lift's cubic vacuum spaces are one-dimensional, so their third indicators lie in $\{2, 2\zeta_3, 2\zeta_3^2\}$. This is stronger than allowing an arbitrary sum of two cube roots.

Applying these discrete possibilities to the three already retained center patterns forces both indicators to be two. Duality, the inner autoequivalence, and a second small discrete check then force

$$\boxed{\nu_3(X_i) = 2 \quad i = 1, 2, 4, 5, 6, 7.}$$

Each of these six cubic spaces is two-dimensional. Since its rotation has order three, trace two means that the rotation is the identity. All 33 excluded discrete branches have exact integer separating certificates; the six retained branch records have exact rational feasibility witnesses. An independent implementation checked them. No center pattern is eliminated outright.

The theorem and solver-free replay are in `r8_1154/ODD_ROTATION_THEOREM.md`; the independent audit is in `audit_parity/AUDIT_R1154_ODD_ROTATION.md`.

A second short argument proves that the integral part of its center is Vec. An integral central simple could forget only to copies of 1 and g, but g is not central in the fusion ring, forcing its coefficient to vanish. The full adjoint subring then makes every central object supported on the unit trivial. This applies directly to arbitrary complex categorifications. It rules out nontrivial finite-group extensions and equivariantizations throughout the Morita class. In the auxiliary rank-ten category, the integral center is exactly double Semion; its unique nontrivial Tannakian subcategory only gives back the already known descent.

Three further direct central-condensation families are ruled out: $\mathrm{Fib}^{\boxtimes 6}$, $GH(C_4, 1) \boxtimes \mathrm{Fib}^{\boxtimes 3}$, and $GH(C_4, 1)^{\boxtimes 2}$, with arbitrary pointed factors. In the last case, the required index-three subgroup contains a central C_4; its condensation would force the impossible rank-two rule $x^2 = 1 + 4x$. These are specified-family obstructions, not a classification of every possible construction.

7. R8_3_1768: a uniform higher-Haagerup obstruction

Put $u = 6 + \sqrt{37}$. The positive source indicators are $(1, -1, 1, 1, 1, 1, 1, 1)$, and its weighted second-indicator total is $6u$. In a category with $GH(C_6, 2)$ fusion rules, the six large simples are selfdual of dimension u. The pointed contribution to that total is $1 + \nu_2(g^3)$. Morita invariance and irrationality force

$$\nu_2(g^3) = -1, \qquad \nu_2(g^j\sigma) = +1 \quad (0 \le j < 6)$$

whenever such a category is Morita equivalent to R1768.

First consider the proposed algebra $A = 1 + g^j\sigma$. The negative invertible of the R1768 dual has underlying dimension $1 + u$ in the higher-Haagerup category. Irrationality forces its underlying object to contain one pointed and one large simple, each once. Its negative pairing cannot restrict to the latter, whose second indicator is positive. Section 2 gives a contradiction. This rules out the route for **every associator**, not only the original arithmetic precursor.

A stronger use of the same parity principle applies on the R1768 side. For a negative selfdual simple in a Morita dual, all six positive nonpointed source constituents must occur evenly. Thus the $\sqrt{37}$ coefficient of its underlying dimension must be even. The complete previously retained list has 21 possible module factors. In 17 of them, some simple module has internal end with an odd such coefficient. Choosing that simple as generator prevents its dual from having a negative-indicator invertible. Each of the other four factors has a dimension-one internal end, so its module category is regular.

It follows that a Morita representative with a negative-indicator invertible must come from the regular module category, and hence have rank eight. In particular **no category with $GH(C_6, 2)$ fusion rules is Morita equivalent to it**. This eliminates the entire higher-Haagerup Morita family, including other algebras in that family. It does not eliminate an unknown R1768 category itself.

A second exact RW arithmetic input was also found during the construction attempts:

$$\gamma = \begin{pmatrix} 85 & 144 \\ 36 & 61 \end{pmatrix}.$$

It supplies another concrete input for a twisted precursor, without proving inequivalence to the old one. The new uniform obstruction explains why that possible change cannot solve R1768 through Morita equivalence. The exact parity filter and full categorical premises are recorded in the `r8_1768/` evidence.

8. R9_2_137: a forced algebra does not supply a new category

In its remaining zero-shift rank-twelve Morita category, let $S = \sum_q X_q$, $d = 4+3\sqrt{2}$, and z be the central involution of the pointed D_8. The canonical étale subalgebra associated to the pointed subcategory is forced to be $1+B$, where B is the Galois-unit simple. Its forgetful algebra has augmentation and splits as

$$F(1+B) \cong 1 \times A, \qquad [A] = 1 + S.$$

Thus this nonpointed algebra is forced, not merely a speculative construction. Its induction Gram, however, gives eight modules underlying $g+S$ and four underlying X_q. Each of the latter has internal end $1+z$. The module category is therefore the one already obtained from the central-involution algebra. This is a Morita loop, not a realization of a presently hypothetical category.

The previously established positive second indicators were then inserted into both remaining TY-sign center systems. Each has an exact nonnegative rational witness, with support 102 and 97, respectively. The complete short tests took under four seconds each; the saved witnesses replay without an optimizer. They show that this combination of constraints still does not exclude the ring.

The proof, source bindings, matrices, and witnesses are in `audit_parity/R137_CONTINUATION.md` and its companion files.

9. Scope of the remaining problems

No based module, modular-data candidate, precursor category, or feasible indicator witness has been promoted to a model of a census ring. The missing ingredients are still actual categorical constructions or complete obstructions for the surviving cases. The report separates accepted theorems, inherited premises, and rejected shortcuts so that the next investigation can start from the stronger frontier without repeating invalid inferences.

Primary references used for the new arguments

- Davydov, Müger, Nikshych, Ostrik, *The Witt group of non-degenerate braided fusion categories*: https://arxiv.org/abs/1009.2117.
- Drinfeld, Gelaki, Nikshych, Ostrik, *On braided fusion categories I*: https://arxiv.org/abs/0906.0620.
- Schaumann, *Traces on module categories over fusion categories*: https://arxiv.org/abs/1206.5716.
- Mulevičius, *Condensation inversion and Witt equivalence via generalised orbifolds*: https://arxiv.org/abs/2206.02611.
- Ng, Schauenburg, *Frobenius-Schur indicators and exponents of spherical categories*: https://arxiv.org/abs/math/0601012.