

Rank twelve, multiplicity one: third offline theoretical continuation

Fusion-ring census research notes

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Abstract

We revisit all 66 rings left unresolved by the previous round. A unified Morita and cohomology argument resolves the fifteen dimension-120 cases, giving three unitary models and twelve general-complex exclusions. Twisted Ising constructions give two further models; relative cocycle adaptation gives another two. A complete central-induction and indicator certificate excludes one quadratic ring. Thus this round resolves twenty cases: thirteen exclusions and seven models. Of the 632 supplied rings, 438 are excluded, 148 have unitary models, and 46 remain unresolved. All results are offline. Independent computational and mathematical checks within the project have been performed; independent human verification remains pending.

1 Results and precise scope

The objects counted here are supplied based fusion rings, identified by their permanent census IDs and SHA256 hashes of their complete tensors. These are not counts of fusion categories up to tensor equivalence. An exclusion concerns every complex fusion categorification. A positive result is an actual unitary category with all $12^3 = 1728$ structure constants matched under a unit-preserving basis bijection.

Stage	Excluded	Modeled	Unresolved
Previous completed round	425	141	66
This round: changes	+13	+7	−20
Current total	438	148	46

The new exclusions are

R12_1_161, R12_1_211, R12_1_262, R12_1_265, R12_1_312, R12_1_350, R12_1_362,
R12_1_388, R12_1_453, R12_1_469, R12_1_472, R12_1_503, R12_1_516.

The new unitary models are

R12_1_142, R12_1_240, R12_1_361, R12_1_408, R12_1_414, R12_1_444, R12_1_531.

No earlier exclusion or model is retracted. The previous round is preserved as a frozen snapshot in the archive. Failed searches, feasible necessary conditions, and based-ring automorphisms without categorical lifts have not been used to change any status.

Family	Entering	Excluded here	Modeled here	Remaining
Dimension 32	20	0	2	18
Dimension 48	22	0	2	20
Dimension 120	15	12	3	0
Dimension 144	4	0	0	4
Quadratic, $\sqrt{13}$	1	0	0	1
Quadratic, $\sqrt{10}$	4	1	0	3
Total	66	13	7	46

2 The dimension-120 family is completely resolved

Theorem 1. *Among the sixteen supplied rings of type $(1^4, 2^4, 5^4)$, the categorifiable ones are precisely $R12_1_25$, $R12_1_142$, $R12_1_361$ and $R12_1_444$. Each has a unitary group-theoretical realization. The first was already modeled; the other three and the twelve exclusions are new decisions.*

2.1 Two forced pointed Morita reductions

Every source has a dimension-20 subring of type $(1^4, 2^4)$. There is a unique common involution b fixing the four dimension-two objects on both sides. On the dimension-five objects it acts freely, with identical left and right permutations. These are exact properties of all sixteen tensors.

A fixed simple trivializes the pointed associator on $\langle b \rangle$, so $A = 1 + b$ is an actual separable algebra. The cyclic stabilizer argument from the previous round makes the dimension-20 subcategory pointed under A -bimodule duality. Briefly, a fixed dimension-two simple has an A -module structure because $H^2(C_2, \mathbb{C}^\times) = 0$. Its internal end has dimension two and contains the unit; hence it is supported on $1 + b$. The same vanishing supplies a commuting left action. The free dimension-five orbits each give two dimension-five bimodules. Thus the full dual $\mathcal{D}$ has type $(1^{20}, 5^4)$.

The invertible group of $\mathcal{D}$ has a unique Sylow subgroup $H = C_5$. It acts trivially on the four other simples on both sides. The algebra $B = \bigoplus_{h \in H} h$ therefore exists. For a simple right B -module with underlying dimension-five object, the internal end has dimension $25/5 = 5$ and contains the unit. It cannot contain a noninvertible simple. Its invertible support is a subgroup of order five and hence equals H . This proves that a commuting left action exists; it does not assume that the original object's full stabilizer had order five.

Each of the four dimension-five objects gives $5 \cdot 5$ invertible bimodules. The pointed sector gives twenty more. Consequently $\mathcal{E} = {}_B \mathcal{D}_B$ is pointed of dimension 120. This proves group-theoreticity for any categorification, with no unitary or spherical hypothesis.

The reverse algebra for the first dual is $R = 1 + c$, where c has order two and lies outside H . The actual free B - R bimodule $B \otimes R$ has ten distinct invertible constituents and transitive support, so is simple. Its internal end in $\mathcal{E}$ has dimension $10^2/(5 \cdot 2) = 10$. The original category therefore has the form

$$\mathcal{C}(G, \omega, K, \psi), \quad |G| = 120, \quad |K| = 10.$$

The subgroup order is obtained from an actual algebra, not guessed from a dimension-divisibility condition.

2.2 The group and subgroup are forced

If $K = C_{10}$, the identity double coset supplies ten invertibles, which is impossible. Therefore $K = D_{10}$, where the subscript denotes group order. Its Schur multiplier is trivial. Each normalizer

coset supplies degrees $(1, 1, 2, 2)$, so the normalizer has order twenty. The two remaining double cosets have size fifty and stabilizer C_2 ; these supply the four dimension-five objects.

If G were solvable, a Hall subgroup of order forty containing K would be a union of K -double cosets. It can contain none of size fifty, and would have to lie in the order-twenty normalizer: a contradiction. The nonsolvable groups of order 120 are $A_5 \times C_2$, S_5 and $SL(2, 5)$. The last has no D_{10} subgroup. In S_5 , or for the trivial graph $D_{10} \subset A_5 \times C_2$, an induced faithful C_2 -grading has neutral type $(1^2, 2^2, 5^2)$. A dimension-five simple times its dual has dimension 25, larger than the sum $2 + 4 + 10 = 16$ of all neutral simple dimensions. Such a product cannot be multiplicity-free. Only the reflection-sign graph remains:

$$G = A_5 \times C_2, \quad K = \{(h, \text{sgn}_D(h)) : h \in D_{10}\}.$$

2.3 Exhaustive cohomology and the four outputs

The admissible classes form the kernel of restriction from $H^3(G, \mathbb{C}^\times)$ to $H^3(K, \mathbb{C}^\times)$. Its 5-primary part vanishes because the subgroup index twelve is prime to five. Its 3-primary part is C_3 and does not change the fusion ring, by the lemma below.

The ambient 2-primary part is C_2^3 . Restriction to $V_4 \times C_2$ is injective, and the normalizer C_3 has a three-dimensional invariant subspace in its degree-three cohomology. Three independent global classes attain this bound: u , the transfer of the anisotropic V_4 cocycle; v , the external C_2 cube; and w , the cup product of the binary-icosahedral Schur cocycle with the external character. Each restricts to the same nonzero class on K . The relative 2-primary part is thus generated by $u + v$ and $w + v$.

Explicit cochain gauges make these two cocycles identically trivial on K , so the subgroup algebra has cochain $\psi = 1$. There is no further subgroup-cochain parameter because $H^2(D_{10}, \mathbb{C}^\times) = 0$.

Coefficient of $u + v$	Coefficient of $w + v$	Exact tensor
0	0	R12_1_25
0	1	R12_1_142
1	0	R12_1_361
1	1	R12_1_444

Lemma 2 (Coprime primary twists preserve the based fusion ring). *Let $p \nmid |K|$. A p -primary change of the associator of a group-theoretical category $\mathcal{C}(G, \omega, K, \psi)$, with a compatible p -primary subgroup trivialization, does not change its based fusion ring.*

Proof. Choose finite roots of unity for the data and a complete strictly henselian discrete valuation ring of mixed characteristic $(0, p)$ containing them. The p -power roots reduce to one. Every double-coset stabilizer has order dividing $|K|$, so its twisted group algebra is separable over this ring and has split semisimple residue fiber. Its simple modules lift with the same ranks. The balanced tensor product and the equivariant Hom spaces are direct summands furnished by averaging idempotents whose denominators divide $|K|$. Their ranks, hence all fusion coefficients, are unchanged under reduction. Both associators have identical reductions. The argument needs $p \nmid |K|$, not $p \nmid |G|$. $\square$

An independent Mackey-formula proof is also included in the archive. The four exact constructions, together with this categorical converse, prove the twelve exclusions. A failed construction search alone would not establish the theorem.

3 Two twisted Ising models in dimension 32

The same construction principle gives R12_1_240 and R12_1_408. Let $\mathcal{I}$ be a unitary Ising category, with objects $1, \psi, \sigma$ and $\sigma^2 = 1 + \psi$. For a group E of order sixteen, a normalized unitary cocycle ω , and a grading $\epsilon : E \rightarrow C_2$, take the matched-grading fusion subcategory

$$\mathcal{B} = \{(X, g) \in \mathcal{I} \boxtimes \text{Vec}_E^\omega : \deg X = \epsilon(g)\}.$$

It has type $(1^{16}, (\sqrt{2})^8)$ and dimension 32. If s is an even involution and ω restricts trivially to $\langle s \rangle$, then $A = 1 + (1, s)$ is a connected separable algebra. The category ${}_A\mathcal{B}_A$ is an actual unitary Morita model.

For R12_1_240, use $E = D_{16}$, with

$$(a, b)(c, d) = ([a + (-1)^b c]_8, b + d), \quad \epsilon(a, b) = a \bmod 2,$$

and $s = (0, 1)$. A suitable cocycle is

$$\omega((a, b), (c, d), (e, f)) = \exp\left(\frac{2\pi i}{8} a(-1)^b \frac{c + (-1)^d e - [c + (-1)^d e]_8}{8}\right).$$

It is the cup product of the crossed rotation coordinate with its twisted integral Bockstein; the two sign coefficient systems cancel. This proves the cocycle identity conceptually. All $16^4 = 65\,536$ identities are also checked exactly.

For R12_1_408, replace the group by the semidihedral group $E = \langle r, s : r^8 = s^2 = 1, srs^{-1} = r^3 \rangle$. The supplied sixteenth-root cocycle is an exact lift of a normalized modulo-eight cocycle. Its full table and all lift data are frozen; every cocycle identity and the restriction to the chosen involution are checked. This uses an actual pointed category and an actual subgroup algebra, not a presumed lift of a fusion-ring symmetry.

Both bimodule categories have type $(1^8, 2^2, (\sqrt{8})^2)$. The primary constructor computes exact balanced products; an independent implementation computes full grade-preserving Hom spaces. Both match all 1728 coefficients to the original tensors. The Ising category supplies the full associator, including its three- σ matrix block, although that block is not needed for these subgroup-bimodule operations.

4 Two models from relative cocycle adaptation

The following elementary device makes previously unusable cocycles usable.

Lemma 3 (Adaptation by a retraction). *Suppose $q : G \rightarrow Q$ admits a group-homomorphic section s with image H , and $p : G \rightarrow L$ is a homomorphism. For a normalized additive three-cocycle v on L , the cochain*

$$\omega = p^*v - (psq)^*v$$

is a normalized three-cocycle identically zero on H . Thus the ordinary subgroup algebra of H exists in $\text{Vec}_G^{\exp(2\pi i\omega)}$.

Proof. Both terms are pullbacks along homomorphisms and hence cocycles. For inputs in H , the retraction sq is the identity, so the terms cancel. The ordinary subgroup multiplication is therefore associative; the usual averaging splitting proves separability. $\square$

For R12_1_414, set

$$G_0 = S_4 \times C_2, \quad H_0 = \langle ((34), 0), ((12), 1) \rangle.$$

The map $(\text{sgn}, \text{id}) : G_0 \rightarrow C_2^2$ restricts to an isomorphism on H_0 . For R12_1_531, set

$$G_1 = S_4 \times_{C_2} C_4, \quad H_1 = \langle ((1324), 1) \rangle.$$

Projection to C_4 restricts to an isomorphism on H_1 . In both cases, use projection to S_4 for p , and a fourth-root cocycle v transferred from a Sylow D_8 subgroup of S_4 . The resulting relative cocycle and the trivial subgroup cochain give the required unitary categories.

The actual cocycle tables, group tables, subgroup embeddings, simple bimodule actions and basis identifications are supplied. Both replay implementations check all $48^4 = 5\,308\,416$ cocycle identities, the subgroup algebra, and all 1728 fusion coefficients. The origin by transfer explains discovery; existence does not depend on an unproved claim that the chosen transfer generators exhaust cohomology.

These use standard categorical constructions. The specific cocycles and census identifications are recorded here without a claim of novelty up to tensor equivalence.

5 A general-complex exclusion for R12_1_350

Theorem 4. *The ring R12_1_350 admits no complex fusion categorification.*

Proof with finite exact certificate. Its basis is $g, g\rho$ for $g \in S_3$, where ρ is central and

$$\rho^2 = 1 + \left(\sum_{g \in S_3} g \right) \rho, \quad d = \text{FPdim}(\rho) = 3 + \sqrt{10}.$$

Published Lemma 3.1 of Schopieray [2] gives a pseudo-unitary Galois conjugate of any categorification whose simple FP dimensions lie in $\{1, d\}$. It does not assume commutativity, rank six, or an initial pivotal structure. We may therefore work with a positive spherical categorification, and write $D = 120 + 36\sqrt{10}$.

The six irreducible representations of the complex based algebra have degrees $(1, 1, 1, 1, 2, 2)$ and formal codegrees

$$D, D' = 120 - 36\sqrt{10}, 12, 12, 6, 6.$$

Accordingly, the six distinct central simples in $I(1)$ have dimensions $1, d^2, 1+3d, 1+3d, 2+6d, 2+6d$ and unit restriction coefficients $1, 1, 1, 1, 2, 2$. Their noninvertible restriction coefficients are constant on the three conjugacy classes of sizes $2, 3, 1$.

The exact induction Gram matrix is $B_{ij} = \sum_{a,k} N_{ai}^k N_{ka*}^j$. Its unit column has value twelve at the unit, zero at other invertibles, and six at each noninvertible. The dimension and unit-column conditions leave exactly thirty ordered low restriction frames. Twenty-six have an exact negative Gram witness. Four survive.

For each survivor, possible remaining restriction columns are bounded using nonnegativity, the full Gram kernel, the rank-one PSD condition, the d -number condition on their central dimensions z , and algebraic integrality of D/z . This yields respectively 30, 43, 43, 28 columns. Centrality justifies compression to the five nonunit conjugacy classes; no unproved invariance of arbitrary noncentral objects is assumed.

The categorical Cauchy theorem restricts twist-exponent primes to $2, 3, 5$, the prime divisors of $DD' = 1440$. The dimension field has conductor forty. The first six normalized relative traces of all allowed roots of unity form a finite set of 45 tuples. Orders dividing 7200, together with the all-zero tuple for higher prime-power layers, cover this entire trace image. This is not a bound imposed on the true exponent.

Duality and adjoint parity fix the second indicators of noninvertibles. The three invertible transpositions retain all eight independent sign choices, even if additional categorical constraints could relate them. The Ng–Schauenburg center trace identities [9] and the rotation-space bounds

$$-\frac{[X^j : 1]}{p-1} \leq \overline{\nu_j(X)} \leq [X^j : 1], \quad j = 3, 4, 5, 6,$$

where p is the least prime divisor of j , give an exact rational nonnegative linear system for every surviving frame and sign assignment.

For each of these $4 \cdot 8 = 32$ systems $Ax = b$, the archive supplies an integer vector y with $A^\top y \geq 0$ and $b^\top y < 0$. Thus no nonnegative solution exists. An independent implementation reconstructs every system and checks every inequality using integers and rational arithmetic, with no optimizer or imported rank-six code. The initial Galois reduction proves the general-complex conclusion. $\square$

6 What remains open, and what was learned

6.1 Dimension 32: eighteen rings

The Ising-pointed construction resolves two cases. Checks of actual soft tensor actions on the Choi–Lu–Sun dimension-16 categories recover known controls, not further targets. Their coherence was checked, but the restricted tensorator ansatz is not a classification of all actions. A possible common isotropic involution reduction does not yet exhaust all Morita sources: the reverse involution can fix noninvertibles and can lie inside their stabilizer. These branches remain in the problem.

6.2 Dimension 48: twenty rings

The untwisted group-theoretical search was expanded to all 52 groups of order 48 and all 1722 conjugacy classes of subgroups. Subgroup-cochain twists and additional relative cocycles were examined. The two models above are the accepted new outputs. This is not a proof that all possible associators were exhausted, nor a theorem that every hypothetical dimension-48 categorification is group-theoretical. The outstanding cohomology and categorical converse gaps are recorded explicitly.

6.3 Dimension 144: one Morita branch remains

All four rings `R12_1_180`, `R12_1_181`, `R12_1_320`, `R12_1_321` remain unresolved. The first actual $1 + z$ condensation has integral type

$$1^6, 3^2, 2^{4t}, 4^{3-t}, \quad 0 \leq t \leq 3.$$

Its pointed group is S_3 . The normal C_3 cannot fix a dimension-two or dimension-four simple, since the corresponding internal end would have nonintegral dimension $4/3$ or $16/3$. Orbit divisibility forces $t = 0$ or 3 .

The $t = 0$ branch is now excluded. A second actual C_3 condensation would give integral type $(1^{24}, 4^3)$ and four odd simples of dimension $3\sqrt{2}$. A formal-codegree argument forces a dimension-24 near-group subcategory of type $\text{NG}(H_8, 2)$. The rank-four odd module has normalized dimensions $\sqrt{6}$. The applicable near-group classification is legitimate here: dimension 24 forces solvability and hence unitarizability [3, 4, 5].

The group-theoretical module formula forces the untwisted D_8 near-group case, with restricted module dual $\text{Rep}(S_4)$. Both subgroup Schur-cochain classes are retained. Clifford theory [6] then makes its full module dual a C_3 -equivariantization of $\text{Rep}(S_4)$. Every such action fixes the five simple isomorphism classes, giving type $(1^6, 2^3, 3^6)$. This contradicts the required dual type $(1^{24}, 4^3)$.

Consequently the unique possible first-dual type is

$$1^6, 2^{12}, 3^2, (3\sqrt{2})^4.$$

The original rank-three subcategory is also proved tensor-equivalent to ordinary $\text{Rep}(S_3)$, so its dimension-six regular algebra exists. A central embedding, the remaining projective stabilizer classes, and the required categorical symmetry lifts are not yet established. Eliminating one Morita branch is not an exclusion of the four rings.

6.4 The four remaining quadratic rings

The unresolved quadratic IDs are R12_1_332, R12_1_561, R12_1_623 and R12_1_628. For the two commutative cases R12_1_561 and R12_1_628, a central Tannakian C_3 lift would yield forbidden rank-four quotients. This rules out that lift; it does not rule out the original rings. For R12_1_623, the higher-indicator calculation forces second indicator -1 for its pointed involution in any pseudo-unitary realization, so the ordinary algebra $1 + z$ cannot supply a model there. The opposite-sign control agrees with the known R12_1_326 category. Surviving indicator branches for R12_1_623 remain. Neither feasible moment data nor failure to find a coherent action has been treated as a categorical model or exclusion.

7 Evidence, replay, and publication boundary

The archive contains the full 632-record ledger, hashes, new decisions, the 46 unresolved tensors, detailed family proofs, exact model data, integer exclusion rays, independent replay programs and their receipts. Every accepted new model is checked against all 1728 input coefficients. The principal replay programs are tested in normal and optimized Python; their checks do not disappear when assertions are disabled.

The finite replays verify the encoded arithmetic. The categorical converses, scope reductions and coverage arguments are supplied as mathematical proofs and were separately reviewed within the project. That distinction is essential: a correct scalar computation cannot replace a missing categorical premise.

The GitHub repositories and public website have not been revised. The prior round remains a frozen baseline; this package is a new offline continuation prepared for review.

References

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