

Rank-twelve multiplicity-one fusion rings

Audit and theoretical continuation

Offline research record

25 October 2026

AI-generated research; independent human verification remains pending.

1 Results and audit conclusion

For the same 632 supplied based rings, the updated decisions are

$$632 = 425 \text{ excluded} + 141 \text{ unitarily modeled} + 66 \text{ unresolved.}$$

The first round recorded 393 exclusions, 127 unitary realizations and 112 unresolved rings. The mathematical and implementation audit found no erroneous first-round decision requiring retraction. The continuation resolves 32 further rings by general complex obstructions and 14 by actual unitary constructions. These counts concern based rings admitting categorifications, not tensor-equivalence classes of categories. No repository or website has been revised.

The original input order and IDs are preserved. The decompressed input SHA-256 is

6949e6fb8afd23f80b54e5a546a0dd34257430fd26c26b7acd8ff8444252f4d7.

The original verification of the fusion axioms and pairwise nonisomorphism is retained. The present evidence includes every one of the 632 updated decisions, complete unresolved tensors, construction data, exclusion certificates and independent audits.

What was checked in the first round

An independent exclusion implementation, importing none of the first-round criterion implementations, recovered exactly the 393 excluded IDs. It recomputed the arithmetic and near-group premises, all 208 cyclic witnesses, the 12 field/order contradictions, the six integral primary forms, all 4012 closed subring embeddings and the inherited negative proofs. In particular, the Liptrap negatives cancel arbitrary nonzero complex variables for every nondegenerate bicharacter assignment; failure of a finite phase search was not used as a nonexistence argument. The rank-eight signed descent was also checked with all 8192 unrestricted sign assignments, before indicator normalization.

The positive audit independently bound all 381 ordinary full-tensor identifications and the 21 Liptrap reconstructions, accounting for all 402 construction records for 127 rings. It examined the categorical premises: actual tensor actions, central lifts, zesting coherence, positive Fourier blocks and balanced bimodule products. This is an audit of the recorded proofs and exact evidence, not a formalization of the cited literature. Normal and assertion-disabled replays pass.

A reference-version issue was checked explicitly: the published two-dimension reduction [4, Lemma 3.1] gives pseudo-unitarity *after Galois conjugation*. Literal pseudo-unitarity of every original category would be false (Yang–Lee is a control). The first round already used the published, correct formulation, so no classification change arose from this issue. Its restricted zest sampler was incomplete as a discovery method, but its accepted models remain valid.

2 A subgroup integrality criterion

The following formulation covers noncyclic stabilizers without assuming that independently chosen projective stabilizer cochains coincide.

Theorem 2.1. *Let $\mathcal{C}$ be a complex fusion category, let X, Y, Z be simple objects, and let H be a finite subgroup of its invertible objects. Suppose*

$$hX \cong X, \quad Yh \cong Y, \quad hZ \cong Zh \cong Z \quad (h \in H), \quad m = [Z : X \otimes Y] > 0.$$

Then $m \text{FPdim}(Z)/|H|$ is an algebraic integer.

Proof. Choose isomorphisms $hX \rightarrow X$. Since X is simple, their composition constants define a connected twisted subgroup algebra A with underlying object $\bigoplus_{h \in H} h$, making X a left A -module. The module pentagon supplies the associativity equation, including the ambient pointed associator. Similarly, Y supplies a possibly different subgroup algebra B and a right B -module. These algebras are separable in characteristic zero and both have FP dimension $|H|$.

The commuting outer actions make $X \otimes Y$ an A – B bimodule. Because H fixes Z on both sides, its full isotypic summand $M = mZ$ is stable under both actions. Taking its Morita dual and the balanced product gives an object of $\text{Bimod}_{\mathcal{C}}(A)$ of FP dimension

$$\frac{\text{FPdim}_{\mathcal{C}}(M \otimes_B M^\vee)}{\text{FPdim}(A)} = \frac{m^2 \text{FPdim}(Z)^2}{\text{FPdim}(A) \text{FPdim}(B)} = \left(\frac{m \text{FPdim}(Z)}{|H|} \right)^2.$$

We use the standard relative dimension formula and bimodule normalization [1, Chapter 7]; see also [3, Corollary 3.32 and Proposition 3.34]. Thus this square is an algebraic integer. The number itself is integral over the algebraic integers, so transitivity of integrality proves the claim. $\square$

For the paired product $X \otimes X^*$ one can use the same algebra on both sides: the transpose action makes X^* a right A -module, and mZ itself has normalized FP dimension $m \text{FPdim}(Z)/|H|$. Neither proof needs a pivotal, spherical or unitary structure. The two-algebra argument explains why the general statement does not require identifying independent noncyclic stabilizer cocycles.

The complete scan of all 632 supplied rings uses the intersection $\text{Stab}_L(X) \cap \text{Stab}_R(Y) \cap \text{Stab}_L(Z) \cap \text{Stab}_R(Z)$. It conflicts with no accepted model and adds precisely the following sixteen exclusions.

Proposition 2.2. *The sixteen rings with suffixes*

$$32, 131, 152, 184, 217, 245, 286, 290, 297, 302, 339, 411, 416, 458, 465, 527$$

have no complex fusion categorification.

Proof. Put $a = 2 \cos(\pi/7)$ and $b = a^2 - 1$. The exact dimension character is $1^{\times 8}, (2a)^{\times 2}, (2b)^{\times 2}$, with $a^3 - a^2 - 2a + 1 = 0$. The certificate selects a simple X of dimension $2b$ whose order-four left stabilizer also fixes it on the right, and with $[X : XX^*] = 1$. Theorem 2.1 would make $b/2$ integral. Its monic minimal polynomial is

$$t^3 - t^2 - \frac{1}{4}t + \frac{1}{8}, \quad N_{\mathbb{Q}(a)/\mathbb{Q}}(b/2) = -\frac{1}{8}.$$

The nonintegral rational norm is impossible for an algebraic integer. $\square$

3 Morita enlargement excludes four more cubic rings

We first prove a small lemma about every category with the indicated fusion rules, rather than imposing equivalence to a particular known model.

Lemma 3.1. *Suppose $\mathcal{B}$ has simples $1, g, T$ with $g^2 = 1$, $gT = Tg = T$ and $T^2 = 1 + g + T$. For the subgroup algebra $A = 1 + g$ supplied by T , the dual category $\text{Bimod}_{\mathcal{B}}(A)$ is pointed of dimension six.*

Proof. The left A -module category has exactly three simples $M_0 = A, M_+, M_-$, all with underlying dimension two. The last two are the two normalized g -actions on T . The two invertible A -bimodules supported on $1 + g$ are A and $u = A_\sigma$, where σ changes the sign on g . The action of u fixes M_0 and swaps M_+, M_- .

Every other simple bimodule is supported on qT and has dimension q in the dual. Since the total dimension is six, the sole nonpointed alternative is a unique dimension-two simple Q . It satisfies $Q^* = Q$ and $uQ = Q$. Therefore $QM_0 = M_+ + M_-$. Reciprocity makes the coefficient of M_0 in QM_+ exactly one. All three module dimensions agree, so QM_+ has total multiplicity two. Its u -invariance would require equal multiplicities of M_+, M_- . The remaining odd total multiplicity cannot be distributed between that pair, a contradiction. $\square$

Proposition 3.2. *The four rings $R12_1_38, R12_1_39, R12_1_218, R12_1_219$ have no complex fusion categorification.*

Proof. Put $b = 2\cos(\pi/9)$, $x = 1 + 2b$ and $y = 2b^2 - 2$, so $b^3 - 3b - 1 = 0$. The source type is $1^{\times 4}, 2^{\times 2}, x^{\times 4}, y^{\times 2}$. Its dimension-twelve integral subcategory $\mathcal{B}_0$ contains the subcategory of Lemma 3.1, with labels

Ring suffix	g	T	h outside $\mathcal{B}$
38,39	11	10	1
218	7	11	3
219	6	11	3

Here h is invertible and normalizes $\langle g \rangle$. Its conjugate subgroup algebra is isomorphic to A : two multiplications on $1 + g$ differ by rescaling g . Thus hA has an invertible bimodule structure outside the pointed dimension-six dual. Consequently $\text{Bimod}_{\mathcal{B}_0}(A)$ is pointed of order twelve.

The six nonintegral source simples are paired freely by g , with equal left/right label actions. Each pair gives two simple bimodules, since its diagonal C_2 stabilizer has only ordinary one-dimensional projective representations over $\mathbb{C}$. Thus the full dual has type $1^{\times 12}, x^{\times 4}, y^{\times 2}$. No multiplicity-one assertion about this dual is needed.

The numbers $1, x, y$ are rationally independent and $y^2 = 6 + 2x + 2y$. For a y -dimensional simple Y , its left and right stabilizers therefore both have order six inside the twelve invertibles. Their intersection contains a subgroup H of order three. Each stabilizer fixes both y -dimensional labels, because their orbits have at most two elements. Some y -dimensional Z occurs in YY^* with multiplicity $m = 1$ or 2 . Theorem 2.1 requires $my/3$ to be integral, whereas

$$N_{\mathbb{Q}(b)/\mathbb{Q}}(y) = -24, \quad N_{\mathbb{Q}(b)/\mathbb{Q}}(my/3) = -8m^3/9 \notin \mathbb{Z}.$$

This contradicts integrality in both cases. $\square$

4 Four quadratic exclusions via a linear sign reduction

The rings R12_1_63, R12_1_64, R12_1_215, R12_1_365 have exact dimensions

$$1^{\times 2}, \quad d^{\times 8}, \quad (d+1)^{\times 2}, \quad d = 4 + 3\sqrt{2}, \quad d^2 = 8d + 2.$$

Their common Casimir polynomial is

$$(t - 36)(t - 18)^8(t - 4)(t^2 - 720t + 2592).$$

We justify sphericity before using any spherical character obstruction.

Lemma 4.1. *In a pivotal category, a self-dual simple with second indicator -1 has even multiplicity in every $Y \otimes Y^*$.*

Proof. The object $M = Y \otimes Y^*$ has the symmetric self-duality $J = j_Y \otimes \text{id}_{Y^*} : M \rightarrow M^*$, where j is the pivotal structure. The identity $J^*j_M = J$ follows from $j_{Y^*} = (j_Y^*)^{-1}$. On a self-dual simple's isotypic part, write $J = \phi \otimes B$. Transposition gives $B^T = \nu_2(\phi)B$. For indicator -1 , the nondegenerate multiplicity pairing is alternating, so its dimension is even. $\square$

Choose dual-compatible lifts in the canonical sphericalization and the positive-second-indicator lift for each self-dual simple. Such a simple has self-dual lifts: exchanging them by duality would equate opposite nonzero spherical dimensions. Tensoring with the central sign object changes the second indicator's sign. Writing the signed quotient coefficient as $T = a_+ - a_-$, Lemma 4.1 gives $N_{Y,Y^*}^S - T_{Y,Y^*}^S \equiv 0 \pmod{4}$ for the selected positive lift S .

Every nonzero coefficient is $(-1)^v$ for a bit v . Cyclic trace and dual reversal identify its variable orbit. Associativity modulo four is linear over $\mathbb{F}_2$: in each nonzero associativity term, add the two coefficient bits, and equate the sum on both sides. Ordinary associativity already equates the term counts. Together with the preceding parity conditions this gives the following short, complete reduction:

Suffix	Variables	Kernel dimension	Linear choices	Associative choices
63	189	1	2	2
64	190	1	2	2
215	179	2	4	4
365	169	4	16	8

Every surviving signed tensor is a change of simple basis signs of N , with unit sign one and equal signs on dual pairs. The exact gauges are recorded. Changing the selected lifts accordingly makes all their products positive. These lifts form a full spherical tensor subcategory equivalent to the original category. Thus any hypothetical categorification is spherical. This uses a rank-independent categorical lemma and new small linear systems; the frozen rank-eight engine is not silently enlarged.

We compute all one-dimensional characters, including in the noncommutative cases. The common annihilator of the regular commutators is exactly the sum of the one-dimensional simple blocks. Restricting the fusion matrices there and choosing a square-free separating element gives every character orbit by exact polynomial coordinates. An independent audit instead uses the commutator-ideal quotient.

A spherical character has no zero simple dimensions and takes equal values on dual pairs. Apart from its FP quadratic orbit, every surviving character field has degree one or three. But all formal codegrees must lie in that field: the distinguished central simple has dimension $\dim(\mathcal{C})/f_E$ and forgetting it is an integer sum of source simple dimensions [2]. The Casimir quadratic $t^2 - 720t + 2592$ cannot split in a field of odd degree. These branches are impossible.

In the remaining quadratic orbit the dimension order is $\mathbb{Z} + \mathbb{Z}d$ and $D = 24 + 84d$. The Casimir root 36 has multiplicity one, so its irreducible representation degree is one and it is the actual formal codegree 36. Its central simple would have dimension

$$D/36 = \frac{2}{3} + \frac{7}{3}d = 10 + 7\sqrt{2} \notin \mathbb{Z} + \mathbb{Z}d.$$

The same order coordinates are nonintegral in the other embedding. This excludes every spherical character, and therefore every complex categorification of the four rings.

5 A complete resolution of the dimension-forty family

All nine remaining rings of type $1^{\times 4}, 2^{\times 4}, (\sqrt{5})^{\times 4}$ are now settled. Three are modeled, and six are excluded by a converse covering every possible associator, rather than by a failed finite construction search.

Lemma 5.1. *Let every noninvertible simple of an integral fusion category have dimension $n > 1$ and the same left and right cyclic stabilizer H of order n . Then its regular subgroup algebra A exists and its A -bimodule dual is pointed.*

Proof. A fixed simple trivializes the pointed associator on H . Also, H is normal, since conjugation sends a common stabilizer to another such stabilizer. Every noninvertible simple X has n simple right A -module structures because $H^2(H, \mathbb{C}^*) = 0$. For any one, M , its internal endomorphism object has FP dimension $n^2/n = n$ and contains the unit once. Since every noninvertible has dimension n , this internal end consists only of invertibles. Each constituent fixes X , hence belongs to H , and occurs at most once. It is therefore exactly $\bigoplus_{h \in H} h$. The cyclic action can be normalized to give a commuting left A -action; its n character choices yield n^2 simple bimodules over X , each of normalized dimension one. An invertible source orbit has size n and cyclic diagonal stabilizer, again giving only normalized dimension-one bimodules. This proves pointedness. The internal-end mechanism also appears in [6, Lemmas 6.1–6.2]. $\square$

In all nine source rings, the integral subcategory $\mathcal{E}$ has type $1^{\times 4}, 2^{\times 4}$ and a common two-element stabilizer $\{1, g\}$ for its dimension-two objects. The element g is central in the entire based ring and acts freely on the four irrational-dimensional objects. Thus $A = 1 + g$ exists, and Lemma 5.1 makes $\mathcal{D} = \text{Bimod}_{\mathcal{C}}(A)$ a category with twenty invertibles and four $\sqrt{5}$ -dimensional simples. It is a genuine generalized TY, or feudal, category. Its noninvertible products are pointed. The reverse Morita algebra is the actual internal end of the simple module A ; its dimension is two, so it is $1 + c$ for an involution c among the twenty invertibles.

Write $S = \text{Irr}(\mathcal{D}_{\text{pt}})$. In the reverse dual, the pointed integral part contributes exactly $|N_S(\langle c \rangle)|$ invertibles. The source has four, so this normalizer has order four. Among the five groups of order twenty, only $F_{20} = C_5 \rtimes_2 C_4$ and $D_{10} \times C_2$ have noncentral involutions. For the latter group, the reverse dual integral ring is commutative for every pointed associator admitting its subgroup algebra: all nonnormal C_2 double cosets are inverse-stable with trivial stabilizer, so their dimension-two simples are self-dual. The normalizer quotient is represented by the extra central involution, giving equal involutive left/right invertible actions. Duality then makes the remaining products commute. All eight noncommutative source rings have noncommutative integral subrings. Their precursor must therefore have $S = F_{20}$.

The common stabilizer is the unique C_5 in F_{20} , and its associator is trivial because it fixes a noninvertible simple. Hence the neutral category for the universal grading is Vec_{C_5} . Its Brauer–Picard group is $O(C_5 \oplus \widehat{C}_5, q) \cong D_8$ [5]. The four invertible components give the faithful order-four diagonal subgroup, through the multiplier two on C_5 ; a rank-one noninvertible component gives a Fourier reflection. These generate the full group. Thus the universal grading is D_8 and its

homomorphism to the Brauer–Picard group is an isomorphism. Changing the chosen noninvertible component accounts for all reflection choices while preserving the distinguished reverse involution $c = r^2$.

There is also a direct Liptrap-data proof of this restriction. If $t_m \in \mathbb{F}_5^*$ is the stabilizer bicharacter parameter, biderivation and quasisymmetry give $t_{r^{-1}m} = 2t_m$ and $t_{m^*} = t_m$. Writing the right quotient action as an involution σ of C_4 and the base noninvertible square as p forces $\sigma = -\text{id}$ and $p = 0$. The argument permits arbitrary nonzero complex coherence variables and variable τ . It is not a bounded phase-search obstruction.

The extension classification [5] now applies to every hypothetical original precursor. The product-system freedom is $H^2(D_8, \text{Inv } Z(\text{Vec}_{C_5})) = H^2(D_8, C_5^2) = 0$ by coprime averaging. An explicit unitary precursor exists by its exact Liptrap data. Therefore all associators form its $H^3(D_8, \mathbb{C}^*)$ torsor. This group has exponent dividing four: restrict to an index-two V_4 , whose third cohomology has exponent two, then corestrict. Consequently every class has a fourth-root representative. The map $H^3(C_2, \mu_4) \rightarrow H^3(C_2, \mathbb{C}^*)$ is an isomorphism, so the actual reverse algebra condition is fully covered by the relative fourth-root classes trivial on $\langle c \rangle$. Its multiplication is unique up to isomorphism, since $H^2(C_2, \mathbb{C}^*) = 0$.

The normalized bar complex has 343 three-cochain variables and 2402 constraints, including the subgroup condition. Exact elimination gives a cocycle kernel of order 2^{87} and a normalized coboundary image of order 2^{83} : precisely sixteen relative fourth-root classes. Independent algorithms check the cardinality and all 120 representative-pair differences. These sixteen classes cover every relative complex class, although different fourth-root classes can become equivalent after extending coefficients.

For every representative the actual $1 + c$ bimodules and balanced products are constructed. Every resulting full tensor is exactly R12_1_156 or R12_1_263. Two independent implementations recompute these sixteen outputs. The coherent C_4 TY action separately realizes the commutative R12_1_449. Therefore

R12_1_309, R12_1_383, R12_1_467, R12_1_470, R12_1_501, R12_1_514 are excluded over $\mathbb{C}$.

This argument explicitly tracks the reverse algebra and all complex associator classes; an arbitrary original category is never identified with one chosen unitary model without the extension-classification proof.

6 The dimension-twenty-four family

Two exclusions follow directly from the general classification of invertible pointed bimodule categories, with no assumption on the pointed associator. Every component of a faithful fusion grading is an invertible bimodule category over its neutral component [5, Theorem 6.1]. Its left and right stabilizers must be isomorphic normal abelian groups: the mixed commutator is a nondegenerate bicharacter between them [7, Theorem 7.9 and Corollary 7.11].

The supplied tensors for R12_1_445 and R12_1_493 have a faithful C_3 grading with pointed neutral component of rank eight. In a nontrivial component, the exact stabilizers are

Ring	Simple	Left stabilizer	Right stabilizer
R12_1_445	3	$\{0, 9, 10, 11\} \cong V_4$	$\{0, 1, 2, 11\} \cong C_4$
R12_1_493	5	$\{0, 9, 10, 11\} \cong V_4$	$\{0, 3, 4, 11\} \cong C_4$

These groups are not isomorphic, so neither ring admits a complex categorification. All subgroup products, element orders and grading products are checked directly.

The remaining four originally unresolved rings in this family are modeled: R12_1_541 by a classical representation category, and R12_1_428, R12_1_513, R12_1_597 by explicit twisted subgroup bimodules in $\text{Vec}_{A_4 \times C_2}^\omega$ with a diagonal subgroup of order two. Two invariant relative trilinear cocycles give a unified construction; their complete values are retained, and every cocycle and subgroup-algebra identity is checked. The independent binary-tetrahedral realization of R12_1_597 is described below. Thus all six cases in the dimension-twenty-four family are settled.

7 Additional unitary models

The following table lists only accepted categorical constructions, each matched against all 1728 input coefficients. A proposed action or algebra whose existence has not been established is not counted.

Suffixes	Construction
25	$\mathcal{C}(A_5 \times C_2, 1, \text{graph } \chi, 1)$
30	$\mathcal{C}(S_4 \times C_2, 1, V_4, 1)$, $V_4 = \langle ((34), 0), ((12), 1) \rangle$
79,80	Full S_3 triality gaugings of $SO(8)_1$
156,263	C_2 Morita duals of explicitly reconstructed D_8 -graded categories over Vec_{C_5}
322,323	Bosonic associative C_2 zests of the two triality gaugings
417	$\mathcal{C}(S_4 \times_{C_2} C_4, 1, C_4, 1)$, $C_4 = \langle ((1324), 1) \rangle$
428,513	$\mathcal{C}(A_4 \times C_2, \omega, H, 1)$ for the stored relative cocycles and diagonal $H = C_2$
449	$\text{TY}(C_5, \chi)^{C_4}$ with the explicit weak inversion action
541	$\text{Rep}(C_3 \rtimes C_8)$ with inversion action
597	Twisted subgroup bimodules for $A_4 \times C_2$; independently also for $SL(2, 3)$

The graph-subgroup model at dimension 120

Let $G = A_5 \times C_2$, let $D_{10} \leq A_5$ be generated by a five-cycle and a reflection, and let $\chi : D_{10} \rightarrow C_2$ be the reflection-sign character. The actual subgroup $H = \{(h, \chi(h)) : h \in D_{10}\}$ has order ten. The connected separable subgroup algebra in Vec_G gives the unitary category $\mathcal{C}(G, 1, H, 1)$, realizing R12_1_25. Its double cosets and stabilizer characters give four invertibles, four dimension-two simples and four dimension-five simples. Direct convolution and an independent construction of all induced graded bimodules and balanced products give the same tensor. Unlike $D_{10} \subset S_5$, this graph subgroup surjects onto the extra C_2 factor; the unwanted quotient grading is absent.

A concrete twisted binary-tetrahedral model

The category realizing R12_1_597 is $\mathcal{C}(SL(2, 3), \omega, C_4, \psi)$. Here is a source-independent specification of the cocycle. Use $SL(2, 3)/\{\pm 1\} \cong V_4 \rtimes C_3$. For $v, w, u \in \mathbb{F}_2^2$ put

$$f(v, w, u) = v_0 w_0 u_0 + v_1 w_1 u_1 + v_0 w_1 u_1, \quad \bar{f} = \sum_{j=0}^2 f(A^j v, A^j w, A^j u),$$

where A is the order-three action. This is an invariant trilinear cocycle. On quotient coordinates $(v, a), (w, b), (u, c)$ define

$$\omega = (-1)^{\bar{f}(v, A^a w, A^{a+b} u)}$$

and inflate it to $SL(2, 3)$. The stored unit-modulus cochain ψ on the actual cyclic subgroup of order four satisfies its algebra equation exactly. Thus the algebra exists without an asserted abstract

classification of $H^3(SL(2, 3))$. Every group cocycle equation, subgroup equation, bimodule action and balanced tensor coefficient is checked. This is a positive construction; it makes no completeness claim about all group cocycles.

Triality, equivariantization and Morita constructions

The full S_3 gaugings of the three-fermion category $SO(8)_1$ give R12_1_79 and R12_1_80 [9, Section 6.2][10, Section VI.D]. These are actual unitary gauging constructions; the modular matrices are used to identify their resulting fusion rules, not to infer existence from modular data. Their sign-representation boson has an actual central lift and lies in the neutral component of the integral/nonintegral C_2 grading. Its associative zest gives R12_1_322 and R12_1_323.

For R12_1_449, inversion of C_5 preserves the bicharacter $\chi(a, b) = \zeta_5^{ab}$ of a unitary TY category. It defines a weak C_4 action with carry coherence $\kappa^{\lfloor (a+b)/4 \rfloor}$, where κ is one on pointed objects and -1 on the TY-odd simple. The strict inversion functor has order two; it is not claimed to have order four on labels. The invariant carry coherence supplies the genuine unitary C_4 action. Its equivariantization has type $1^{\times 4}, 2^{\times 4}, (\sqrt{5})^{\times 4}$ and the full target tensor.

The additional dimension-40 Morita models use a unitary Liptrap reconstruction over C_5 [8, Theorem 4 and Lemma 5.4], optionally twisted by a stored unit-modulus grading cocycle. The required order-two subgroup algebra and all associators entering its bimodule products are supplied explicitly. Independent checks evaluate the scalar pentagons, Fourier unitarity and balanced Hom spaces.

Finally, R12_1_541 is classical: $\text{Rep}(C_3 \rtimes C_8)$, with inversion action. There are eight linear characters r_k and four induced degree-two characters X_j , with

$$r_k r_l = r_{k+l}, \quad r_k X_j = X_{j+k}, \quad X_i X_j = r_{i+j} + r_{i+j+4} + X_{i+j}.$$

Subscripts on r and X are taken modulo eight and four, respectively. The source-to-model permutation is $(0, 2, 6, 1, 7, 9, 11, 5, 3, 8, 10, 4)$.

8 Remaining cases and precise limits

FP dimension multiset	Cases
$1^4, 2^4, 5^4$	15
$1^8, 2^2, 4^2$	22
$1^8, 2^2, (2\sqrt{2})^2$	20
$1^2, 2, 3^2, 4^3, (3\sqrt{2})^4$	4
$1^6, (3 + \sqrt{10})^6$	4
$1^6, ((3 + \sqrt{13})/2)^6$	1
Total	66

All 66 remaining IDs and tensors are supplied separately. Every case received structural analysis and relevant model/criterion attempts. This remains a partial categorification result, not a completed rank-twelve classification.

The dimension-48 central-quotient calculations identify compatible smaller rings among the supplied data. They do not prove uniqueness of their categorifications or tensor actions, and were not used as negative completeness claims. The four dimension-144 rings with suffixes 180,181,320,321 admit exact graded twists at the based-ring level, but the necessary coherent tensor actions have not been established. They remain unresolved.

For the five two-dimension cases with suffixes 332,350,561,623,628, the short primary probes gave no negative certificate. In case 332, the trivial outer automorphism group of a familiar unitary Haagerup model does not classify arbitrary original categorifications; twisted pointed associators must also be covered. In case 350, a conditional subgroup-dual reduction cannot be applied until its algebra exists in every relevant pointed associator class. Similar central-lift and algebra existence gaps are stated in the research notes. No such gap has been hidden by a successful scalar calculation or a failed construction search.

Known and proposed constructions are distinguished throughout. “New” here concerns the recorded argument or explicit construction, when it has not to our knowledge been explicitly recorded in the literature; no novelty up to tensor equivalence or exoticness is asserted. Independent human verification remains pending.

9 Reproducibility

The complete offline package preserves the first-round evidence and adds the independent audit, all new accepted proofs and constructions, and the updated JSON/CSV ledger. The top-level replay checks source bindings and every new decision; the independent mathematical audits and constructor-specific replays are included. Normal and optimized Python verification pass. Short exhaustive linear/cohomological systems are explicitly distinguished from unsuccessful or incomplete discovery attempts. No long frame search or numerical pentagon solving was used. Verification of identities is not presented as a substitute for the categorical existence and scope arguments above.

References

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