

Rank twelve, multiplicity one: fresh audit and fourth theoretical continuation

Fusion-ring census research notes

Offline working report

Abstract

We audit all 438 exclusions and 148 unitary realizations retained in the preceding round, and revisit its 46 unresolved based fusion rings. No accepted classification is withdrawn. A complete Morita and extension-theory argument resolves the remaining dimension-32 family, with two new unitary models and sixteen exclusions. A new inner-conjugation equivariantization argument excludes R12_1_332 over the complex numbers. A uniform integral-Lagrangian criterion, together with complete quotient reconstructions, excludes all twenty remaining dimension-48 rings. An alternate Brauer–Picard extension resolves the four dimension-144 cases. The accepted totals are 478 excluded, 151 unitarily realized, and 3 unresolved, among 632 inputs. Additional theoretical reductions for the unresolved families are stated with their precise limitations. An erroneous blanket near-group assertion in an earlier exploratory note is corrected; no accepted decision depended on it. All work remains offline. The proofs and exact computations have received independent checks within the assistant team; independent human verification remains pending.

1 Results, identifiers, and scope

We count supplied *based fusion rings*, not tensor-equivalence classes of categories. Every input is bound to its complete multiplication tensor and SHA256 digest. A negative decision excludes arbitrary complex fusion categories. Every positive decision supplies an actual unitary category and a unit-preserving bijection checking all $12^3 = 1728$ coefficients.

Stage	Excluded	Unitary models	Unresolved
Previous accepted ledger	438	148	46
New decisions in this report	40	3	-43
Current accepted ledger	478	151	3

The new models are R12_1_320, R12_1_419, and R12_1_530. “New” here means new explicit realizations in this project; no novelty up to tensor equivalence or exoticness is claimed. The new exclusions are:

Family	IDs, with prefix R12_1_
Dimension 32	128, 144, 147, 173, 237, 275, 285, 294, 301, 318, 412, 415, 422, 431, 524, 536
Dimension 48	130, 146, 151, 175, 216, 239, 244, 277, 289, 296, 303, 319, 407, 410, 421, 424, 433, 523, 526, 538
Dimension 144	180,181,321
Quadratic	332

The proofs below separate categorical existence/converse arguments from finite arithmetic and reconstruction checks. An associative candidate table, a based-ring automorphism, or a failed construction search is not treated as a categorical realization or exclusion.

2 Audit of the preceding 586 settled decisions

2.1 Input, exclusions, and general-complex scope

The fresh audit binds all 632 tensors to their original physical input lines, verifies their fusion-ring axioms, and checks all 4012 recorded closed-subring embeddings. Each of the 438 exclusion routes was reviewed and replayed with explicit checks active under optimized Python.

Particular attention was paid to four possible sources of overstatement. First, the published version of Schopieray’s Lemma 3.1 gives a *pseudo-unitary Galois conjugate* for a two-FP-value categorification; it does not assert pseudo-unitarity of the original category. Second, unitary obstruction theorems were used only after a valid bridge to unitarity: the relevant prime-power-product dimensions are solvable by ENO and hence completely unitary by Galindo–Hong–Rowell. Third, Liptrap negative certificates retain arbitrary nonzero complex variables, including variable τ ; their integer Laurent relations cancel every unknown, rather than rejecting a bounded phase search. Fourth, the inherited rank-eight signed obstruction enumerates all 8192 unrestricted oriented sign tables. All sixteen associative survivors are actual sign gauges, and every possible spherical dimension-character orbit is obstructed.

The twenty-four inherited exclusions are freshly mapped to six complete lower-rank source proofs. The twelve field/order positivity checks were strengthened to use rational isolating intervals. The dimension-120 premise checker was also strengthened: it now explicitly checks the *complete left and right stabilizer* of every dimension-two simple, not just the common involution. The stronger hypothesis holds for all sixteen inputs. The pointed Morita converse, finite-group restriction, relative cohomology exhaustion, and coprime-twist lemma were separately reread. All four resulting models were freshly reconstructed, and all 64 model/input comparisons were rechecked.

For R12_1_350, the portable independent verifier reconstructs thirty low induction frames, four PSD survivors, 45 complete joint cyclotomic trace types, and 32 exact Farkas systems. It imports neither an optimizer nor the earlier rank-six implementation. The codegree of each two-dimensional matrix block is correctly 6, not its Casimir eigenvalue 12. The cyclotomic trace box is not an assumed bound on the categorical exponent.

2.2 Models, categorical actions, and unitary source tables

The positive audit checks 381 ordinary first-round witnesses and 21 independently reconstructed Liptrap witnesses on 127 distinct rings, then the specialized witnesses for the additional twenty-one rings. It checks actual source categories, tensor actions, algebra objects, complete simple-module lists, and full tensor identifications. The checks distinguish a central lift from a commuting basis element.

A delicate inherited premise was strengthened constructively. For both actual pseudo-PSU associator tables used in the rank-eight source constructions, the audit finds an exact positive diagonal gauge. Every nonzero coefficient satisfies

$$\frac{(F^{-1})_{fe}}{F_{ef}} = \frac{w_{abe}w_{ecd}}{w_{bcf}w_{afd}}, \quad w_{ijk} > 0.$$

The weights are rational powers of 2 and $1 + \sqrt{2}$. Thus all 136 associator blocks become unitary. The two sets of 2688 pentagon identities and the corresponding half-braiding hexagons are checked directly. This avoids identifying a saved pentagon solution with a published unitary example merely because their fusion rings agree.

2.3 A genuine correction, with no classification change

An earlier exploratory dimension-48 note called every near-group quotient $\text{NG}(G, 2)$ with $|G| = 8$ impossible. This omits the nonabelian possibilities D_8 and Q_8 , which do admit such categories. No accepted exclusion used that assertion. The new analysis restores these branches. Exact orbit-ring checks allow the D_8 branch for R12_1_216 and both D_8, Q_8 branches for R12_1_239; known models provide additional controls. The analogous particular branches for R12_1_175, R12_1_319, R12_1_407, and R12_1_523 fail the complete nonabelian orbit-ring comparison. These are restrictions on possible central lifts, not whole-ring exclusions.

3 Complete resolution of the remaining dimension-32 family

Theorem 1. *For the verified input profile $1^8, 2^2, (\sqrt{8})^2$, with adjoint series of dimensions 32, 16, 4, 1, the only categorifiable supplied based rings are*

R12_1_28, R12_1_183, R12_1_240, R12_1_405, R12_1_408, R12_1_419, R12_1_522, R12_1_530.

All eight have unitary models. Consequently the other sixteen remaining inputs in this family are excluded over $\mathbb{C}$.

This is a classification for the stated, exactly checked profile; it is not a claim about every fusion category of dimension 32.

3.1 A forced Morita reduction

Let $\mathcal{C}$ be a hypothetical categorification. Its dimension-sixteen integral part has eight invertibles and two dimension-two simples X, Y , with common abelian stabilizer H of order four. Write $Y = gX$ and $P = \text{Ad}_g|_H$. Liptrap's identities give the nondegenerate bicharacter relations

$$B_Y(a, b) = B_X(Pa, b), \quad B_{X^*}(a, b) = B_X(b, a), \quad P^2 = 1.$$

There is a nontrivial involution $h \in H$ fixed by P and isotropic for both forms. For $H = C_4$, take its unique involution. For $H = C_2^2$, a symmetric nondegenerate form has an isotropic vector, and a nontrivial orthogonal involution has an isotropic fixed line. If duality exchanges X, Y , the cosquare relation forces $P = 1$ in dimension two.

The rank-one action on X trivializes the pointed associator on H . Thus $A = 1 + h$ is an actual separable algebra. The isotropy identities make the integral A -bimodule dual pointed of dimension sixteen. Its whole Morita dual $\mathcal{E}$ has one of the two types

$$1^{16}, (\sqrt{8})^2, \quad 1^{16}, (\sqrt{2})^8.$$

The inverse algebra is an actual $1 + c$, with c a noncentral involution in $G(\mathcal{E})$, and the original integral type forces $|C_{G(\mathcal{E})}(c)| = 8$.

3.2 The two-square-root-eight source branch

The common stabilizer has order eight. If the universal grading is C_4 , the associated involutive cosquare is the identity for every abelian group of order eight; the invertible group would be abelian, a contradiction. For universal group C_2^2 , the inverse involution cannot lie in the stabilizer. Indeed, for an orthogonal involution P and a moved order-two element c , the two required commutators $B(c, c) = B(Pc, c) = -1$ are incompatible. For C_2^3 this follows from the rank-one symmetric form associated to $P - 1$; the C_8 and $C_4 \times C_2$ cases have no suitable moved involution. Complete finite form checks independently verify these elementary lemmas.

Therefore the source is a genuine crossed product of a TY category on an abelian group of order eight. Its inverse algebra yields a coherent C_2 equivariantization. Orthogonal involutions fixing four group elements and the two parity-square choices exhaust the possibilities. They yield only the already known rings R12_1_28, R12_1_183, R12_1_405, R12_1_522.

3.3 The generalized-Ising sources and the missed nonsplit class

For the other branch, $\mathcal{E}_{\text{ad}} = \text{Vec}_{C_2}$ and its universal grading group U has order sixteen. Put $K = G(\mathcal{E})/\langle h \rangle$. The inverse involution has a noncentral image $\bar{c} \in K$: a central image would force a dimension-two object times an odd object to have multiplicity two. Thus $K = D_8$. Write

$$G(\mathcal{E}) = \langle R, c, h \mid R^4 = c^2 = h^2 = 1, \ h \text{ central}, \ cRc = R^{-1}h^b \rangle, \quad b = 0, 1.$$

The centralizer condition forces $R^4 = 1$. Both classes $b = 0, 1$ must be retained; the nonsplit class $b = 1$ is the new source of the two models. The condition that no odd degree centralizes $\bar{c}$ leaves $U = D_{16}$ or SD_{16} . There are therefore four reference families.

Their categorical exhaustion follows from ENO extension theory, not from a search for associators. The coefficient module is

$$M = \text{Inv } Z(\text{Vec}_{C_2}) = C_2^2 = \text{Ind}_K^U C_2, \quad H^2(U, M) \cong H^2(K, C_2).$$

The Shapiro projection is restriction to K followed by forgetting the half-braiding. It is exactly the central-extension class of the invertible group displayed above, so no additional H^2 datum is omitted. Actual Liptrap reference categories exist for all four possibilities; all others are U -three-cocycle twists.

The LHS bounds for $|H^3(U, \mathbb{C}^\times)|$ are 32 for D_{16} and 16 for SD_{16} . Explicit rotation, reflection, and (in the dihedral case) mixed cocycles attain these bounds. The inverse C_2 algebra exists precisely when the restriction is trivial; its algebra cochain is then unique up to isomorphism. All rotation powers, admissible mixed twists, and both inverse lifts are included.

The exact catalogue comprises 96 generalized-Ising constructions and 46 TY constructions. An independent implementation recomputes all 142 full fusion tensors by monomial Hom equations. A separate graph-isomorphism check proves that none of the sixteen excluded inputs is isomorphic to an allowed output. The new nonsplit rotation twists give R12_1_419 and R12_1_530. Their actual source equations, Fourier unitarity, cocycle identities, subgroup algebra, and complete bimodule tensors are also audited separately.

4 A new symmetry obstruction: R12_1_332

Lemma 2 (Inner conjugation). *Let H be a subgroup of invertible isomorphism classes of a fusion category, preserving a fusion subcategory $\mathcal{D}$ by conjugation. Then the conjugation functors give a coherent categorical H action on $\mathcal{D}$, even when the pointed associator on H is nontrivial.*

Proof. Choose representatives u_h and multiplication isomorphisms $m_{h,k} : u_h \otimes u_k \rightarrow u_{hk}$. Conjugation is $A_u(X) = u \otimes X \otimes u^*$, with tensorator supplied by evaluation. An isomorphism $f : u \rightarrow v$ induces $A_f = f \otimes \text{id} \otimes (f^{-1})^*$. The two composites of multiplication maps on three invertibles differ by the scalar associator. Conjugation applies this scalar and its inverse on the two outer factors, so the discrepancy cancels. The induced composition constraints therefore satisfy the action pentagon. No algebra $\bigoplus_h u_h$ is asserted by this argument. $\square$

Theorem 3. *The fusion ring R12_1_332 has no complex categorification.*

Proof. Pass to a pseudo-unitary Galois conjugate using published Schopieray Lemma 3.1. The exact source contains a Haagerup- C_3 subcategory with

$$\alpha^3 = 1, \quad \rho\alpha = \alpha^{-1}\rho, \quad \rho^2 = 1 + \rho + \alpha\rho + \alpha^2\rho, \quad d = \frac{3 + \sqrt{13}}{2}.$$

An existing invertible involution b conjugates α to α^{-1} and fixes ρ . The lemma gives a genuine C_2 action. Its equivariantization is pseudo-unitary and has six simples $(1, g, U, X, Y, V)$ of dimensions $(1, 1, 2, d, d, 2d)$, with forgetful images

$$1, 1, \alpha + \alpha^2, \rho, \rho, \alpha\rho + \alpha^2\rho.$$

Here g is the canonical sign object, $Y = gX$, and $gU = U, gV = V$. The tensor images force $U^2 = 1 + g + U$, $UX = UY = V$, and $UV = X + Y + V$. Initially $X^2 = a + W + V$ with $a \in \{1, g\}$ and $W \in \{X, Y\}$. If $a = g$, duality exchanges X, Y , but $Y^2 = X^2$ by centrality of g , contradicting the unique X/Y summand. Thus $a = 1$, and relabeling permits $W = X$. All remaining products are forced:

$$\begin{aligned} X^2 = Y^2 &= 1 + X + V, & XY &= g + Y + V, \\ XV = YV &= U + X + Y + 2V, & V^2 &= 1 + g + U + 2X + 2Y + 4V. \end{aligned}$$

This covers arbitrary action tensorators and coherence, not just one chosen equivariant structure. The global dimension is $D = 39 + 9\sqrt{13}$. The character

$$(1, g, U, X, Y, V) \mapsto (1, -1, 0, \phi, -\phi, 0), \quad \phi = \frac{1 + \sqrt{5}}{2},$$

has formal codegree $f = 5 + \sqrt{5}$. Central induction would supply an object of dimension D/f , an algebraic integer. But

$$N_{\mathbb{Q}(\sqrt{13}, \sqrt{5})/\mathbb{Q}}(D/f) = \frac{468^2}{20^2} = \frac{13689}{25} \notin \mathbb{Z}.$$

This is impossible. The norm argument is independent of the project's older exclusion of the isomorphic rank-six ring. $\square$

5 A uniform exclusion of the dimension-48 family

Theorem 4. *None of the twenty previously unresolved inputs of type $1^8, 2^2, 4^2$ listed above admits a complex categorification.*

The proof has two independent parts: a categorical converse forcing any hypothetical category to be group-theoretical, and complete exhaustion of that group-theoretical construction. Neither a failed search nor an assumed complete census of possible quotient rings is used.

5.1 Complete group-theoretical parameters

Double-coset dimensions and normal generation restrict the possible finite-group/subgroup pairs. LHS and Kunneth bounds, attained by explicit cocycles, give complete relative associator coverage for $S_4 \times C_2$ and $A_4 \rtimes C_4$. For $GL_2(3)$ and the binary octahedral group, the sphere-bundle Gysin sequence and $p_1(\text{Ad}) = -4c_2$ identify the periodic generators and their restriction kernels. All subgroup Schur classes are retained. There are 432 exact restriction certificates.

The 92 admissible order-four/order-eight subgroup constructions give 36 wrong ranks, sixteen tables with a coefficient larger than one, and forty copies of R12_1_30, R12_1_414, R12_1_417, R12_1_531. The remaining twelve order-sixteen subgroup constructions give six wrong ranks and six copies of R12_1_30 or R12_1_417. Independent code certifies the complete projective simple lists and reconstructs all relevant fusion tensors. Thus the twenty targets have no group-theoretical realization.

5.2 An integral-Lagrangian criterion

Lemma 5. *Every integral 2-group extension of Vec_G^ω is group-theoretical when $G = A_4 \times C_2$ or $SL(2, 3)$, for arbitrary ω .*

Proof. A Lagrangian in $Z(\text{Vec}_G^\omega)$ is described by a normal abelian subgroup N and an admissible invariant twisted alternating bicharacter. Its corresponding module category has uniform normalized simple dimension $\sqrt{|N|}$. Call it integral when $|N|$ is a square.

An integral Morita component preserves these Lagrangians. Indeed canonical module dimensions are multiplicative under the balanced tensor functor (ENO, Propositions 2.2 and 4.13). If the three module stabilizer orders are n_0, n, n' , a product of two simples gives $\sqrt{n_0}\sqrt{n} = k\sqrt{n'}$ with k a positive integer. When n_0 is a square, n, n' have the same rational square class.

For $A_4 \times C_2$, the square-order normal abelian subgroups are exactly $1, V_4$. The first supplies one Lagrangian. The admissible forms on V_4 are either absent or a torsor under its two invariant alternating forms. There are therefore one or three integral Lagrangians. For $SL(2, 3)$, only the trivial normal abelian subgroup has square order, giving one. A finite 2-group acting on an odd set fixes an element; the invariant Lagrangian criterion then proves the claim. $\square$

Integrality is preserved by the Morita operations used here. One direct proof is that $Z(\mathcal{C})$ is integral when $\mathcal{C}$ is integral. For a Morita equivalent $\mathcal{D}$, induction gives $\text{FPdim } I_{\mathcal{D}}(X) = \text{FPdim}(\mathcal{D}) \text{FPdim}(X) \in \mathbb{Z}$. Thus $\text{FPdim}(X)$ is rational and an algebraic integer, hence an integer.

5.3 All possible central quotients

Each target has trivial universal grading. Solvability forces a final C_2 equivariantization: a final extension is impossible, and a final C_3 equivariantization would put an order-three subgroup among its eight invertibles. Every possible central involution is retained. Its fixed simple objects give four kinds of dimension-24 quotient.

Rank-twelve quotients. A new bounded reconstruction starts with all five groups of order eight, all eligible involutions, all left and right actions on four dimension-two objects, all dualities, and all remaining fusion coefficients. The invariant orbit products reduce the last step to at most three binary variables. It finds exactly the two quotient rings R12_1_351, R12_1_513; 704 full tables and every returned identification were independently checked. This establishes completeness without relying on the supplied rank-twelve list being exhaustive.

Both quotients are C_3 extensions of their pointed adjoint category of dimension eight. For R12_1_351, the group is C_2^3 and the distinct left and right order-four stabilizers force its associator to be trivial: the full bimodule stabilizer trivializes $\omega \boxtimes \omega^{-1}$, and restriction to its order-two elements kills all seven Boolean cohomology coordinates. An order-three isometry of the resulting six-dimensional hyperbolic quadratic space has moving dimension two or four. Moving dimension four has an invariant Lagrangian. Moving dimension two forces the left and right stabilizers to coincide, contrary to the actual tensor. Thus the category is group-theoretical. For R12_1_513, the two cyclic stabilizers in $C_4 \times C_2$ leave only the trivial associator or the inflated C_2 class. Their centers have respectively ten and four Lagrangians. A C_3 action fixes one.

The graded pointed Morita presentation has group $\Gamma = A \rtimes C_3$, $|A| = 8$. Nontrivial actions give only $A_4 \times C_2$ or $SL(2, 3)$. A trivial action would split the category as its original pointed neutral component times Vec_{C_3} , contradicting the dimension-two objects. Apply the integral-Lagrangian lemma to the Morita transport of the final C_2 crossed product. Thus these quotient branches yield only group-theoretical parents.

Near-group quotients. The restored $\text{NG}(D_8, 2)$ and $\text{NG}(Q_8, 2)$ branches are treated in full. Solvability and complete unitarizability justify the unitary classification. Each is Morita equivalent to $\text{Vec}_{S_4}^\omega$

with ω inflated from S_3 . Its center has exactly three Lagrangians, so every C_2 action fixes one. Consequently every such equivariantization is group-theoretical.

Free rank-six quotients. Their type is $1^4, 2, 4$ and their universal grading is trivial. Solvability gives a further C_2 equivariantization of dimension twelve. A complete 556-table elementary ring check, together with the required swap of the two dimension-two objects and the Jordan–Larson theorem, forces the predecessor ring R_{3,V_4} . The outer action fixes the unique central lift of the inner boson; this flattens the two equivariantizations into one by a group P of order four. There are three group-theoretical predecessors, with exactly three central Lagrangians each, so their P equivariantizations are group-theoretical. For the three nongroup-theoretical predecessors, the exact soft-kernel calculation gives an embedding of tensor autoequivalences into S_3 . An order-four action reduces to the trivial action or the reflection, and in both cases a fusion coefficient two survives. Thus none produces a target of multiplicity one. All categorical action coherences are retained; the reduction is not a based-ring automorphism assumption.

Rank-nine quotients. A separate direct reconstruction retains all five dimension-two objects, their actions and dualities, and all bounded integer coefficient assignments. Two of the four orbit-table classes have no associative lift. Each of the other two has one lift, with trivial universal grading and a unique central invertible involution h ; h fixes all five noninvertibles. Solvability therefore forces an h -equivariantization of a pointed dimension-twelve category. The outer action fixes h , and its half-braiding is unique because the universal grading is trivial. The central Tannakian subcategories combine to $\text{Rep}(P)$, $|P| = 4$. The original category is a P equivariantization of that pointed category, and is group-theoretical. In particular this argument does *not* assume a C_3 grading for a rank-nine quotient; some of the surviving based-ring lifts have no such grading.

These four branches cover all possible central involutions and contradict the group-theoretical exhaustion above.

6 Complete resolution of the dimension-144 family

Theorem 6. *$R_{12_1_320}$ has a unitary model. The rings $R_{12_1_180}$, $R_{12_1_181}$, $R_{12_1_321}$ have no complex categorification.*

The actual rank-three subcategory is $\text{Rep}(S_3)$, so its regular algebra $A = 1 + z + 2E$ exists. The strengthened Morita argument proves that the complete A -bimodule dual has type

$$1^{24}, 2^{12}, (\sqrt{2})^{36}, \quad \mathcal{D}_{\text{ad}} = \text{Vec}_{V_4}.$$

In particular the dimension-24 subcategory with $\text{Rep}(S_4)$ fusion rules has pointed dual $\text{Vec}_{S_4}^\omega$, with $\omega|_{A_4} = 0$. It is never identified with ordinary $\text{Rep}(S_4)$ just from its fusion ring. The proof uses internal ends of transported modules, full stabilizers, and the commuting left/right S_4 actions. These are spelled out in `d144/new_morita_reduction.md`.

The universal group has order 36 and embeds in $\text{BrPic}(\text{Vec}_{V_4}) = (S_3 \times S_3) \rtimes C_2$. Here the quadratic space is the sum of two anisotropic planes, with diagonal electric Lagrangian; its pointed parabolic is $\text{diag}(S_3) \times C_2$. Only two index-two subgroups have the necessary six pointed grades:

$$U_{\text{base}} = S_3 \times S_3, \quad U_{\text{alt}} = \{(a, b, \epsilon) : \text{sgn}(a) \text{sgn}(b) = (-1)^\epsilon\} \cong C_3^2 \rtimes C_4.$$

The twelve admissible marked base extensions are exhausted by the six triality gaugings and their central zests. An explicit scalar cup cocycle proves that these zests stay within the base subgroup; all give previously modeled rings, none of the four targets.

For the alternate subgroup, put $M = \text{Inv } Z(\text{Vec}_{V_4})$. The normal C_3^2 has no invariants in M , so $H^i(U_{\text{alt}}, M) = 0$. The obstruction O_3 vanishes and the multiplication datum is unique. The obstruction O_4 restricts trivially to C_3^2 , using the existing base precursor, and to the Sylow C_4 ,

whose fourth cohomology with $\mathbb{C}^\times$ coefficients vanishes. Transfer kills O_4 entirely. Actual alternate extensions therefore exist. Restriction of their H^3 torsor onto the reverse S_3 subgroup is surjective, with kernel of order two, so both admissible classes carry an actual separable S_3 algebra. This proves existence before computing fusion rules.

The absolute three-primary parameter needs care. A Galois automorphism fixing $\mathbb{Q}(\zeta_8)$ and inverting ζ_3 acts affinely by -1 on the C_3^2 extension torsor, so has a unique fixed point. The complete marked first-axis triality table at $m = 0$ is defined over $\mathbb{Q}(\zeta_8)$. Its first-axis restriction and trivial reverse diagonal restriction make its torsor point fixed. The alternate extension has the same property, since its invariant line $x^2 + y^2$ restricts injectively to the diagonal. The remaining two-primary integral ambiguity is detected on the reverse S_3 subgroup and vanishes because that restriction is fixed as trivial. Hence its integral reverse fusion ring is exactly the $m = 0$ ring. This excludes R12_1_181, R12_1_321: their three dimension-four simples have $N_{WW}^W = 1$, whereas the forced ring has zero.

Finally the alternate Sylow C_4 restriction has a unique admissible Brauer–Picard class up to pointed-parabolic conjugacy, and $H^2(C_4, M) = 0$. A concrete unitary Liptrap source realizes it; its neutral V_4 associator is explicitly trivialized by a two-cochain. Thus the local fusion tensor of the actual extension is forced. The actual reverse S_3 bimodule orbits give four nonselfdual odd objects of dimension $3\sqrt{2}$. This excludes R12_1_180, which has two self-dual odd objects. A short complete NIMrep argument then forces the whole tensor: four of eight candidate tables are associative, and each matches all 1728 coefficients of R12_1_320. An independent implementation reconstructs those tables. This closes the gap between precursor existence and an actual model of the requested ring. Unitarizability follows from solvability; no hypothetical lift of a ring automorphism is needed.

7 The remaining quadratic cases and further reductions

The unresolved quadratic rings are R12_1_561, R12_1_623, R12_1_628. The new inner-action method produces stronger finite section and induction constraints. It must not be conflated with de-equivariantization by a Tannakian subcategory. A braided central lift need not be Tannakian in a nonbraided ambient fusion category. In particular the assertion that every nondegenerate central pointed subcategory splits off as a tensor factor is false in that generality. No exclusion in this report uses that assertion. The partial constraints and the exact remaining categorical gaps are retained in the evidence notes.

For R12_1_561 and R12_1_628, the coherent inner C_3 action and complete 72-section arithmetic leave only subgroup sections. They imply an actual central braided C_3 lift, but do not force its braiding to be isotropic. The three surviving sections are compatible with the three possible braided diagonals. In the nondegenerate cases, the actual regular C_3 algebra gives an Azumaya Morita autoequivalence; it does not give a rank-four tensor quotient or a different excluded ring.

For R12_1_623, the corresponding inner C_2 reduction forces a central semionic lift. Its square-root twists and duality greatly reduce the possible central-induction data. Exact dimension and Gram constraints leave thirteen integer profiles, and the strengthened semion trace conditions eliminate one of them. Twelve remain unseparated. Neither a Tannakian quotient nor an algebra $1 + z$ is asserted for the semionic object.

As a separate uniform result, complete exact TY-center calculations give Lagrangian counts 7, 5, 2, 2, 1, 1, 1, 1 for the eight dimension-eight Tambara–Yamagami categories. Thus every C_3 extension of such a category is group-theoretical. This useful result is not used to assign a false C_3 grading to the surviving rank-nine quotient tables.

The September 2026 Haagerup–Izumi construction and classification were checked against the remaining inputs. Their odd-order and stated Q-system hypotheses do not cover these three even-order rings. The paper’s general inner-conjugation proposition independently supports the action lemma used above, but supplies no missing classification here.

8 Remaining IDs and reproducibility

Exactly three rings remain unresolved: R12_1_561, R12_1_623, R12_1_628. No failed search is promoted to an exclusion. All other forty-three formerly unresolved cases have either a general-complex obstruction or an actual unitary realization.

The package contains the updated ledger, input hashes, full proof notes, portable accepted constructions and verifiers, separate independent receipts, and the frozen preceding package. Discovery failures and conditional reductions are labeled separately from accepted decisions. No repository commit, push, website edit, or deployment was made.

Selected primary references

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