

Rank-twelve multiplicity-one fusion rings

Screening by established theoretical methods

Offline research record

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AI-generated research; independent human verification remains pending.

1 Result and scope

For the supplied list of 632 based rings, the present established-method pass gives

$$632 = 393 \text{ excluded} + 127 \text{ unitarily realized} + 112 \text{ unresolved.}$$

Every accepted exclusion concerns arbitrary fusion categories over $\mathbb{C}$. Every positive record supplies a categorical construction and an exact identification of all $12^3 = 1728$ structure constants. The numbers count based rings admitting a categorification, not tensor-equivalence classes of fusion categories. A necessary-test pass or an unsuccessful construction search is never counted as a realization.

No repository or website has been revised. The input census was not rerun. IDs R12_1_i refer to one-based physical input lines; basis labels in the code are zero-based, with unit label zero. The uncompressed input has SHA-256

6949e6fb8afd23f80b54e5a546a0dd34257430fd26c26b7acd8ff8444252f4d7.

All fusion axioms and pairwise nonisomorphism were checked. There are 374 commutative and 258 noncommutative rings, 204 weakly integral rings, and five pointed rings. A separate C++ verifier also found exactly 632 distinct associative rings and no duplicates.

Disjoint accounting of exclusions

Overlapping witnesses are retained. The following display assigns each excluded ring to its first listed route so that every ring is counted once.

Route	New exclusions
Arithmetic / near-group	234
Cyclic isotypic summands	113
Zero spectrum	0
Central field / dimension order	12
Primary-3 with solvability gate	6
Liptrap monomial obstruction	4
Replayed lower-rank exclusions	24
Total	393

The lower-rank transfer step replays six additional accepted results: the three Liptrap obstructions R10_1_118, 122, 180, the dimension-field/order obstructions R6_1_12, 13, and the general signed proof for R8_1_17. Each transfer checks product and dual closure and the full embedded tensor.

2 Theoretical exclusion checks

The inventory covers the established criterion families used here and explains their scope. The complete attempt records also retain tests that gave no conclusion.

Arithmetic and support.

Formal codegrees must be d-numbers [1]. The regular Casimir has eigenvalues $e_E f_E$, where e_E is an irreducible representation degree; multiplication by a rational integer preserves the d-number property. The code checks the monic minimal polynomial divisibilities exactly. Cyclotomicity is used for simple FP dimensions and, on commutative input, all fusion-matrix eigenvalues. For a noncommutative ring we do not assume that every regular-matrix eigenvalue is cyclotomic. Near-group and two-dimension restrictions use their precise hypotheses [3]. The zero- and one-spectrum tests are support consequences of the pentagon equation [4]; all accepted witnesses are reevaluated with integers.

Cyclic isotypic integrality.

Let g have order n , with $gX = X$, $Yg = Y$, and $gZ = Zg = Z$. If $m = N_{XY}^Z > 0$, then $m \text{FPdim}(Z)/n$ is an algebraic integer. Indeed, a fixed simple object trivializes the associator class on $\langle g \rangle$; its regular algebra A exists. The Z -isotypic summand of $X \otimes Y$ is an A -bimodule, whose FP dimension is $m \text{FPdim}(Z)/\text{FPdim}(A)$. This argument uses no pivotality. The present multiplicities are one. The checker isolates the Perron factor and proves nonintegrality of the scaled monic polynomial. There are 208 witnesses, including overlaps.

Subcategory index and central dimension order.

For every proper based subring, $\text{FPdim}(R)/\text{FPdim}(S)$ must be an algebraic integer [2]. Exact positive dimension characters and all 4012 proper embeddings are retained. For weakly integral rings, or rings with exactly two FP values after the Galois-pseudo-unitary reduction of [3, Lemma 3.1], a distinguished central simple has dimension D/f_E . Its forgotten dimension lies in the integer span of the source dimensions. Thus every Casimir root must belong to that dimension field. For a root $\lambda = e_E f_E$, the candidate $e_E D/\lambda$ must lie in the dimension order and have unit multiplicity e_E . Hermite normal form checks the order exactly; noncommutative regular multiplicities must be sums of allowed squares e_E^2 . No pseudo-unitary conclusion is drawn without an explicit bridge.

Unitary tests and their bridge.

The primary-3 matrix is

$$P_3 = \sum_i \frac{N_i \otimes N_i \otimes N_i}{\text{FPdim}(i)}.$$

It must be positive semidefinite for a unitary category [5]. Six accepted negative integer quadratic forms occur at IDs 91, 92, 197, 198, 220 and 221. These rings have FP dimension 216. Any categorification is solvable by [6, Theorem 1.6], hence unitarizable by [7, Theorem 5.20]. This is why the exclusions are general. The coefficients of each quadratic form are recomputed by independent integer tensor contractions; rational square-root intervals certify strict negativity. Floating-point eigensolvers only discover the vectors.

The inherited full- $I(1)$ quadratic-compression, reciprocal-codegree, global twist-moment, and short involution-module checks produced no additional disjoint exclusions. Their exact passes or no-conclusion records remain available. No Isaacs conjecture, braided-only classification, or proposed numerical modular datum is used as a general categorification obstruction.

3 Models and categorical hypotheses

The preferred recorded constructions are distributed as follows; alternative models are also retained. The replay checks 402 recorded model certificates/identifications.

Constructor	Rings
Deligne products	52
Associative C_2 zesting	23
Generalized Tambara–Yamagami (Liptrap)	14
Graded pullbacks with Ising	13
Twisted group-theoretical categories	5
TY equivariantizations	4
$SU(2)$ and $PSU(2)$	3
Pointed categories: nonabelian groups	3
Dihedral/dicyclic representation categories	2
Half even-metaplectic category	1
Odd metaplectic category	1
Metaplectic crossed product	1
Untwisted group-theoretical category	1
Generalized Haagerup C_6	1
Pointed equivariantization	1
Pointed category: abelian group	1
Tambara–Yamagami category	1
Total	127

The standard families include pointed categories for all five groups of order 12, $SU(2)_{11}$, $PSU(2)_{22}$, $PSU(2)_{23}$, dihedral and dicyclic representation categories, odd metaplectic and half even-metaplectic categories [11], Tambara–Yamagami categories, and all relevant products of the accepted lower-rank multiplicity-one models.

Equivariantization and zesting.

Pointed actions are actual group automorphisms. TY actions preserve an explicitly displayed nondegenerate symmetric bicharacter; both allowed parity coherences are checked. Fusion coefficients are computed by exact stabilizer Fourier projection [9]. A based-ring automorphism alone is not accepted as a tensor action. For associative C_2 zesting [10], the order-two current has a specified central lift and lies in the neutral component. The 2-cocycle is h^{ab} . For self-braiding +1 the scalar associator correction is trivial; for -1 it is i^{abc} , whose coboundary cancels $(-1)^{abcd}$. The retained records identify the relevant boson or fermion and the actual grading.

Graded pullbacks and crossed products.

For an accepted unitary rank-eight category with a C_2 grading having four simples in each component, take the full subcategory of its product with Ising on pairs of matching degrees. This gives twelve simples. Its existence follows directly from closure under tensor product and duality. Additional crossed products start with $TY(C_5)^{C_2}$: multiplying equivariance by the nontrivial character on the TY-odd component is an actual strict order-two tensor autoequivalence. The canonical central $\text{Rep}(C_2)$ boson is preserved, permitting the recorded crossed products and zests.

Group-theoretical categories.

The constructions are $\mathcal{C}(G, \omega, H, 1)$, the bimodule categories of the separable subgroup algebra in Vec_G^ω . Quotient cocycles are checked before use. For $H = C_2$, explicit graded bimodule actions and balanced relative tensor quotients are constructed with fourth-root phases. All left, right, compatibility, and quotient-descent equations are checked. Stabilizers are trivial or C_2 , so their one or two projective characters exhaust the simples. An independent rank-eight twisted-bimodule

control is reproduced exactly. The untwisted convolution and balanced-bimodule implementations also agree on their common cases.

Generalized Haagerup.

R12_1_326 is the standard generalized Haagerup C_6 based ring: $g_ag_b = g_{a+b}$, $g_aX_b = X_{a+b}$, $X_ag_b = X_{a-b}$, and $X_aX_b = g_{a-b} + \sum_c X_c$. Existence of actual unitary categories is recorded in [12, Example 4.5]; the numerical modular-data conjecture in that paper is not needed. The full input tensor is matched exactly. The two categorical examples in that reference count as one based ring here.

4 What remains unresolved

All 112 remaining cases are listed below by their exact FP dimension pattern. Superscripts indicate multiplicities of dimensions. Put $a = 2\cos(\pi/7)$ and $b = 2\cos(\pi/9)$, the largest roots of $t^3 - t^2 - 2t + 1$ and $t^3 - 3t - 1$, respectively. These patterns were verified as exact positive dimension characters, not inferred from rounded eigenvalues.

FP dimensions	Cases
$1^{\times 4}, 2^{\times 4}, 5^{\times 4}$	16
$1^{\times 8}, 2^{\times 2}, 4^{\times 2}$	24
$1^{\times 8}, (2a)^{\times 2}, (2a^2 - 2)^{\times 2}$	16
$1^{\times 4}, 2^{\times 2}, (1 + 2b)^{\times 4}, (2b^2 - 2)^{\times 2}$	4
$1^{\times 2}, (4 + 3\sqrt{2})^{\times 8}, (5 + 3\sqrt{2})^{\times 2}$	4
$1^{\times 2}, 2, 3^{\times 2}, 4^{\times 3}, (3\sqrt{2})^{\times 4}$	8
$1^{\times 8}, 2^{\times 2}, (2\sqrt{2})^{\times 2}$	20
$1^{\times 8}, 2^{\times 4}$	6
$1^{\times 4}, 2^{\times 4}, (\sqrt{5})^{\times 4}$	9
$1^{\times 6}, (3 + \sqrt{10})^{\times 6}$	4
$1^{\times 6}, ((3 + \sqrt{13})/2)^{\times 6}$	1
Total	112

The corresponding suffixes i in R12_1_i are:

Family key	Input suffixes
D120	25, 142, 161, 211, 262, 265, 312, 361, 362, 388, 444, 453, 469, 472, 503, 516
D48	30, 130, 146, 151, 175, 216, 239, 244, 277, 289, 296, 303, 319, 407, 410, 414, 417, 421, 424, 433, 523, 526, 531, 538
cubic_C7	32, 131, 152, 184, 217, 245, 286, 290, 297, 302, 339, 411, 416, 458, 465, 527
cubic_C9	38, 39, 218, 219
quadratic_C2	63, 64, 215, 365
D144	79, 80, 180, 181, 320, 321, 322, 323
D32	128, 144, 147, 173, 237, 240, 275, 285, 294, 301, 318, 408, 412, 415, 419, 422, 431, 524, 530, 536
D24	428, 445, 493, 513, 541, 597
D40	156, 263, 309, 383, 449, 467, 470, 501, 514
quadratic_C6_sqrt10	350, 561, 623, 628
quadratic_C6_sqrt13	332

The JSON and CSV ledgers supply every ID, tensor hash and route; the separate unresolved tensor file is ready for further work. There are only two simple based rings in this input: R12_1_116

is realized by $\text{PSU}(2)_{23}$ and R12_1_118 fails the d-number test (factor $t^2 - 935t + 1785$). This observation is not a proof of an unrestricted-rank classification.

5 Verification and explicit limitations

`python verify.py` and `python -O verify.py` both pass. All 632 input bindings, 4012 closed subring embeddings, accepted exclusion witnesses and all 402 recorded model certificates/identifications are checked. The accepted replay uses Python, NumPy, SymPy and NetworkX, takes seconds on the recorded machine, and requires neither a network service nor a discovery solver. The independent C++ input audit is retained separately. Algebraic-number arithmetic is shared infrastructure; this is not a Lean formalization or a substitute for independent human verification of the categorical references and inherited proofs.

The short module scan uses fixed rank at most four, at most two generators and at most 4096 generator assignments, with a short per-involution time limit. The exploratory commutative low-induction scan has column/combination node limits and a one-second continuation limit; an interrupted initial attempt is retained as research history, not a proof. Numerical primary searches can also give no conclusion. The inherited complete oriented signed engine has rank at most eight and is used only inside an already proved rank-eight subring, not silently extended to twelve.

No long induction-frame, conductor, modular-data, higher-indicator or pentagon search was undertaken. The inventory distinguishes these uncompleted systems from exact passes. The remaining list means unresolved by this established-method pass under the requested short-computation constraint. It is not a claim that every consequence of the literature or every possible established construction has been exhausted. The constructions here use established methods; no novelty up to tensor equivalence is claimed.

References

- [1] V. Ostrik, *On formal codegrees of fusion categories*, Math. Res. Lett. 16 (2009), 895–901. [arXiv:0810.3242](#).
- [2] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories*, AMS, 2015, especially Chapters 7–8.
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- [5] L. Huang, Z. Liu, S. Palcoux and J. Wu, *Complete Positivity of Comultiplication and Primary Criteria for Unitary Categorification*. [arXiv:2210.00792](#).
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- [7] C. Galindo, S.-M. Hong and E. C. Rowell, *Generalized and quasi-localizations of braid group representations*. [arXiv:1105.5048](#), Theorem 5.20.
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- [9] S. Burciu and S. Natale, *Fusion rules of equivariantizations of fusion categories*. [arXiv:1206.6625v2](#).
- [10] C. Delaney, C. Galindo, J. Y. Plavnik, E. C. Rowell and Q. Zhang, *Braided zesting and its applications*. [arXiv:2005.05544](#).
- [11] P. Gustafson, E. C. Rowell and Y. Ruan, *Metaplectic Categories, Gauging and Property F*. [arXiv:1808.00698](#), Sections 3.1 and 3.3.
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